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Some new results for power means

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Abstract

In this paper, we establish some new inequalities for power means with n positive numbers. Moreover, some new properties of $p \mapsto M_n(\mathbf{a}, p)$ are obtained, where $M_n(\mathbf{a}, p)$ denotes the p-th power mean of first n entry of vector $\mathbf{a}$.

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1. Introduction and Preliminaries

Given n positive numbers $a_1, a_2, \cdots, a_n$, denote $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, we recall that the classical arithmetic mean $A_n(\mathbf{a})$, the geometric mean $G_n(\mathbf{a})$, the harmonic mean $H_n(\mathbf{a})$, and finally $M_n(\mathbf{a}, p)$, the p-th power mean, are respectively defined by

$$A_n(\mathbf{a}) = \frac{\sum_{k=1}^{n} a_k}{n}, \quad G_n(\mathbf{a}) = \sqrt[n]{\prod_{k=1}^{n} a_k}, \quad H_n(\mathbf{a}) = \frac{n}{\sum_{k=1}^{n} \frac{1}{a_k}},$$

and

$$M_n(\mathbf{a}, p) = \begin{cases} \left(\frac{1}{n}\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}, & p \neq 0, \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & p = 0. \end{cases} \tag{1.1}$$

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We can see that $M_n(\mathbf{a},-1) = H_n(\mathbf{a})$, $M_n(\mathbf{a},0) = G_n(\mathbf{a})$, $M_n(\mathbf{a},1) = A_n(\mathbf{a})$. Then, multivariate means of classical arithmetic, geometric and harmonic are special cases of power mean, and the relations of these means can be written by next inequalities

$$H_n(\mathbf{a}) = M_n(\mathbf{a},-1) \le G_n(\mathbf{a}) = M_n(\mathbf{a},0) \le A_n(\mathbf{a}) = M_n(\mathbf{a},1) \tag{1.2}$$

To date, some excellent methods have been proposed to prove and establish inequalities. For example, in [9], Ibrahim and Dragomir established inequalities by utilizing power series and Young's inequality. In [10], Kouba used classical analysis to obtain some new inequalities. In [12], V. Mascioni discover new inequalities by the differences of some Stolarsky means. Due to the importance of power mean in modern mathematics, it has also been given considerable attention by mathematicians. Many remarkable results for power mean have been presented in the literature (see, for example, [2, 3, 4, 5, 6, 11, 14, 15, 16, 17] and the references cited therein).

The main purpose of this paper is to establish some new inequalities for $M_n(\mathbf{a},p)$ and to give some new properties for $p \mapsto \lim_{n\to\infty} M_n(\mathbf{a},p)/M_n(\mathbf{a},p+1)$ and $p \mapsto \lim_{n\to\infty} M_n(\mathbf{a},\frac{1}{p})/M_n(\mathbf{a},\frac{1}{p+1})$ in the case when $\mathbf{a}$ is an arithmetic sequence.

2. Some known results

In this section we restate some Lemmas and Theorems, which relate to our main results.

Lemma 2.1 (Hölder's inequality, [8]). *Let $a_k, b_k \ge 0$, for $k = 1,2,\cdots,n$, and let $\frac{1}{p}+\frac{1}{q} = 1$ with $p,q > 1$. Then*

$$\sum_{k=1}^{n} a_k b_k \le \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}}, \tag{2.1}$$

with equality if and only if $a_k^p = \tau b_k^q$ $(k = 1,2,\cdots,n)$, τ is constant.

The p-th power mean type Hölder's inequality can be described by the following Theorem.

Theorem 2.2 (Hölder's inequality [1], p.211). *If $a_k, b_k > 0$ $(k = 1,2,\cdots,n)$, and $\frac{1}{p}+\frac{1}{q} = 1$ with $p > 1$. Then*

$$M_n(\mathbf{ab},1) \le M_n(\mathbf{a},p) M_n(\mathbf{b},q), \tag{2.2}$$

where $\mathbf{a} = (a_1,a_2,\cdots,a_n)$, $\mathbf{b} = (b_1,b_2,\cdots,b_n)$, $\mathbf{ab} = (a_1b_1,a_2b_2,\cdots,a_nb_n)$.

Lemma 2.3 ([1], p.31). *Let $\psi(x)$ be twice differentiable and $\psi''(x) \ge 0$. Then*

$$\psi\Big(\frac{1}{n}\sum_{k=1}^{n} x_k\Big) \le \frac{1}{n}\sum_{k=1}^{n}\psi(x_k). \tag{2.3}$$

Lemma 2.4 (Minkowski's inequality, [13]). *Let $a_k, b_k \ge 0$ $(k = 1,2,\cdots,n)$ and $p > 1$. Then*

$$\Big[\sum_{k=1}^{n}(a_k+b_k)^p\Big]^{\frac{1}{p}} \le \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} b_k^p\Big)^{\frac{1}{p}}. \tag{2.4}$$

The inequality is reversed for $p < 1$ $(p \ne 0)$. In each case, the sign of equality holds if and only if $a_k = b_k$ $(k = 1,2,\cdots,n)$.

Lemma 2.5 (Jensen's inequality[7], p.28). *Let $a_k > 0$ $(k = 1,2,\cdots,n)$, $0 < r < s$. Then*

$$\Big(\sum_{k=1}^{n} a_k^s\Big)^{\frac{1}{s}} < \Big(\sum_{k=1}^{n} a_k^r\Big)^{\frac{1}{r}}. \tag{2.5}$$

Theorem 2.6 ([7], p.26). *Let $a_1, a_2, \cdots, a_n$ be a positive sequence, $n \in \mathbb{N}$. If $p \in \mathbb{R}$, then $M_n(\mathbf{a}, p)$ is increasing for fixed $a_1, a_2, \cdots, a_n$; if $p \in (-\infty, 0) \cup (0, +\infty)$, then $M_n(\mathbf{a}, \frac{1}{p})$ is decreasing for fixed $a_1, a_2, \cdots, a_n$.*

Lemma 2.7 ([1], p.26). *Let Φ be a differentiable function defined on D, if*

$$\Psi(x) = \frac{\Phi(x) - \Phi(x_0)}{x - x_0}, \quad x, x_0 \in D, \quad x \neq x_0.$$

Then $\Phi(x)$ is (strictly) convex if and only if $\Psi(x)$ is (strictly) increasing on D.

3. Some new inequalities for power means

In this section, we establish some new inequalities for $M_n(\mathbf{a}, p)$ and $M_n(\mathbf{a}, \frac{1}{p})$.

Theorem 3.1. *Let $a_k > 0$, $k = 1, 2, \cdots, n$, $p \in [1, +\infty)$. Then*

$$\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^{\frac{1}{p}}\Big)^p \leq \Big(\frac{1}{n}\sum_{k=1}^{n} a_k^{p}\Big)^{\frac{1}{p}} \leq \frac{\max\limits_{1\leq k\leq n}\{a_k\}}{\sqrt[n]{\prod_{k=1}^{n} a_k}}\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^{\frac{1}{p}}\Big)^p, \tag{3.1}$$

the sign of equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Proof. Let

$$F(p) = \frac{M_n(\mathbf{a}, p)}{M_n(\mathbf{a}, \frac{1}{p})}. \tag{3.2}$$

By Theorem 2.6, we know that $F(p)$ is an increasing function for $p \in (0, +\infty)$.
Note that

$$F(1) = 1, \tag{3.3}$$

$$\lim_{p\to+\infty} F(p) = \frac{\lim\limits_{p\to+\infty} M_n(\mathbf{a}, p)}{\lim\limits_{p\to+\infty} M_n(\mathbf{a}, \dfrac{1}{p})} = \frac{\max\limits_{1\leq k\leq n}\{a_k\}}{\sqrt[n]{\prod_{k=1}^{n} a_k}}. \tag{3.4}$$

Then, for $p \in [1, +\infty)$, we have

$$1 \leq F(p) \leq \frac{\max\limits_{1\leq k\leq n}\{a_k\}}{\sqrt[n]{\prod_{k=1}^{n} a_k}}. \tag{3.5}$$

Combining equations (3.2), (3.3) and inequality (3.5) lead to inequality (3.1) immediately. The proof of Theorem 3.1 is completed. □

Corollary 3.2. *Let $a_1, a_2, \cdots, a_n$ be an increasing arithmetic sequence in inequality (3.1), $p \geq 1$. Then we obtain*

$$\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^{\frac{1}{p}}\Big)^p \leq \Big(\frac{1}{n}\sum_{k=1}^{n} a_k^{p}\Big)^{\frac{1}{p}} < e\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^{\frac{1}{p}}\Big)^p, \tag{3.6}$$

and the bound e for the right-side of inequality (3.6) is optimal.

Remark 3.3. Let $a_k = k$ in inequality (3.6), $p \geq 1$. Then we have

$$\begin{aligned}\Big(\frac{1}{n}\sum_{k=1}^{n} k^{\frac{1}{p}}\Big)^p &\leq \Big(\frac{1}{n}\sum_{k=1}^{n} k^{p}\Big)^{\frac{1}{p}} \\ &\leq \frac{n}{\sqrt[n]{n!}}\Big(\frac{1}{n}\sum_{k=1}^{n} k^{\frac{1}{p}}\Big)^p < e\Big(\frac{1}{n}\sum_{k=1}^{n} k^{\frac{1}{p}}\Big)^p.\end{aligned} \tag{3.7}$$

Using Lemma 2.3, we can obtain the following Theorem easily.

Theorem 3.4. *Let $a_k > 0$ $(k = 1, 2, \cdots, n)$, $p > 1$. Then*

$$M_n(\mathbf{a}, \frac{1}{p}) \le M_n^{\frac{1}{p}}(\mathbf{a}^p, \frac{1}{p}), \tag{3.8}$$

The inequality is reversed for $0 < p < 1$, where $\mathbf{a}^p = (a_1^p, a_2^p, \cdots, a_n^p)$.

Corollary 3.5. *Let $a_k, b_k > 0$ $(k = 1, 2, \cdots, n)$, $p, q > 1$. Then*

$$M_n(\mathbf{a}, \frac{1}{p})M_n(\mathbf{b}, \frac{1}{q}) \le M_n^{\frac{1}{p}}(\mathbf{a}^p, \frac{1}{p})M_n^{\frac{1}{q}}(\mathbf{b}^q, \frac{1}{q}), \tag{3.9}$$

Theorem 3.6. *Let $a_k, b_k > 0$ $(k = 1, 2, \cdots, n)$, and let $\frac{1}{p} + \frac{1}{q} = 1$ with $p, q > 1$. Then*

$$\max\{M_n(\mathbf{ab}, 1), M_n^p(\mathbf{a}^{\frac{1}{p}}, p)M_n^q(\mathbf{b}^{\frac{1}{q}}, q)\} \le M_n(\mathbf{a}, p)M_n(\mathbf{b}, q). \tag{3.10}$$

Proof. By Theorem 2.2, we can obtain

$$M_n(\mathbf{ab}, 1) \le M_n(\mathbf{a}, p)M_n(\mathbf{b}, q). \tag{3.11}$$

And by Theorem 3.4, we have

$$M_n(\mathbf{a}, \frac{1}{p'}) \ge M_n^{\frac{1}{p'}}(\mathbf{a}^{p'}, \frac{1}{p'}), \quad 0 < p' < 1. \tag{3.12}$$

Let $p' = \frac{1}{p}$ in inequality (3.12). Then

$$M_n(\mathbf{a}, p) \ge M_n^p(\mathbf{a}^{\frac{1}{p}}, p), \quad p > 1. \tag{3.13}$$

Similarly,

$$M_n(\mathbf{b}, q) \ge M_n^q(\mathbf{b}^{\frac{1}{q}}, q), \quad q > 1. \tag{3.14}$$

Combining equations (3.11), (3.13) and (3.14) lead to equation (3.10) easily. □

Remark 3.7. Let $0 < a_1 < a_2 < \cdots < a_n, 0 < b_1 < b_2 < \cdots < b_n$ in Theorem 3.6. Then we have

$$M_n^p(\mathbf{a}^{\frac{1}{p}}, p)M_n^q(\mathbf{b}^{\frac{1}{q}}, q) \le M_n(\mathbf{ab}, 1) \le M_n(\mathbf{a}, p)M_n(\mathbf{b}, q). \tag{3.15}$$

Theorem 3.8. *Let a_k, $b_k \ge 0$ $(k = 1, 2, \cdots, n)$, $\lambda, \mu > 0$, and let $p \ge 1$. Then*

$$M_n(\lambda\mathbf{a} + \mu\mathbf{b}, p) \le \lambda M_n(\mathbf{a}, p) + \mu M_n(\mathbf{b}, p). \tag{3.16}$$

The inequality is reversed for $p < 1$.

Proof. Using Minkowski inequality in Lemma 2.4, we have

$$\begin{aligned} M_n(\lambda\mathbf{a} + \mu\mathbf{b}, p) &= \Big(\frac{1}{n}\sum_{k=1}^{n}(\lambda\mathbf{a} + \mu\mathbf{b})^p\Big)^{\frac{1}{p}} \\ &\le \frac{1}{n^{1/p}}\Big[\Big(\sum_{k=1}^{n}(\lambda\mathbf{a})^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n}(\mu\mathbf{b})^p\Big)^{\frac{1}{p}}\Big] \\ &= \lambda M_n(\mathbf{a}, p) + \mu M_n(\mathbf{b}, p). \end{aligned}$$

Therefore, we obtain the desired result (3.16). □

Remark 3.9 (Minkowski's inequality [7], p.30). Let $\lambda = \mu = 1$ in Theorem 3.8. Then we can easily obtain the p-th power mean type Minkowski's inequality

$$M_n(\mathbf{a}+\mathbf{b},p) \le M_n(\mathbf{a},p) + M_n(\mathbf{b},p). \tag{3.17}$$

The inequality is reversed for $p < 1$.

Theorem 3.10 (Young's inequality). *Let $a_k,\ b_k \ge 0$ $(k = 1, 2, \cdots, n)$, $\lambda + \mu = 1, \lambda > 0$, and let $p \ge 1$. Then*

$$M_n(\mathbf{a}^\lambda \mathbf{b}^\mu, p) \le \lambda M_n(\mathbf{a},p) + \mu M_n(\mathbf{b},p). \tag{3.18}$$

Proof. By using inequality

$$A^\lambda B^\mu \le \lambda A + \mu B, \quad \lambda + \mu = 1, A, B, \lambda > 0,$$

we get

$$M_n(\mathbf{a}^\lambda \mathbf{b}^\mu, p) \le M_n(\lambda\mathbf{a} + \mu\mathbf{b}, p). \tag{3.19}$$

Then, combining inequalities (3.16) and (3.19), we obtain the desired result (3.18). □

Theorem 3.11. *If $a_k \ge 0$ $(k = 1, 2, \cdots, n)$, $0 < \rho \le \nu$. Then*

$$\frac{M_n(\mathbf{a},\frac{1}{\nu})}{M_n(\mathbf{a},-\frac{1}{\rho})} \ge \frac{1}{n^{\nu-\rho}}\left[\frac{M_n(\mathbf{a},\frac{1}{\nu})}{M_n(\mathbf{a},-\frac{1}{\nu})}\right]^{\frac{\rho}{\nu}} \ge \frac{1}{n^{\nu-\rho}}. \tag{3.20}$$

Proof. Using Jensen inequality in Lemma 2.5, and simple computations lead to

$$\begin{aligned}
\left(\frac{1}{n}\sum_{k=1}^{n} a_k^{\frac{1}{\nu}}\right)^{\nu}\cdot\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k^{\frac{1}{\rho}}}\right)^{\rho} &= \frac{1}{n^{\nu+\rho}}\left[\sum_{k=1}^{n}\frac{1}{a_k^{\frac{1}{\rho}}}\left(\sum_{k=1}^{n} a_k^{\frac{1}{\nu}}\right)^{\frac{\nu}{\rho}}\right]^{\rho} \\
&= \frac{1}{n^{\nu+\rho}}\left[\sum_{k=1}^{n}\left(\frac{1}{a_k^{\frac{1}{\nu}}}\sum_{k=1}^{n} a_k^{\frac{1}{\nu}}\right)^{\frac{\nu}{\rho}}\right]^{\rho} \ge \frac{1}{n^{\nu+\rho}}\left[\sum_{k=1}^{n}\left(\frac{1}{a_k^{\frac{1}{\nu}}}\sum_{k=1}^{n} a_k^{\frac{1}{\nu}}\right)\right]^{\rho} \\
&\ge \frac{n^{2\rho}}{n^{\nu+\rho}} = \frac{1}{n^{\nu-\rho}}.
\end{aligned} \tag{3.21}$$

The proof of Theorem 3.11 is completed. □

4. Some known results

In this section, we give some new properties for the ratio of power means.

Theorem 4.1. *Let $\{a_n\}$ be an increasing arithmetic sequence with $a_1 > 0$, $n \in \mathbb{N}$, denote $C_n(\mathbf{a},p) = \frac{M_n(\mathbf{a},p)}{M_n(\mathbf{a},p+1)}$, and $C(p) = \lim_{n\to\infty} C_n(\mathbf{a},p)$. Then*

$$C(p) = \begin{cases} \dfrac{\sqrt[p+1]{p+2}}{\sqrt[p]{p+1}}, & p \in (-1,0)\cup(0,\infty), \\ \dfrac{2}{e}, & p = 0, \\ 0, & p \le -1, \end{cases} \tag{4.1}$$

and $C(p)$ is strictly increasing on $(-1,+\infty)$.

Proof. Denote $a = \frac{d}{a_1}$ (d is tolerance). Then simple computation leads to

$$
\begin{aligned}
C_n(\mathbf{a}, p) = \frac{M_n(\mathbf{a}, p)}{M_n(\mathbf{a}, p+1)} &= \frac{\left(\frac{1}{n}\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}}}{\left(\frac{1}{n}\sum_{k=1}^{n} a_k^{p+1}\right)^{\frac{1}{p+1}}} \\
&= \frac{\left(\frac{1}{n}\sum_{k=1}^{n}[1+(k-1)\frac{d}{a_1}]^p\right)^{\frac{1}{p}}}{\left(\frac{1}{n}\sum_{k=1}^{n}[1+(k-1)\frac{d}{a_1}]^{p+1}\right)^{\frac{1}{p+1}}} \\
&= \frac{\left(\frac{1}{n}\sum_{k=1}^{n}[1+(k-1)a]^p\right)^{\frac{1}{p}}}{\left(\frac{1}{n}\sum_{k=1}^{n}[1+(k-1)a]^{p+1}\right)^{\frac{1}{p+1}}} \\
&= \frac{\left(\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{1+(k-1)a}{n}\Big]^p\right)^{\frac{1}{p}}}{\left(\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{1+(k-1)a}{n}\Big]^{p+1}\right)^{\frac{1}{p+1}}}.
\end{aligned}
\tag{4.2}
$$

Let $\psi(x) = x^p$. Then applying Lagrange mean value theorem for $\psi(x)$ on interval $\left[\frac{(k-1)a}{n}, \frac{1+(k-1)a}{n}\right]$, we have

$$\Big[\frac{1+(k-1)a}{n}\Big]^p - \Big[\frac{(k-1)a}{n}\Big]^p = p\xi_k^{p-1}\cdot\frac{1}{n},$$

where

$$\frac{(k-1)a}{n} < \xi_k < \frac{1+(k-1)a}{n}.$$

Denote

$$r_n = \frac{1}{n}\sum_{k=1}^{n}\left(\Big[\frac{1+(k-1)a}{n}\Big]^p - \Big[\frac{(k-1)a}{n}\Big]^p\right). \tag{4.3}$$

Now, we prove that

$$\lim_{n\to\infty} r_n = 0. \tag{4.4}$$

We divide three cases to prove equation (4.4).

Case I. When $p \geq 1$, we get

$$
\begin{aligned}
r_n &= \frac{1}{n^{1+p}} + \frac{1}{n^2}\sum_{k=2}^{n} p\xi_k^{p-1} \leq \frac{1}{n^{1+p}} + \frac{p}{n^2}\sum_{k=1}^{n-1}\Big(\frac{1+ka}{n}\Big)^{p-1} \\
&< \frac{1}{n^{1+p}} + \frac{\alpha}{n}\cdot\frac{1}{n}\sum_{k=1}^{n-1}\Big(\frac{k}{n}\Big)^{p-1}, \qquad \alpha = p(a+1)^{p-1}.
\end{aligned}
\tag{4.5}
$$

Moreover,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1}\big(\frac{k}{n}\big)^{p-1} = \int_0^1 x^{p-1}dx = \frac{1}{p}, \tag{4.6}$$

Thus, inequality (4.5) and equation (4.6) lead to equation (4.4) immediately.

Case II. When $0 < p < 1$, simple computation leads to

$$\begin{aligned} r_n &= \frac{1}{n^{1+p}} + \frac{p}{n^2}\sum_{k=2}^{n}\xi_k^{p-1} = \frac{1}{n^{1+p}} + \frac{p}{n^2}\sum_{k=2}^{n}(\frac{1}{\xi_k})^{1-p} \\ &< \frac{1}{n^{1+p}} + \frac{p}{n^2}\sum_{k=1}^{n-1}\Big(\frac{n}{ka}\Big)^{1-p} = \frac{1}{n^{1+p}} + \frac{\beta}{n^2}\sum_{k=1}^{n-1}\Big(\frac{n}{k}\Big)^{1-p} \\ &< \frac{1}{n^{p+1}} + \frac{\beta}{n^p} \to 0 \quad (n\to\infty), \end{aligned} \tag{4.7}$$

where $\beta = pa^{p-1}$.

Obviously, inequality (4.7) is equivalent to inequality (4.4)

Case III. When $-1 < p < 0$, denote $t = -p$ $(0 < t < 1)$. Then

$$\begin{aligned} \Big|r_n\Big| &= \Big|\frac{1}{n^{1-t}} - \frac{t}{n^2}\sum_{k=2}^{n}\xi_k^{-t-1}\Big| \\ &< \frac{1}{n^{1-t}} + \frac{t}{n^2}\sum_{k=1}^{n-1}(\frac{1}{\xi_k})^{t+1} \\ &< \frac{1}{n^{1-t}} + \frac{t}{n^2}\sum_{k=1}^{n-1}(\frac{n}{ka})^{t+1} \\ &< \frac{1}{n^{1-t}} + \frac{\gamma}{n^{1-t}}\sum_{k=1}^{n}\frac{1}{k}, \quad \gamma = \frac{t}{a^{t+1}}. \end{aligned} \tag{4.8}$$

Note that

$$\lim_{n\to\infty}\frac{1}{n^{\varepsilon}}\sum_{k=1}^{n}\frac{1}{k} = 0, \quad \varepsilon > 0. \tag{4.9}$$

Thus, inequality (4.8) and equation (4.9) lead to equation (4.4) immediately.

Equations (4.3) and (4.4) imply that

$$\begin{aligned} \lim_{n\to\infty} C_n(\mathbf{a},p) &= \lim_{n\to\infty}\frac{\left(\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{1+(k-1)a}{n}\Big]^p\right)^{\frac{1}{p}}}{\left(\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{1+(k-1)a}{n}\Big]^{p+1}\right)^{\frac{1}{p+1}}} \\ &= \frac{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{(k-1)a}{n}\Big]^p\right)^{\frac{1}{p}}}{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{(k-1)a}{n}\Big]^{p+1}\right)^{\frac{1}{p+1}}} \\ &= \frac{\big(\int_0^1 (ax)^p dx\big)^{\frac{1}{p}}}{\big(\int_0^1 (ax)^{p+1} dx\big)^{\frac{1}{p+1}}} = \frac{\sqrt[p+1]{p+2}}{\sqrt[p]{p+1}}. \end{aligned} \tag{4.10}$$

Hence,

$$C(p) = \frac{\sqrt[p+1]{p+2}}{\sqrt[p]{p+1}}, \quad p \in (-1,0)\cup(0,+\infty). \tag{4.11}$$

Note that

$$C(0) = \lim_{n\to\infty} \frac{M_n(\mathbf{a},0)}{M_n(\mathbf{a},1)} = \lim_{n\to\infty} \frac{\sqrt[n]{\prod_{k=1}^{n} a_k}}{\frac{1}{n}\sum_{k=1}^{n} a_k} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}\sum_{k=1}^{n} ln\frac{a_k}{a_n}}}{\frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{a_n}} = \frac{2}{e}, \tag{4.12}$$

and

$$\lim_{p\to 0} C(p) = \lim_{p\to 0} \frac{\sqrt[p+1]{p+2}}{\sqrt[p]{p+1}} = e^{\lim_{p\to 0}\left[\frac{1}{p+1}ln(p+2) - \frac{1}{p}ln(p+1)\right]} = \frac{2}{e}. \tag{4.13}$$

Thus, $C(p)$ is continuous at $p = 0$, and we obtain

$$C(p) = \begin{cases} \frac{\sqrt[p+1]{p+2}}{\sqrt[p]{p+1}}, & p \in (-1,0)\cup(0,+\infty), \\ \frac{2}{e}, & p = 0. \end{cases} \tag{4.14}$$

Similarly, when $p \le -1$, we can obtain

$$C(p) = 0. \tag{4.15}$$

Combining equations (4.14) and (4.15) lead to equation (4.1) immediately.

Next, we prove that $C(p)$ is strictly increasing on $(-1, +\infty)$.

Denote

$$f(p) = lnC(p) = \frac{g(p+1) - g(p)}{p+1-p}, (p > -1), \tag{4.16}$$

where

$$g(p) = \frac{1}{p} ln(p+1), \quad g(0) = 1. \tag{4.17}$$

By the Lemma 2.7, we know that $f(p)$ is strictly increasing on $(-1, +\infty)$ if and only if $g(p)$ is strictly convex on $(-1, +\infty)$.

Simple computations lead to

$$g'(p) = \frac{\frac{p}{(p+1)} - ln(p+1)}{p^2}, \quad p \in (-1,0)\cup(0,+\infty), \tag{4.18}$$

$$g'(0) = \lim_{p\to 0} g'(p) = -\frac{1}{2}, \tag{4.19}$$

$$g''(p) = \frac{h(p)}{p^3}, \quad p \in (-1,0)\cup(0,+\infty), \tag{4.20}$$

where

$$h(p) = 2ln(1+p) - \frac{3p^2+2p}{(p+1)^2}. \tag{4.21}$$

Simple computations give

$$\lim_{p\to -1^+} h(p) = -\infty, \quad h(0) = 0, \tag{4.22}$$

$$g''(0) = \lim_{p\to 0} g''(p) = \frac{2}{3}, \tag{4.23}$$

$$h'(p) = \frac{2p^2}{(p+1)^3} \ge 0, \quad p \in (-1, +\infty). \tag{4.24}$$

Hence, $h(p)$ is increasing on $(-1, +\infty)$. It follows from (4.21) and (4.22) together with the monotonicity of $h(p)$ that $h(p) < 0$ when $p \in (-1, 0)$, and $h(p) > 0$ when $p \in (0, +\infty)$. Combining equations (4.20) and (4.23) lead to $g''(p) > 0$ immediately. Hence $g(p)$ is strictly convex on $(-1, +\infty)$.

Therefore, $C(p)$ is strictly increasing on $(-1, +\infty)$. The proof of Theorem 4.1 is completed. □

Theorem 4.2. *Let $\{a_n\}$ be an increasing arithmetic sequence with $a_1 > 0$, $n \in \mathbb{N}$, denote $E_n(\mathbf{a}, p) = \frac{M_n(\mathbf{a}, \frac{1}{p})}{M_n(\mathbf{a}, \frac{1}{p+1})}$, and $E(p) = \lim_{n\to\infty} E_n(\mathbf{a}, p)$. Then*

$$E(p) = \begin{cases} \dfrac{p^p(p+2)^{p+1}}{(p+1)^{2p+1}}, & p > 0, \\ \dfrac{(-p)^p(-p-2)^{p+1}}{(-p-1)^{2p+1}}, & p < -2. \end{cases} \tag{4.25}$$

Moreover, $E(p)$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, -2)$.

Proof. Denote $a = \frac{d}{a_1}$ (d is tolerance), $p = -t$ ($t > 2$). When $p \in (-\infty, -2)$, simple computation leads to

$$\begin{aligned} E_n(\mathbf{a}, p) = E_n(\mathbf{a}, -t) &= \frac{M_n(\mathbf{a}, -\frac{1}{t})}{M_n(\mathbf{a}, \frac{1}{1-t})} \\ &= \frac{\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k^{-\frac{1}{t}}\right)^{-t}}{\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} a_k^{-\frac{1}{t-1}}\right)^{1-t}} \\ &= \frac{\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \left[\dfrac{n}{1+(k-1)\frac{d}{a_1}}\right]^{\frac{1}{t-1}}\right)^{t-1}}{\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \left[\dfrac{n}{1+(k-1)\frac{d}{a_1}}\right]^{\frac{1}{t}}\right)^{t}} \\ &= \frac{\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \left[\dfrac{n}{1+(k-1)a}\right]^{\frac{1}{t-1}}\right)^{t-1}}{\left(\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \left[\dfrac{n}{1+(k-1)a}\right]^{\frac{1}{t}}\right)^{t}}. \end{aligned} \tag{4.26}$$

Let $\varphi(x) = x^{-\frac{1}{t}}$, then $\varphi'(x) = -\frac{1}{t}x^{-\frac{1}{t}-1}$. Applying Lagrange mean value theorem for $\varphi(x)$ on interval $\left[\frac{(k-1)a}{n}, \frac{1+(k-1)a}{n}\right]$, we have

$$\left[\frac{1+(k-1)a}{n}\right]^{-\frac{1}{t}} - \left[\frac{(k-1)a}{n}\right]^{-\frac{1}{t}} = -\frac{1}{t}\xi_k^{-\frac{1}{t}-1} \cdot \frac{1}{n},$$

where $\frac{(k-1)a}{n} < \xi_k < \frac{1+(k-1)a}{n}$.
Denote

$$\delta_n = \Big|\frac{1}{n^{1-\frac{1}{t}}} + \frac{1}{n}\sum_{k=2}^{n}\left(\left[\frac{1+(k-1)a}{n}\right]^{-\frac{1}{t}} - \left[\frac{(k-1)a}{n}\right]^{-\frac{1}{t}}\right)\Big|. \tag{4.27}$$

Now, we prove that

$$\lim_{n\to\infty} \delta_n = 0. \tag{4.28}$$

Simple computations lead to

$$\begin{aligned} \delta_n &\le \frac{1}{n^{1-\frac{1}{t}}} + \frac{1}{tn^2}\sum_{k=2}^{n}\left(\frac{1}{\xi_k}\right)^{1+\frac{1}{t}} \\ &< \frac{1}{n^{1-\frac{1}{t}}} + \frac{1}{tn^2}\sum_{k=1}^{n-1}\left(\frac{n}{ka}\right)^{1+\frac{1}{t}} \\ &= \frac{1}{n^{1-\frac{1}{t}}} + \frac{1}{cn^{1-\frac{1}{t}}}\sum_{k=1}^{n-1}\left(\frac{1}{k}\right)^{1+\frac{1}{t}}, \end{aligned} \tag{4.29}$$

where $c = ta^{1+1/t}$.

By the property of p-series, we know that $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n-1}(\dfrac{1}{k})^{1+\frac{1}{t}}$ is convergent for $t > 2$.

Combining equation (4.27) and inequality (4.29) lead to equation (4.28). Thus

$$\begin{aligned}\lim_{n\to\infty} E_n(\mathbf{a},-t) &= \frac{\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\sum\limits_{k=1}^{n-1}\left(\dfrac{ka}{n}\right)^{-\frac{1}{t-1}}\right)^{t-1}}{\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\sum\limits_{k=1}^{n-1}\left(\dfrac{ka}{n}\right)^{-\frac{1}{t}}\right)^{t}} \\ &= \frac{\left(\int_0^1 (ax)^{-\frac{1}{t-1}}dx\right)^{t-1}}{\left(\int_0^1 (ax)^{-\frac{1}{t}}dx\right)^{t}} \\ &= \frac{(t-1)^{2t-1}}{t^t(t-2)^{t-1}}, \quad (t>2).\end{aligned} \tag{4.30}$$

Let $t = -p$ in the last equation of (4.30), we obtain

$$E(p) = \frac{(-p-1)^{-2p-1}}{(-p)^{-p}(-p-2)^{-p-1}} = \frac{(-p)^p(-p-2)^{p+1}}{(-p-1)^{2p+1}}. \tag{4.31}$$

Next, we prove $E(p)$ is increasing on $(-\infty,-2)$.
When $p < -2$, denote

$$\phi(p) = lnE(p) = pln(-p) + (p+1)ln(-p-2) - (2p+1)ln(-p-1). \tag{4.32}$$

Simple computations lead to

$$\phi'(p) = ln(-p) + ln(-p-2) - 2ln(-p-1) + \frac{1}{p+1} - \frac{1}{p+2}, \tag{4.33}$$

$$\phi''(p) = \frac{3p+4}{p(p+1)^2(p+2)^2} > 0. \tag{4.34}$$

Then, $\phi'(p)$ is increasing on $(-\infty,-2)$.
Note that

$$\lim_{p\to-\infty}\phi'(p) = \lim_{p\to-\infty}\left[ln\frac{p(p+2)}{(p+1)^2} + \frac{1}{(p+1)(p+2)}\right] = 0.$$

Thus

$$\phi'(p) > 0, \tag{4.35}$$

Therefore, $\phi(p)$ is increasing on $(-\infty,-2)$.

When $p \in (0,+\infty)$. By the same method, we can obtain

$$E(p) = \frac{p^p(p+2)^{p+1}}{(p+1)^{2p+1}}.$$

And $E(p)$ is decreasing on $(0,+\infty)$. The proof of Theorem 4.2 is completed. □

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References

[1] P. S. Bullen, Mitrinović, P. M. Vasić, *Means and Their Inequalities*, Kluwer, (2003). 2.2, 2.3, 2.7

[2] Y. M. Chu, W. F. Xia, *Two sharp inequalities for power mean, geometric mean, and harmonic mean*, J. Inequal. Appl., **2009**, (2009), 6 pages. 1

[3] I. I. Costin, G. H. Toader, *Optimal evaluations of some Seiffert-type means by power means*, Appl. Math. Comput., **219** (2013), 4745–4754. 1

[4] A. Čižmešija, *A new sharp double inequality for generalized heronian, harmonic and power means*, Comput. Math. Appl., **64** (2012), 664–671. 1

[5] A. Čižmešija, *The optimal power mean bounds for two convex combinations of A-G-H means*, J. Math. Inequal., **6** (2012), 33–41. 1

[6] H. Y. Gao, J. L. Guo, W. G. Yu, *Sharp bounds for power mean in terms of generalized heronian mean*, Abstr. Appl. Anal., **2011** (2011), 9 pages. 1

[7] G. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, UK, (1952). 2.5, 2.6, 3.9

[8] O. Hölder, Üeber einen Mittelwerthabsatz. Göttinger. Nachr, (1889), 38–47. 2.1

[9] A. Ibrahim, S. S. Dragomir, M. Darus, *Power series inequalities via Young's inequality with applications*, J. Inequal. Appl., **2013** (2013), 13 pages. 1

[10] O. Kouba, *Bounds for the ratios of differences of power means in two arguments*, Math. Inequal. Appl., **17** (2014), 913–928. 1

[11] B. Y. Long, Y. M. Chu, *Optimal power mean bounds for the weighted geometric mean of classical means*, J. Inequal Appl., **2010** (2010), 6 pages. 1

[12] V. Mascioni, *A note on stolarsky, arithmetic and logarithmic means*, Math. Inequal. Appl., **15** (2012), 581–590. 1

[13] H. Minkowski, Geometrie der Zahlen, Teubner-Verlag, Leipzig, (1896). 2.4

[14] J. J. Wen, W. L. Wang, *The optimization for the inequalities of power means*, J. Inequal. Appl., **2006** (2006), 1–25. 1

[15] S. H. Wu, *Generalization and sharpness of the power means inequality and their applications*, J. Math. Anal. Appl., **312** (2005), 637–652. 1

[16] W. F. Xia, W. Janous, Y. M. Chu, *The optimal convex combination bounds of arithmetic and harmonic means in terms of power mean*, J. Math. Inequal., **6** (2012), 241–248. 1

[17] Z. H. Yang, *Sharp bounds for Seiffert mean in terms of weighted power means of arithmetic mean and geometric mean*, Math. Inequal. Appl., **17** (2014), 499–511. 1