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Sequentially injective and complete acts over a semigroup

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

In this paper using the notion of a sequentially dense monomorphism we consider sequential injectivity (s-injectivity) for acts over a semigroup S. We show that s-injectivity, s-absolutely retract, and sequential compactness are equivalent.

Keywords: sequential injective, completeness, absolute retract.

1. Introduction

One of the very useful notions in many branches of mathematics as well as in computer sciences is the notion of acts of a semigroup or a monoid on a set. Recall that a (right) S-*act* or S-*system* is a set A together with a function $\lambda : A \times S \to A$, called the *action* of S (or the S-action) on A, such that for $a \in A$ and $s, t \in S$ (denoting $\lambda(a, s)$ by as) $a(st) = (as)t$. If S is a monoid with identity e, we add the condition $xe = x$.

We call an S-act A *separated* if for each $a \neq b$ in A there exists $s \neq e \in S$ such that $as \neq bs$.

A morphism $f : X \to Y$ from an S-act A to an S-act B is called an S-*map* if, for each $a \in A$, $s \in S$, $f(as) = f(a)s$.

Since id_A and the composite of two S-maps are S-maps, we have the category **Act-S** of all S-acts and S-maps between them.

The class of S-acts is an equational class, and so the category **Act-S** is complete (has all products and equalizers) and cocomplete (has all coproducts and coequalizers). In fact, limits and colimits in this category

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are computed as in the category **Set** of sets and equipped with a natural action. Also, monomorphisms of this category are exactly one-one act maps.

An S-act B containing (an isomorphic copy of) an S-act A as a sub-act is called an *extension* of A.

The S-act A is said to be a *retract* of its extension B if there exists a homomorphism $f : B \to A$ such that $f \restriction_A = id_A$:

$$\begin{array}{cccc} & B & \overset{i}{\hookrightarrow} & A \\ id_A & \downarrow & \swarrow & f \\ & A & & \end{array}$$

in which case f is said to be a *retraction*.

The S-act A is called *absolute retract* if it is a retract of each of its extensions.

An S-act A is said to be *injective* if for every S-monomorphism $h : B \to C$ and every S-map $f : B \to A$ there exists an S-map $g : C \to A$ such that $gh = f$:

$$\begin{array}{cccc} & B & \overset{h}{\rightarrowtail} & C \\ f & \downarrow & \swarrow & g \\ & A & & \end{array}$$

We have the following result from [3] or [1].

Theorem 1.1. *The category* **Act**-S *has enough injectives, and for any S-act A the following conditions are equivalent:*

(i) A *is injective.*

(ii) A *is an absolute retract.*

(iii) A *has no proper essential extension.*

2. Cauchy completeness

The notion of a Cauchy sequence is used in [7] and [6] for projection algebras. We generalize this notion to an arbitrary S-act to study s-injectivity.

To make the main notions and so the results about s-injectivity and s-completeness non trivial, from now on we take S to be a semigroup without identity. Of course one can always adjoin an identity e to S making it a monoid.

Definition 2.1. By a *Cauchy sequence* over an S-act A we mean a family $(a_s)_{s\in S}$ of elements of A with $a_s t = a_{st}$ for all $s, t \in S$.

By a *limit* of a Cauchy sequence $(a_s)_{s\in S}$ over A in some extension B of A we mean an element $b \in B$ such that $bs = a_s$ for all $s \in S$.

Lemma 2.2. *A sequence $(a_s)_{s\in S}$ over an S-act A has a limit in some extension B of A if and only if it is a Cauchy sequence.*

Proof. Take b as a limit of $(a_s)_{s\in S}$. Then for $s, t \in S$, $bs = a_s$ implies $a_{st} = b(st) = (bs)t = a_s t$.

Conversely, let $(a_s)_{s\in S}$ be a Cauchy sequence over A. Then $B = A \cup \{(a_s)_{s\in S}\}$ with the action $(a_s)_{s\in S}.t = a_t$ for $t \in S$ is an extension of A, and $b = (a_s)_{s\in S}$ is a limit of $(a_s)_{s\in S}$ in B. □

Note that limits of a Cauchy sequences over A is not necessarily unique, unless A is separated.

Denoting the set of all Cauchy sequences over A by $\mathcal{C}(A)$, we have

Theorem 2.3. *For an S-act A, the set $\mathcal{C}(A)$ of all Cauchy sequences over A is an S-act with the action of S on it given by $(a_s)_{s\in S}.t = (a_{ts})_{s\in S}$, for $t \in S$. Also, it is separated if S is idempotent.*

Proof. Take $t, t' \in S$, and a Cauchy sequence $(a_s)_{s\in S}$. Then we have

$$((a_s)_{s\in S}.t).t' = (a_{ts})_{s\in S}.t' = (a_{t(t's)})_{s\in S} = (a_{(tt')s)})_{s\in S} = (a_s)_{s\in S}.(tt').$$

To see that $\mathcal{C}(A)$ is separated, take Cauchy sequences $\gamma = (a_s)_{s\in S}$ and $\gamma' = (a'_s)_{s\in S}$ with $\gamma.t = \gamma'.t$ for all $t \in S$. This means that $a_{ts} = a'_{ts}$ for all $t, s \in S$, and in particular $a_s = a_{ss} = a'_{ss} = a'_s$ which means $\gamma = \gamma'$. □

Definition 2.4. An S-act A is said to be *Cauchy complete* or *sequentially complete* or simply *s-complete* if any Cauchy sequence over A has a limit in A.

Theorem 2.5. *If S is idempotent, then for each S-act A, $\mathcal{C}(A)$ is an s-complete S-act.*

Proof. Take a Cauchy sequence $(\gamma_s)_{s\in S}$ over $\mathcal{C}(A)$ with $\gamma_s = (a^s_t)_{t\in S}$ for $s \in S$. Then since it is Cauchy, $\gamma_s t' = \gamma_{st'}$ and hence

$$a^s_{t't} = a^{st'}_t, \quad \forall s, t, t' \in S \qquad (1)$$

On the other hand, since γ_s is a Cauchy sequence over A,

$$a^s_t t' = a^s_{tt'} \qquad (2)$$

Now, the sequence $\gamma = (a^s_s)_{s\in S}$ is in $\mathcal{C}(A)$. Since using (2), repeatedly (1), and that S is idempotent, we get

$$a^s_s t = a^s_{st} = a^s_{stst} = a^{ss}_{tst} = a^s_{tst} = a^{st}_{st}. \quad (3)$$

The sequence γ is a limit of $(\gamma_s)_{s\in S}$. This is because $\gamma.s = (a^t_t)_{t\in S}.s = (a^{st}_{st})_{t\in S}$ and using (1) and (3), the tth component of γ_s is $a^s_t = a^{ss}_t = a^s_{st} = a^{st}_{st}$. □

Remark 2.6. For an S-act A and $a \in A$ the convergent Cauchy sequence $(as)_{s\in S}$ is denoted by $\lambda(a)$, and the set of all $\lambda(a)$ for $a \in A$ is denoted by $\lambda(A)$. It is clear that $\lambda(A)$ is a subact of $\mathcal{C}(A)$ and the assignment $\lambda : a \mapsto \lambda(a)$ is an S-map. Further, λ is one-one if and only if A is separated, and in this case $A \cong \lambda(A)$. Moreover, it is clear that A is s-complete if and only if $\mathcal{C}(A) = \lambda(A)$.

3. s-injectivity verses s-completeness

Here, as in [4], we define a closure operator C_s, and then discuss injectivity with respect to C_s-dense monomorphisms. Then, we show that the notions of s-injectivity, s-absolutely retract, and s-completeness coincide.

Definition 3.1. For an S-act B, and a subact A of B, by the *s-closure* of A in B we mean $C_s(A) = \{b \in B : bs \in A, \ \forall s \in S\}$.

We say that A is *s-closed* in B if $C_s(A) = A$, and A is *s-dense* in B if $C_s(A) = B$.

An S-map $f : A \to B$ is said to be s-dense (s-closed) if $f(A)$ is an s-dense (C-closed) subact of B.

Note that, Some properties of s-closure are as follows:
(Extensive) $A \leq C_s(A)$,
(Monotonicity) $A_1 \subseteq A_2$ implies $C_s(A_1) \subseteq C_s(A_2)$,
(Continuity) $f(C_s(A)) \leq C_s(f(A))$, for all S-maps f from B.
Also, it has the following property if S is idempotent:
(idempotency) $C_s(C_s(A)) = C_s(A)$.

Lemma 3.2. *If S is idempotent, then the composition of s-dense act maps is s-dense. Moreover, each S-map $f : A \to B$ has an s-dense-s-closed factorization.*

Proof. Consider the following factorization: $A \to C_s(f(A)) \hookrightarrow B$. □

Definition 3.3. An S-act A is called:
(1) *Sequentially injective* or *s-injective* if it is injective with respect to s-dense monomorphisms.
(2) *Sequentially absolute retract* or *s-absolute retract* if it is a retract of each of its s-dense extensions.

Remark 3.4. If A is an injective S-act then it is s-injective, but the converse is not necessarily true. For example, let S be a group then it is s-injective as an S-act (see the following theorem) but it is not injective, since it does not have a zero element.

Lemma 3.5. (1) *A retract of an s-injective act is s-injective.*
(2) *The product of s-injective acts is s-injective.*

Proof. (1) Let the S-act A be a retract of the S-act D with retraction $l : D \to A$, and D be s-injective. Let $h : B \rightarrowtail C$ be an s-dense monomorphism, and $f : B \to A$ be an S-map. Then considering the diagram

$$\begin{array}{ccc} & B & \overset{h}{\rightarrowtail} & C \\ f & \downarrow & & \\ & A & \rightleftarrows & D \end{array}$$

since D is s-injective we get an S-map $g : C \to D$ such that $gh = if$, and so $lg : C \to A$ satisfies $(lg)h = f$.
(2) Let $\{A_i : i \in I\}$ be a family of s-injective acts, $h : B \rightarrowtail C$ be an s-dense monomorphism, and $f : B \to \prod A_i$ be an S-map. Consider the diagram

$$\begin{array}{cccc} & B & \overset{h}{\rightarrowtail} & C \\ f & \downarrow & & \\ & \prod A_i & \overset{p_i}{\to} & A_i \end{array}$$

for $i \in I$, where p_i is the ith projection map. Since each A_i is s-injective there exist $g_i : C \to A_i$ for $i \in I$ such that $g_i h = p_i f$. Then the map $g : C \to \prod A_i$ which exists by the universal property of products, that is, $p_i g = g_i$ for each $i \in I$, satisfies $gh = f$. □

Proposition 3.6. *The following are equivalent:*
(i) *All right S-acts are s-injective.*
(ii) *S as an S-act is s-injective.*
(iii) *The identity map on S belongs to $\lambda(S)$.*
(iv) *S has a left identity element.*
(v) *S is generated by an idempotent element.*

Proof. (i)⇒(ii)⇒(iii), and (iv)⇒(v) are clear. To get (iii)⇒(iv), assuming $id_S = \lambda_e$, e would be a left identity of S. Finally, to see (v)⇒(i), taking $e \in S$ idempotent and $eS^1 = S$, for any Cauchy sequence $(a_s)_{s\in S}$ over an S-act A, we have $(a_s)_{s\in S} = \lambda(a_e)$. □

It is very interesting that the notion of s-completeness defined in the last section is the same as s-injectivity.

Theorem 3.7. *For any S-act A, the following are equivalent:*
(i) *A is s-complete.*
(ii) *A is s-injective.*
(iii) *A is s-absolute retract.*

Proof. (i)⇒(ii) Let $f : B \to C$ be an s-dense monomorphism, taking it as an inclusion, and let $g : B \to A$ be an S-map. Then, since f is s-dense, for every $c \in C$, $cs = b_s$ for some $b_s \in B$. Since for every $t \in S$ we have $g(b_s)t = g(cst)$, $(g(cs))_{s\in S}$ is a Cauchy sequence over A. But A is s-complete and therefore there exists $a_c \in A$ such that $a_c s = g(cs)$. Define $h : C \to A$ with $h\restriction_B = g$ and $h(c) = a_c$ for $c \in C - B$. To see that h is an S-map, let $c \in C - B,\ t \in S$. Then $h(ct) = g(ct) = a_c t = h(c)t$.

(ii)⇒(iii) is clear.

(iii)⇒(i) Let A be an s-absolute retract S-act and $(a_s)_{s\in S}$ be a Cauchy sequence over A. Consider the S-act $B = A \cup \{b\}$ with $bs = a_s, \forall s \in S$. Then the inclusion map $f : A \to B$ is an s-dense monomorphism. So there exists an S-map $g : B \to A$ such that $g\restriction_A = f$. Now $g(b)$ is a limit point of the Cauchy sequence $(a_s)_{s\in S}$. □

Now, applying the above theorem and Remark 2.6, we have

Corollary 3.8. *An S-act A is s-injective if and only if every Cauchy sequence is of the form $\lambda(a)$ for some $a \in A$.*

To close the paper we see how close is s-injectivity to ideal injectivity. Recall that

Definition 3.9. An S-act A is said to be

(i) *ideal injective*, if every S-map $f : I \to A$ from a right ideal I of S can be represented as $\lambda_a : s \mapsto as$, for some $a \in A$.

(ii) *weakly injective*, if every S-map $f : I \to A$ from a right ideal I of S can be extended to an S-map $\overline{f} : S \to A$.

Theorem 3.10. *An S-act A is ideal injective if and only if it is s-injective and weakly injective.*

Proof. It follows using Corollary 3.8 that ideal injectivity implies s-injectivity. This is because, every Cauchy sequence $(a_s)_{s\in S}$ represents an S-map $f : S \to A$ with $s \mapsto a_s$. Also, ideal injectivity gives weak injectivity, because any S-map of the form $\lambda_a : I \to A$ can be clearly extended to S.

Conversely, let $f : I \to A$ be an S-map. Then f can be extended to S assuming that A is weakly injective. Now, $(f(s))_{s\in S}$ is a Cauchy sequence. Assuming that A is s-injective, it is also s-complete by Theorem 3.7. So the above sequence is of the form $\lambda(a)$, for some $a \in A$. This means that $f = \lambda_a$. □

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