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Generalized order of entire monogenic functions of slow growth

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

In the present paper we study the generalized growth of entire monogenic functions having slow growth. The characterizations of generalized order of entire monogenic functions have been obtained in terms of their Taylor's series coefficients.

Keywords: Clifford algebra, Clifford analysis, Generalized Cauchy-Riemann system, Entire monogenic function, Generalized order.
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1. Introduction

Clifford analysis offers possibility of generalizing complex function theory to higher dimensions. It considers Clifford algebra valued functions that are defined in open subsets of $\mathbb{R}^n$ for arbitrary finite $n \in \mathbb{N}$ and that are solutions of higher dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions.
In order to make calculations more concise we use the following notations, where $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{N}_0^n$ is the $n-$ dimensional multi-index and $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{x}^{\mathbf{m}} = x_1^{m_1}...x_n^{m_n}\ , \quad \mathbf{m!} = m_1!...m_n!\ , \quad |\mathbf{m}| = m_1 + ... + m_n.$$

Following Constales, Almeida and Krausshar (see [1] and [2]), we give some definitions and associated properties.

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By $\{e_1, e_2, ..., e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The associated real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo $\mathbf{x}^2 = -||\mathbf{x}||^2 e_0$, where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0\,, \quad i, j = 1, 2, ..., n\,,$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, 2, ..., n\}\}$ with $e_A = e_{l_1} e_{l_2} ... e_{l_r}$, where $1 \leq l_1 < l_2 < ... < l_r \leq n$, $e_\phi = e_0 = 1$. Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. The conjugation in Clifford algebra Cl_{0n} is defined by $\bar{a} = \sum_A a_A\, \bar{e}_A$, where $e_A = \bar{e}_{l_r}\, \bar{e}_{l_{r-1}} ... \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$ for $j = 1, 2, ... n$, $\bar{e}_0 = e_0 = 1$. The linear subspace $\text{span}_R\{1, e_1, ..., e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_{0n}$ is the so called space of paravectors $z = x_0 + x_1 e_1 + x_2 e_2 + ... + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. Here $x_0 = \text{Sc}(z)$ is scalar part and $\mathbf{x} = x_1 e_1 + x_2 e_2 + ... + x_n e_n = \text{Vec}(z)$ is vector part of paravector z. The Clifford norm of an arbitrary $a = \sum_A a_A\, e_A$ is given by

$$||a|| = \left(\sum_A |a_A|^2 \right)^{1/2} .$$

Each paravector $z \in \mathbb{R}^{n+1} \backslash \{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ which can be represented in the form $z^{-1} = \overline{z}/||z||^2$.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} .$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, then a function $g : U \to Cl_{0n}$ is called left (right) monogenic at a point $z \in U$ if $D\, g(z) = 0$ $(g\, D(z) = 0)$. The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Let A_{n+1} be the n-dimensional surface area of $n+1$-dimensional unit ball and $q_{\mathbf{0}}(z) = \overline{z}/||z||^{n+1}$ be Cauchy kernel function. Then every function g which is monogenic in a neighborhood of closure $\overline{G}$ of domain G satisfies the following equation (see [2], p. 766)

$$g(z) = \frac{1}{A_{n+1}} \int_{\partial G} q_{\mathbf{0}}(z - \zeta)\; d\tau(\zeta)\, g(\zeta) \;\; \text{, for all} z \in G,$$

where

$$d\tau(\zeta) = \sum_{j=0}^{n} (-1)^j e_j \widehat{d\zeta_j}$$

with

$$\widehat{d\zeta_j} = d\zeta_0 \,\wedge\, ... \,\wedge\, d\zeta_{j-1} \,\wedge\, d\zeta_{j+1} \,\wedge\, ... \,\wedge\, d\zeta_n$$

is the oriented outer normal surface measure. Following [1] and [2], we define the Fueter polynomials $V_{\mathbf{m}}(z)$ as

$$V_{\mathbf{m}}(z) = \frac{\mathbf{m!}}{|\mathbf{m}|!} \sum_{\pi \in perm(\mathbf{m})} z_{\pi(m_1)} ... z_{\pi(m_n)},$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1, m_2 ..., m_n)$ and $z_i = x_i - x_0 e_i$ for $i = 1, ..., n$ and $V_{\mathbf{0}}(z) = 1$. If g is a left monogenic function in a ball $||z|| < R$, then for all $||z|| < r$ with $0 < r < R$, the Taylor series expansion of $g(z)$ is given by (see [1] and [2])

$$g(z)=\sum_{|\mathbf{m}|=0}^{\infty} V_{\mathbf{m}}(z)\,a_{\mathbf{m}}\,. \tag{1.1}$$

In (1.1), $\{a_{\mathbf{m}}\}$ are Clifford numbers which are defined by

$$a_{\mathbf{m}}=\frac{1}{\mathbf{m}!\,A_{n+1}}\int_{||\zeta||<r} q_{\mathbf{m}}(\zeta)\;d\tau(\zeta)\,g(\zeta)$$

and satisfy the inequality

$$||a_{\mathbf{m}}|| \;\le\; c(n,\mathbf{m})\frac{M(r,g)}{r^{|\mathbf{m}|}}.$$

Here $M(r,g)=\max\limits_{||z||=r}\{||g(z)||\}$ denotes the maximum modulus of the function g in the closed ball of radius r and

$$q_{\mathbf{m}}(z)=\frac{\partial^{m_0+m_1+...+m_n}}{\partial x_0^{m_0}\partial x_1^{m_1}...\partial x_n^{m_n}}q_{\mathbf{0}}(z),\quad c(n,\mathbf{m})=\frac{n(n+1)...(n+|\mathbf{m}|-1)}{\mathbf{m}!}.$$

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [4], Kapoor and Nautiyal [3]. Hence, let L^0 denote the class of functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a,\infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x\to\infty$,

(ii) $\lim\limits_{x\to\infty}\frac{h[\{1+1/\psi(x)\}x]}{h(x)}=1$, for every function $\psi(x)$ such that $\psi(x)\to\infty$ as $x\to\infty$. The functions of the form $f(x)=ax+b, 0<a<\infty, 0<b<\infty$ are in class L^0.

Let Λ denote the class of functions h satisfying conditions (i) and

(iii) $\lim\limits_{x\to\infty}\frac{h(cx)}{h(x)}=1$, for every $c>0$, that is $h(x)$ is slowly increasing. The functions of the form $f(x)=\log{(ax)}\,, 0<a<\infty$, are in class Λ.

Let Ω be the class of functions $h(x)$ satisfying conditions (i) and

(iv) there exist a function $\delta(x)\in\Lambda$ and constants x_0 , K_1 and K_2 such that

$$0<K_1\le\frac{d\{h(x)\}}{d\{\delta(\log x)\}}\le K_2<\infty\quad ,$$

for all $x>x_0$. The functions of the form $f(x)=\delta\left(\log x\right),\delta\in\Lambda$ are in class Ω. (see [3])

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v) $\lim\limits_{x\to\infty}\frac{d\{h(x)\}}{d(\log x)}=K$, $\quad 0<K<\infty$. The functions of the form $f(x)=\log x+a\left(\log\log x\right)^b, 0<a<\infty, 0<b<\infty$ are in class $\overline{\Omega}$. (see [3])

Kapoor and Nautiyal [3] showed that classes Ω and $\overline{\Omega}$ are contained in Λ and $\Omega\bigcap\overline{\Omega}=\phi$.
For an entire monogenic function $g(z)$ and functions $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, we define the generalized order $\rho(\alpha,g)$ of $g(z)$ as

$$\rho(\alpha,g)=\lim_{r\to\infty}\sup\frac{\alpha\left[\log M(r,g)\right]}{\alpha\left(\log r\right)}\,. \tag{1.2}$$

2. Main results

We now prove

Theorem 2.1. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} V_{\boldsymbol{m}}(z)\, a_{\boldsymbol{m}}$. If $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, then the generalized order $\rho(\alpha, g)(1 < \rho(\alpha, g) < \infty)$ of $g(z)$ is given as*

$$\rho(\alpha, g) - 1 = \lim_{|\boldsymbol{m}|\to\infty} \sup \frac{\alpha(|\boldsymbol{m}|)}{\alpha\left\{\log ||a_{\boldsymbol{m}}/c(n, \boldsymbol{m})||^{-1/|\boldsymbol{m}|}\right\}}. \tag{2.1}$$

Proof. Write

$$\theta = \lim_{|\mathbf{m}|\to\infty} \sup \frac{\alpha(|\mathbf{m}|)}{\alpha\left\{\log ||a_{\mathbf{m}}/c(n, \mathbf{m})||^{-1/|\mathbf{m}|}\right\}}.$$

Now first we prove that $\rho - 1 \geq \theta$. The coefficients of a monogenic Taylor's series satisfy Cauchy's inequality, that is

$$||a_{\mathbf{m}}/c(n, \mathbf{m})|| \ \leq \ r^{-|\mathbf{m}|}\ M(r, g). \tag{2.2}$$

Also from (1.2), for $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$M(r, g) \leq \exp\left[\alpha^{-1}\left\{\overline{\rho}\,\alpha(\log r)\right\}\right]\ ,$$

where $\overline{\rho} = \rho + \varepsilon$ provided r is sufficiently large. So from (2.2), we get

$$||a_{\mathbf{m}}/c(n, \mathbf{m})|| \ \leq\ r^{-|\mathbf{m}|} \exp\left[\alpha^{-1}\left\{\overline{\rho}\,\alpha(\log r)\right\}\right]$$

or

$$||a_{\mathbf{m}}/c(n, \mathbf{m})|| \quad \leq \quad \exp[-|\mathbf{m}| \log r + \alpha^{-1}\{\overline{\rho}\alpha(\log r)\}]\,. \tag{2.3}$$

Since $\alpha(x)$ is an increasing function of x, we define $r = r(|\mathbf{m}|)$ as the unique root of the equation

$$\alpha\left[\frac{|\mathbf{m}|\ \log r}{\overline{\rho}}\right] = \overline{\rho}\,\alpha(\log r). \tag{2.4}$$

For large values of $|\mathbf{m}|$, we have

$$\begin{aligned}
&\alpha\left(c\,|\mathbf{m}|\right) \simeq \alpha\left(|\mathbf{m}|\right)\\
&\Rightarrow \alpha\left(c\,|\mathbf{m}|\right) \simeq \alpha\left(|\mathbf{m}|\right)\{1 + o(1)\}\\
&\Rightarrow \alpha\left(c\,|\mathbf{m}|\right) \simeq \alpha\left(|\mathbf{m}|\right)\left\{1 + \tfrac{\alpha(c)}{\alpha(|\mathbf{m}|)}\right\}\\
&\Rightarrow \alpha\left(c\,|\mathbf{m}|\right) \simeq \alpha\left(|\mathbf{m}|\right) + \alpha\left(c\right).
\end{aligned}$$

Thus for large values of $|\mathbf{m}|$ from equation (2.4), we have

$$\overline{\rho}\,\alpha\left(\log r\right) \simeq \alpha\left(|\mathbf{m}|\right) + \alpha\left(\log r\right) - \alpha\left(\overline{\rho}\right)$$

or

$$\alpha\left(\log r\right) \simeq \frac{\alpha\left(|\mathbf{m}|\right)}{\left(\overline{\rho}\ - 1\right)}\left\{1 - \frac{\alpha\left(\overline{\rho}\right)}{\alpha\left(|\mathbf{m}|\right)}\right\}.$$

or

$$\alpha\left(\log r\right) \simeq \frac{\alpha\left(|\mathbf{m}|\right)}{\left(\overline{\rho}\ - 1\right)}\{1 + o(1)\}$$

or

$$\log r \simeq \alpha^{-1}\left\{\frac{1}{\overline{\rho}-1}\alpha(|\mathbf{m}|)\right\} = F\left(|\mathbf{m}|, \frac{1}{\overline{\rho}-1}\right). \tag{2.5}$$

Using (2.4) and (2.5) in (2.3), we get

$$||a_{\mathbf{m}}/c(n,\mathbf{m})|| \leq \exp[-|\mathbf{m}|\,F + (\,|\mathbf{m}|/\overline{\rho})\,F]$$

or

$$\frac{\overline{\rho}}{\overline{\rho}-1}\log\{||a_{\mathbf{m}}/c(n,\mathbf{m})||\,\}^{-1/|\mathbf{m}|} \geq \alpha^{-1}\left\{\frac{1}{\overline{\rho}-1}\alpha(|\mathbf{m}|)\right\}$$

or

$$\frac{\alpha(|\mathbf{m}|)}{\alpha\left[\frac{\overline{\rho}}{\overline{\rho}-1}\log\{||a_{\mathbf{m}}/c(n,\mathbf{m})||\,\}^{-1/|\mathbf{m}|}\right]} \leq \overline{\rho}-1$$

or

$$\frac{\alpha(|\mathbf{m}|)}{\alpha\left[\log\{||a_{\mathbf{m}}/c(n,\mathbf{m})||\,\}^{-1/|\mathbf{m}|}\right]} \leq (\overline{\rho}-1)\times \times\frac{\alpha\left[\frac{\overline{\rho}}{\overline{\rho}-1}\log\{||a_{\mathbf{m}}/c(n,\mathbf{m})||\,\}^{-1/|\mathbf{m}|}\right]}{\alpha\left[\log\{||a_{\mathbf{m}}/c(n,\mathbf{m})||\,\}^{-1/|\mathbf{m}|}\right]}.$$

Since $\alpha(cx) \simeq \alpha(x)$ as $x\to\infty$, proceeding to limits as $|\mathbf{m}|\to\infty$ we get

$$\theta \leq \overline{\rho}-1.$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\theta \leq \rho - 1. \tag{2.6}$$

Now we will prove that $\theta \geq \rho-1$. If $\theta = \infty$, then there is nothing to prove. So let us assume that $0 \leq \theta<\infty$. Therefore, for all ε >0 there exist $n_0 \in N$ such that for all multi-indices m with $|\mathbf{m}| >n_0$, we have

$$0 \leq \frac{\alpha(|\mathbf{m}|)}{\alpha\left[\log\{||a_{\mathbf{m}}/c(n,\mathbf{m})||\,\}^{-1/|\mathbf{m}|}\right]} < \theta+\varepsilon=\overline{\theta}$$

or

$$||a_{\mathbf{m}}/c(n,\mathbf{m})|| \leq \exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right].$$

Now from the property of maximum modulus, we have

$$M(r,g) \leq \sum_{|\mathbf{m}|=0}^{\infty}||a_{\mathbf{m}}||r^{|\mathbf{m}|}$$

or

$$M(r,g) \leq \sum_{|\mathbf{m}|=0}^{n_0}||a_{\mathbf{m}}||r^{|\mathbf{m}|} + \sum_{|\mathbf{m}|=n_0+1}^{\infty}c(n,\mathbf{m})\,r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right].$$

Now for $r=\max\left\{1, \exp\left(\alpha^{-1}\left(\frac{\alpha(n_0+1)}{\overline{\theta}}\right)/(n+1)\right)\right\}$, we have

$$M(r,g) \leq A_1 r^{n_0} + \sum_{|\mathbf{m}|=n_0+1}^{\infty}c(n,\mathbf{m})\,r^{|\mathbf{m}|}\exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right], \tag{2.7}$$

where A_1 is a positive real constant. We take

$$N(r) = \left[\alpha^{-1} \left\{ \overline{\theta}\, \alpha[\log\{(n+1)r\}] \right\} \right],$$

where $[x]$ denotes the integer part of $x \geq 0$. Since $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$, the integer $N(r)$ is well defined. Now if r is sufficiently large, then from (2.7) we have

$$\begin{aligned} M(r,g) \leq A_1 r^{n_0} + r^{N(r)} \times \\ \times \textstyle\sum_{n_0+1 \leq |\mathbf{m}| \leq N(r)} c(n,\mathbf{m}) \exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right] \\ + \textstyle\sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right] \end{aligned}$$

or

$$\begin{aligned} M(r,g) \leq A_1 r^{n_0} + r^{N(r)} \times \\ \times \textstyle\sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right] \\ + \textstyle\sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right] . \end{aligned} \tag{2.8}$$

Now the first series in (2.8) can be rewritten as

$$\sum_{p=1}^{\infty} \left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m}) \right) \exp\left[-p\alpha^{-1}\left\{\alpha(p)/\overline{\theta}\right\}\right]. \tag{2.9}$$

Now from ([2], Lemma 1), we have

$$\lim_{p\to\infty} \sup \left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m}) \right)^{1/p} = n.$$

Hence we have

$$\begin{aligned} &\lim_{p\to\infty} \sup \left[\left(\textstyle\sum_{|\mathbf{m}|=p} c(n,\mathbf{m}) \right) \exp\left[-p\alpha^{-1}\left\{\alpha(p)/\overline{\theta}\right\}\right] \right]^{1/p} \\ &= n \lim_{p\to\infty} \sup \exp\left[-\alpha^{-1}\left\{\alpha(p)/\overline{\theta}\right\}\right] = 0. \end{aligned}$$

Hence the series (2.9) converges to a positive real constant A_2. So from (2.8), we get

$$\begin{aligned} M(r,g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \\ + \textstyle\sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}|\alpha^{-1}\left\{\alpha(|\mathbf{m}|)/\overline{\theta}\right\}\right] \end{aligned}$$

or

$$\begin{aligned} M(r,g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \\ + \textstyle\sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m})\, r^{|\mathbf{m}|} \exp\left[-|\mathbf{m}| \log\{(n+1)r\}\right] \end{aligned}$$

or

$$M(r,g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|>N(r)} c(n,\mathbf{m}) \left(\frac{1}{n+1} \right)^{|\mathbf{m}|}$$

or

$$M(r,g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{|\mathbf{m}|=1}^{\infty} c(n,\mathbf{m}) \left(\frac{1}{n+1} \right)^{|\mathbf{m}|}. \tag{2.10}$$

The series in (2.10) can we rewritten as

$$\sum_{p=1}^{\infty}\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right)\left(\frac{1}{n+1}\right)^{p}. \tag{2.11}$$

So we have

$$\lim_{p\to\infty}\sup\left[\left(\sum_{|\mathbf{m}|=p} c(n,\mathbf{m})\right)\left\{\frac{1}{n+1}\right\}^{p}\right]^{1/p} = \frac{n}{n+1}<1.$$

Hence the series (2.11) converges to a positive real constant A_3. Therefore from (2.10), we get

$$M(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + A_3.$$

Since $N(r) \to \infty$ as $r \to \infty$ so we can write above inequality as

$$\log M(r,g) \le \ [1+o(1)]\ N(r)\log r$$

or

$$\begin{aligned}\log M(r,g) \le\ & [1+o(1)]\ \left[\alpha^{-1}\left\{\overline{\theta}\,\alpha[\log\{(n+1)r\}]\right\}\right]\ \log r\\ &\le [1+o(1)]\ \left[\alpha^{-1}\left\{\overline{\theta}\,\alpha[\log\{(n+1)r\}]\right\}\right]\ \times\\ &\quad\times\left[\alpha^{-1}\left\{\alpha[\log\{(n+1)r\}]\right\}\right]\\ &\le [1+o(1)]\ \left[\alpha^{-1}\left\{\left(\overline{\theta}+1\right)\ \alpha[\log\{(n+1)r\}]\right\}\right]\end{aligned}$$

or

$$\alpha[\log M(r,g)] \le \left(\overline{\theta}+1\right)\ \alpha[\log\{(n+1)r\}]$$

or

$$\frac{\alpha[\log M(r,g)]}{\alpha(\log r)} \le \left(\overline{\theta}+1\right)\ \frac{\alpha[\{1+o(1)\}\log r]}{\alpha(\log r)}\ .$$

Proceeding to limits as $r \to \infty$ and using properties of $\alpha(x)$, we get

$$\rho \le \overline{\theta}+1.$$

Since $\varepsilon > 0$ is arbitrarily small, we finally get

$$\rho - 1 \le \theta. \tag{2.12}$$

Combining (2.6) and (2.12), we get (2.1). Hence Theorem 2.1 is proved. □

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References

[1] D. Constales, R. De Almeida and R. S. Krausshar, *On the growth type of entire monogenic functions*, Arch. Math. **88** (2007), 153-163. 1

[2] D. Constales, R. De Almeida and R. S. Krausshar, *On the relation between the growth and the Taylor coefficients of entire solutions to the higher dimensional Cauchy-Riemann system in* $\mathbb{R}^{n+1}$, J. Math. Anal. App. **327** (2007), 763-775. 1, 2

[3] G. P. Kapoor and A. Nautiyal, *Polynomial approximation of an entire function of slow growth*, J. Approx. Theory **32** (1981), 64-75. 1

[4] M. N. Seremeta, *On the connection between the growth of the maximum modulus of an entire function and the moduli of the coefficients of its power series expansion*, Amer. Math. Soc. Trans. **88** (1970), 291-301. 1