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Blow-up for a degenerate and singular parabolic equation with nonlocal boundary condition

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Abstract

The purpose of this work is to deal with the blow-up behavior of the nonnegative solution to a degenerate and singular parabolic equation with nonlocal boundary condition. The conditions on the existence and non-existence of the global solution are given. Further, under some suitable hypotheses, we discuss the blow-up set and the uniform blow-up profile of the blow-up solution.

Keywords: Degenerate and singular parabolic equation, global existence, blow-up, blow-up set, uniform blow-up profile.
2010 MSC: 35K50, 35K55, 35K65.

1. Introduction and main results

In this article, we consider the blow-up phenomenon of the following nonlinear degenerate and singular parabolic equation with nonlocal boundary condition

$$\begin{cases} u_t = (x^\alpha u_x)_x + \int_0^l u^p dx, & (x,t) \in (0,l) \times (0,+\infty), \\ u(0,t) = \int_0^l f(x) u^q(x,t) dx, & t \in (0,+\infty), \\ u(l,t) = \int_0^l g(x) u^q(x,t) dx, & t \in (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in [0,l], \end{cases} \tag{1.1}$$

where $0 \leq \alpha < 1$, p and q are positive parameters, the weight functions $f(x)$ and $g(x)$ are nonnegative continuous on $[0,l]$ and not identically zero, and $u_0(x)$ satisfies the following assumption:
(H_1) $u_0(x) \in C^{2+\chi}(0,l) \cap C[0,l]$ with $0 < \chi < 1$.

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(H$_2$) $u_0(x) > 0$ in $(0, l)$, $u_0(0) = \int_0^l f(x)\, u_0^q dx$ and $u_0(l) = \int_0^l g(x)\, u_0^q dx$.
(H$_3$) $(x^\alpha u_{0x})_x + \int_0^l u_0^p dx \geq 0$ for $x \in (0, l)$.

The equation in (1.1) arises in large numbers of physical phenomena. For example, it can be used to describe the conduction of heat related to the geometric shape of the body (see [2] and the references therein for more details of the physical background). It is necessary to point out that problem (1.1) is singular and degenerate because the coefficients of u_x and u_{xx} tend to ∞ and 0 as $x \to 0$, respectively.

Blow-up singularity, as one of the most remarkable properties that distinguish nonlinear parabolic problems from the linear ones, attracted extensive attention of mathematicians in the past few decades. There are many works focused on the global existence and the blow-up property of various degenerate and singular parabolic equations (or systems) with homogenous Dirichlet boundary conditions (see [1, 3, 4, 5, 18, 23, 24, 25] and the references therein).

On the other hand, parabolic equations with nonlocal (or nonlinear nonlocal) boundary conditions come from applied science, for instance, in the study of the heat conduction with thermoelastcity, Day [6, 7] derived a class of heat equation with nonlocal boundary in one-dimension space. In this model, the solution $u(x, t)$ describes entropy per volume of the material. Motivated by the works of Day, a lot of mathematicians devoted to studying the blow-up behaviours of different kinds of parabolic equations with nonlocal boundary conditions in the past few years. In particular, Lin and Liu [17] considered problem (1.1) with $\alpha = 0$ and $q = 1$ in multidimensional space. They obtained some results on the existence and nonexistence of the global solutions, and derived the uniform blow-up profile estimate under some assumptions. For other works on this topic, we refer the readers to [8, 9, 10, 19, 21] and the references therein.

However, as far as we know, there were only few articles which concerned with the blow-up behaviors of solutions for parabolic equations coupled with nonlocal nonlinear boundary condition. Gladkov and Kim [13, 14] considered a semilinear heat equation as the form

$$\begin{cases} u_t = \Delta u + c(x,t)\, u^p, & (x,t) \in \Omega \times (0, \infty), \\ u(x,t) = \int_\Omega \varphi(x,y,t)\, u^l(y,t)\, dy, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

where p, $l > 0$ and Ω is a bounded open domain in $\mathbb{R}^N$. First, they obtained the uniqueness and the non-uniqueness of the local solution (see [14]), then according to the different behaviors of the coefficient functions $c(x,t)$ and $\varphi(x,y,t)$ as t tends to infinity, they gave some criteria for the existence of the global solutions as well as for finite time blow-up solutions (see [13]). Recently, Gladkov and Guedda studied problem (1.2) with $c(x,t)\, u^p$ replaced by $-c(x,t)\, u^p$. The authors showed the existence, uniqueness and non-uniqueness of local solution (see [12]). What is more, they gave the critical blow-up exponent (see [11]).

The main goal of this article is to understand the effects of α, p, q and the weight functions $f(x)$ and $g(x)$ in problem (1.1) on the global existence and blow-up singularity of the solution to problem (1.1). Compared with [13] and [17], we need more skills to handle the difficulties, which are produced by the degeneration and singularity of problem (1.1) and the appearance of the nonlinear nonlocal boundary condition.

Throughout this article, we denote

$$\mathcal{N} = \max\left\{ \int_0^l f(x) dx, \int_0^l g(x) dx \right\},$$

and let λ_1 be the first eigenvalue and $\zeta(x)$ be the corresponding eigenfunction of the following eigenvalue problem

$$-(x^\alpha \zeta_x)_x = \lambda_1 \zeta, \quad 0 < x < l; \quad \zeta(0) = \zeta(l) = 0. \tag{1.3}$$

In fact, from [4, 20], we know that the principle eigenvalue λ_1 of the eigenvalue problem (1.3) is the first zero of $J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha} x^{\frac{2-\alpha}{2}}\right)$, where $J_{\frac{1-\alpha}{2-\alpha}}$ is Bessel function of the first kind of orders $\frac{1-\alpha}{2-\alpha}$. In addition, we know that $\zeta(x)$ is a positive smooth function in $(0, l)$ and can be expressed in an explicit form as follows

$$\zeta(x) = a x^{\frac{1-\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}}\right), \tag{1.4}$$

where a is an arbitrary positive parameter. Here, for the sake of convenience, we normalize $\zeta(x)$ in L^1-norm by choosing an appropriate value for parameter a.

The first part of our main results is on the existence and nonexistence of the global solution.

Theorem 1.1. *Assume that $p < 1$ and $q = 1$. Then for any nonnegative initial datum $u_0(x)$, the solution of problem (1.1) exists globally provided that $\mathcal{N} < 1$.*

Theorem 1.2. *Assume that $\min\{p, q\} > 1$. Then for any nonnegative weight functions $f(x)$ and $g(x)$, the solution of problem (1.1) exists globally provided that $u_0(x)$ is sufficiently small.*

Remark 1.3. In fact, when $p > 1$ and $q = 1$ (or $q > 1$ and $p = 1$), we can also prove that the solution of problem (1.1) exists globally for sufficiently small initial data. In the case $p > 1$ and $q < 1$ (or $q > 1$ and $p < 1$), we guess that the solution of problem (1.1) exists globally for small initial data, but we can not give a proof for this conjecture by the methods used in this paper. We hope to address this question in the future.

Theorem 1.4. *Assume that $\max\{p, q\} > 1$.*
(i) If $p = \max\{p, q\}$, then for any nonnegative weight functions $f(x)$ and $g(x)$, the solution of problem (1.1) blows up in finite time provided that $u_0(x)$ satisfies

$$\int_0^l u_0(x)\,\zeta(x)\,dx > \left[\lambda_1 \max_{x\in[0,l]} \zeta(x)\right]^{\frac{1}{p-1}}.$$

(ii) If $q = \max\{p, q\}$, then for any $f(x) \geq 0$ and $g(x) > 0$, the solution of problem (1.1) blows up in finite time provided that $u_0(x)$ satisfies

$$\int_0^l u_0(x)\,\zeta(x)\,dx > \left[\frac{\max\limits_{x\in[0,l]} \zeta(x)}{\min\limits_{x\in[0,l]} g(x)}\right]^{\frac{1}{q-1}}.$$

The second part of our main results is on the blow-up set and the uniform blow-up profile of the blow-up solution. In this part, we need the following two additional assumptions on initial datum $u_0(x)$.
(H$_4$) $\lim\limits_{x\to 0^+} \left[(x^\alpha u_{0x})_x + \int_0^l u_0^p dx\right] = \lim\limits_{x\to l^-} \left[(x^\alpha u_{0x})_x + \int_0^l u_0^p dx\right] = 0.$
(H$_5$) $(x^\alpha u_{0x})_x \leq 0$ in $(0, l)$.

Theorem 1.5. *Suppose that hypotheses $(H_1) - (H_5)$ hold, and assume that $p > 1$, $q = 1$ and $\mathcal{N} \leq 1$. Then the blow-up set of the blow-up solution $u(x,t)$ of problem (1.1) is the whole interval $(0, l)$.*

Theorem 1.6. *Under the assumptions of Theorem 1.5, we have*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e.\ in\ (0,l)\ as\ t \to T,$$

where T is the blow-up time.

Remark 1.7. There are many functions which satisfy the condition

$$\mathcal{N} = \max\left\{\int_0^l f(x)dx, \int_0^l g(x)dx\right\} \leq 1$$

in Theorems 1.1, 1.5 and 1.6. For example,

$$f(x) = g(x) = \frac{1}{l}\cos\frac{\pi}{2l}x.$$

The rest of this paper is organized as follows. In Section 2, we state the comparison theorem, the existence and uniqueness result on the local solution of problem (1.1) as preliminaries. Section 3 is mainly about the existence and nonexistence of the global solution and the proofs of Theorems 1.1, 1.2 and 1.4. The blow-up set and the uniform blow-up profile of the blow-up solution are considered in section 4.

2. Preliminaries

In this section, we will establish a suitable comparison principle for problem (1.1) and state the existence and uniqueness result on the local solution. For the sake of simplify, we first denote $I_T = (0,l) \times (0,T)$ and $\overline{I}_T = [0,l] \times [0,T)$. We begin with the definitions of the super-solution and sub-solution to problem (1.1).

Definition 2.1. A nonnegative function $\overline{u}(x,t)$ is called a super-solution of problem (1.1) if $\overline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfies

$$\begin{cases} \overline{u}_t \geq (x^\alpha \overline{u}_x)_x + \int_0^l \overline{u}^p dx, & (x,t) \in I_T, \\ \overline{u}(0,t) \geq \int_0^l f(x) \overline{u}^q(x,t) dx, & t \in (0,T), \\ \overline{u}(l,t) \geq \int_0^l g(x) \overline{u}^q(x,t) dx, & t \in (0,T), \\ \overline{u}(x,0) \geq \overline{u}_0(x), & x \in [0,l]. \end{cases} \tag{2.1}$$

Similarly, $\underline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ is called a sub-solution of problem (1.1) if it satisfies all the reversed inequalities in (2.1). We say that $u(x,t)$ is a solution of problem (1.1) if it is both a sub-solution and a super-solution of problem (1.1).

Now, by making use of the similar arguments as those in [8], we can prove the following maximum principle, which plays a critical role in the discussions of the blow-up set and the uniform blow-up profile of the blow-up solution.

Lemma 2.2. *Let $\omega(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfy*

$$\begin{cases} \omega_t - (x^\alpha \omega_x)_x \geq \int_0^l \theta_1(x,t) \omega(x,t) dx, & (x,t) \in I_T, \\ \omega(0,t) \geq \int_0^l \theta_2(x) \omega(x,t) dx, & t \in (0,T), \\ \omega(l,t) \geq \int_0^l \theta_3(x) \omega(x,t) dx, & t \in (0,T), \end{cases} \tag{2.2}$$

where $\theta_i(x,t)$, $i = 1,2,3$, are bounded functions, $\theta_1(x,t)$ is nonnegative for $(x,t) \in I_T$, $\theta_2(x)$ and $\theta_3(x)$ are nonnegative, nontrivial in $(0,l)$. Then $\omega(x,0) > 0$ in $[0,l]$ implies that $\omega(x,t) > 0$ for $(x,t) \in I_T$. Moreover, if one of the following conditions holds, (a) $\theta_2(x) = \theta_3(x) \equiv 0$ for $x \in (0,l)$; (b) $\theta_2(x), \theta_3(x) \geq 0$ for $x \in (0,l)$ and $\max\left\{\int_0^l \theta_2(x) dx, \int_0^l \theta_3(x) dx\right\} \leq 1$, then $\omega(x,0) \geq 0$ in $[0,l]$ leads to $\omega(x,t) \geq 0$ for $(x,t) \in I_T$.

By using the idea in [16], we can show the following comparison principle, which plays an important part in investigating the existence of the global solution for problem (1.1).

Proposition 2.3 (Comparison principle)**.** *Let $\overline{u}(x,t)$ and $\underline{u}(x,t)$ be a nonnegative super-solution and sub-solution of problem (1.1), respectively. Suppose that either $\overline{u}(x,t) > 0$ or $\underline{u}(x,t) > 0$ if $\min\{p,q\} < 1$. Then $\overline{u}(x,t) \geq \underline{u}(x,t)$ holds in $\overline{I}_T$ if $\overline{u}(x,0) \geq \underline{u}(x,0)$ for $x \in [0,l]$.*

Next, we state the result on the existence and uniqueness of the local solution of problem (1.1) at the end of this section.

Theorem 2.4 (Local existence and uniqueness)**.** *Assume that (H_1), (H_2) and (H_3) hold, then there exists a small positive real number T such that problem (1.1) admits a nonnegative solution $u(x,t) \in C(\overline{I}_T) \cap C^{2,1}(I_T)$. Furthermore, assume that the initial datum $u_0(x)$ is positive for the case $\min\{p,q\} < 1$, then the local solution of problem (1.1) is unique.*

Remark 2.5. We can get the proof of Theorem 2.4 by using regularization method and Schauder's fixed point theorem. For more details, we refer the readers to [4, 22]. Moreover, for the case $\min\{p,q\} \geq 1$, the uniqueness of the local solution holds without the restrictive condition $u_0 > 0$.

3. Global existence and blow-up in finite time

In this section, first of all, by constructing some appropriate global super-solutions and employing comparison principle, we investigate the existence of the global solutions for problem (1.1), and give the proofs of Theorems 1.1 and 1.2, respectively.

Proof of Theorem 1.1. Putting

$$\sigma(x) = \frac{l\epsilon_0}{2-\alpha}x^{1-\alpha} - \frac{\epsilon_0}{2-\alpha}x^{2-\alpha} + \mathcal{N}, \quad x \in [0,l],$$

where

$$\epsilon_0 \in \left(0, \frac{(1-\mathcal{N})(2-\alpha)^{3-\alpha}}{l^{2-\alpha}(1-\alpha)^{1-\alpha}}\right) \tag{3.1}$$

is a given constant. Then, it is not difficult to verify that

$$\begin{cases} -(x^\alpha \sigma_x)_x = \epsilon_0, & 0 < x < l, \\ \sigma(0) = \sigma(l) = \mathcal{N}, \end{cases} \tag{3.2}$$

and

$$\min_{x\in[0,l]} \sigma(x) = \mathcal{N}, \quad \max_{x\in[0,l]} \sigma(x) = \mathcal{N} + \frac{\epsilon_0 l^{2-\alpha}(1-\alpha)^{1-\alpha}}{(2-\alpha)^{3-\alpha}} < 1. \tag{3.3}$$

Defining

$$v_1(x,t) = \epsilon_1 \sigma(x),$$

where

$$\epsilon_1 = \max\left\{\mathcal{N}^{-1}\max_{x\in[0,1]}(u_0(x)+1), \left[l\epsilon_0\left(\max_{x\in[0,l]}\sigma(x)\right)^p\right]^{\frac{1}{1-p}}\right\}. \tag{3.4}$$

Calculating directly, one has

$$\begin{aligned} \mathcal{L}(v_1) &:\equiv v_{1t} - (x^\alpha v_{1x})_x - \int_0^l v_1^p dx \\ &= \epsilon_0\epsilon_1 - \epsilon_1^p \int_0^l \sigma^p dx \\ &\geq \epsilon_0\epsilon_1 - l\left[\epsilon_1 \max_{x\in[0,l]} \sigma(x)\right]^p. \end{aligned}$$

By $p < 1$ and the choice of the value for ϵ_1, for any $x \in (0,l)$ and $t \in (0,\infty)$, we can easily deduce that

$$\mathcal{L}(v_1) \geq 0. \tag{3.5}$$

On the other hand, for $x = 0$, we have that

$$\begin{aligned} v_1(0,t) = \epsilon_1\sigma(0) &\geq \int_0^l \epsilon_1 f(x)\, dx \geq \int_0^l \epsilon_1 f(x)\sigma(x)\, dx \\ &= \int_0^l f(x) v_1(x,t)\, dx. \end{aligned} \tag{3.6}$$

By the similar argument, for $x = l$, we can claim that

$$v_1(l,t) \geq \int_0^l g(x) v_1(x,t)\, dx. \tag{3.7}$$

From (3.4), (3.5), (3.6) and (3.7), we can infer that $v_1(x,t)$ is a global super-solution of problem (1.1). And hence, by comparison principle, we know that the solution $u(x,t)$ of problem (1.1) exists globally. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Taking a bounded open interval $(a,b)\subset\mathbb{R}$ such that $(0,l)\subset\subset(a,b)$. Letting $\widetilde{\lambda}_1$ be the first eigenvalue and $\widetilde{\zeta}(x)$ be the associated eigenfunction of the following eigenvalue problem

$$-\left(x^{\alpha}\widetilde{\zeta}_x\right)_x=\widetilde{\lambda}_1\widetilde{\zeta},\quad a<x<b;\quad \zeta(a)=\zeta(b)=0. \tag{3.8}$$

It is clear that there exists a constant $\mu_1\in(1,\infty)$ such that

$$\max_{x\in[a,b]}\widetilde{\zeta}(x)<\mu_1\min_{x\in[0,l]}\widetilde{\zeta}(x). \tag{3.9}$$

Putting

$$\widehat{\zeta}(x)=\frac{\mu_1\mu_2}{\max\limits_{x\in[a,b]}\widetilde{\zeta}(x)}\widetilde{\zeta}(x),$$

where μ_2 will be specialized later. Then it is easy to verify that

$$\max_{x\in[a,b]}\widehat{\zeta}(x)=\mu_1\mu_2$$

and

$$\frac{\max\limits_{x\in[a,b]}\widehat{\zeta}(x)}{\min\limits_{x\in[0,l]}\widehat{\zeta}(x)}=\frac{\max\limits_{x\in[a,b]}\widetilde{\zeta}(x)}{\min\limits_{x\in[0,l]}\widetilde{\zeta}(x)}<\mu_1. \tag{3.10}$$

Furthermore, $\widehat{\zeta}(x)$ also satisfies problem (3.8). From (3.9), we have that

$$\min_{x\in[0,l]}\widehat{\zeta}(x)>\mu_2. \tag{3.11}$$

Setting

$$h(t)=e^{-\widetilde{\lambda}_1 t}\left[1+\frac{l\mu_1^p\mu_2^{p-1}}{\widetilde{\lambda}_1e^{(p-1)\widetilde{\lambda}_1 t}}\right]^{-\frac{1}{p-1}}, \tag{3.12}$$

and

$$v_2(x,t)=\widehat{\zeta}(x)h(t).$$

A Simple computation shows that

$$\begin{aligned}\mathcal{L}(v_2)&=\widehat{\zeta}(x)h'(t)+\widetilde{\lambda}_1\widehat{\zeta}(x)h(t)-h^p(t)\int_0^l\widehat{\zeta}^p(x)\,dx\\&\geq\widehat{\zeta}(x)\left[h'(t)+\widetilde{\lambda}_1h(t)-l\mu_1^p\mu_2^{p-1}h^p(t)\right]\\&\geq 0.\end{aligned} \tag{3.13}$$

Choosing $\mu_2\in\left(0,(\mathcal{N}\mu_1^q)^{-\frac{1}{q-1}}\right)$, and noticing that $h(t)\in(0,1)$, then we have that

$$v_2(0,t)\geq\mu_2h(t)\geq\mathcal{N}\mu_1^q\mu_2^qh^q(t)\geq\int_0^l f(x)v_2^q(x,t)\,dx, \tag{3.14}$$

and

$$v_2(l,t)\geq\int_0^l g(x)v_2^q(x,t)\,dx. \tag{3.15}$$

By exploiting (3.13), (3.14) and (3.15), we can conclude that $v_2(x,t)$ is a global super-solution of problem (1.1) provided that

$$u_0(x)\leq\mu_2\left(\frac{\widetilde{\lambda}_1+l\mu_1^p\mu_2^{p-1}}{\widetilde{\lambda}_1}\right)^{-\frac{1}{p-1}}. \tag{3.16}$$

That is to say, the solution $u(x,t)$ of problem (1.1) exists globally if $u_0(x)$ fulfill (3.16). The proof Theorem 1.4 is complete. □

Now, by using a slight variant of the eigenfunction method (Kaplan's Method), which is introduced by Kaplan in [15], we will discuss the blow-up singularity in finite time for problem (1.1) with $\max\{p,q\}>1$ and sufficiently large initial data.

Proof of Theorem 1.4. Letting an auxiliary function $\Pi(t)$ be defined as

$$\Pi(t)=\int_0^l u(x,t)\,\zeta(x)\,dx.$$

Multiplying both sides of the equation in problem (1.1) by $\zeta(x)$, where $\zeta(x)$ is given by (1.4), and integrating from 0 to l, one has

$$\begin{aligned}\Pi'(t)&=\int_0^l\left[(x^\alpha u_x)_x+\int_0^l u^p dx\right]\zeta(x)\,dx\\&=-\lambda_1\Pi+\int_0^l u^p dx+\lambda_1\int_0^l g(x)\,u^q dx.\end{aligned}\tag{3.17}$$

When $p=\max\{p,q\}>1$, then it follows from (3.17) and Jensen's inequality that

$$\Pi'(t)\geq-\lambda_1\Pi+\frac{1}{\max\limits_{x\in[0,l]}\zeta(x)}\Pi^p.\tag{3.18}$$

Solving (3.18), we obtain

$$\Pi(t)\geq\left\{\frac{\lambda_1\max\limits_{x\in[0,l]}\zeta(x)}{1-\left[1-\lambda_1\Pi(0)^{1-p}\max\limits_{x\in[0,l]}\zeta(x)\right]e^{\lambda_1(p-1)t}}\right\}^{\frac{1}{p-1}}.\tag{3.19}$$

From (3.19), we know that if

$$\Pi(0)=\int_0^l u_0(x)\,\zeta(x)\,dx>\left[\lambda_1\max_{x\in[0,l]}\zeta(x)\right]^{\frac{1}{p-1}},$$

then

$$\lim_{t\to T}\Pi(t)=\infty,$$

where

$$T=\frac{1}{\lambda_1(p-1)}\ln\frac{\Pi(0)^{p-1}}{\Pi(0)^{p-1}-\lambda_1\max\limits_{x\in[0,l]}\zeta(x)}.$$

When $q=\max\{p,q\}>1$, we can deduce from (3.17) that

$$\Pi'(t)\geq-\lambda_1\left[\Pi+\frac{\min\limits_{x\in[0,l]}g(x)}{\max\limits_{x\in[0,l]}\zeta(x)}\Pi^q\right],$$

which implies that, for any positive weight function $g(x)$, $\Pi(t)$ tends to infinity in a finite time provided that

$$\Pi(0)>\left[\frac{\max\limits_{x\in[0,l]}\zeta(x)}{\min\limits_{x\in[0,l]}g(x)}\right]^{\frac{1}{q-1}}.$$

The proof of Theorem 1.4 is complete. □

4. Blow-up set and uniform blow-up profile

In this section, we will discuss the blow-up set and the uniform blow-up profile of the blow-up solution for problem (1.1). Throughout this section, we assume that $p > 1$, $q = 1$ and $\mathcal{N} \leq 1$. From Theorem 1.4, we see that the solution $u(x,t)$ of problem (1.1) blows up in finite time for sufficiently large initial data. In addition, we denote T the blow-up time.

From $(\mathrm{H}_1) - (\mathrm{H}_5)$, we know that there exist a sufficiently small positive constant ε_1 and a nonnegative function $w_{0\varepsilon}(x)$ such that

(1) $w_{0\varepsilon} \in C^{2+\chi}(\varepsilon, l-\varepsilon) \cap C[\varepsilon, l-\varepsilon]$ with $\chi \in (0,1)$ and $\varepsilon \in (0, \varepsilon_1]$.

(2) $w_{0\varepsilon}(\varepsilon) = \int_\varepsilon^{l-\varepsilon} f(x) w_{0\varepsilon}(x)\,dx$ and $w_{0\varepsilon}(l-\varepsilon) = \int_\varepsilon^{l-\varepsilon} g(x) w_{0\varepsilon}(x)\,dx$.

(3) $w_{0\varepsilon}(x) < u_0(x)$ for $x \in (\varepsilon, 2\varepsilon) \cup (l-2\varepsilon, l-\varepsilon)$, and $w_{0\varepsilon}(x) = u_0(x)$ for $x \in [2\varepsilon, l-2\varepsilon]$.

(4) $(x^\alpha w_{0\varepsilon x})_x \leq 0$ for $x \in (\varepsilon, l-\varepsilon)$.

(5) $w_{0\varepsilon}$ is non-increasing with respect to ε in $(0, \varepsilon_1]$. Moreover

$$\lim_{x\to\varepsilon^+}\left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx\right] = \lim_{x\to(l-\varepsilon)^-}\left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx\right] = 0.$$

(6) $(x^\alpha w_{0\varepsilon x})_x + \int_0^l w_{0\varepsilon}^p dx \geq 0$ for $\varepsilon \in (0, \varepsilon_1]$ and $x \in (\varepsilon, l-\varepsilon)$.

It is obvious that

$$\lim_{\varepsilon\to0^+} w_{0\varepsilon}(x) = u_0(x).$$

Now, we consider the following regularized problem

$$\begin{cases} w_{\varepsilon t} = (x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx, & (x,t) \in (\varepsilon, l-\varepsilon) \times (0, +\infty), \\ w_\varepsilon(\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t)dx, & t \in (0,+\infty), \\ w_\varepsilon(l-\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} g(x) w_\varepsilon(x,t)dx, & t \in (0,+\infty), \\ w_\varepsilon(x,0) = w_{0\varepsilon}(x), & x \in [0,l]. \end{cases} \tag{4.1}$$

Then it is not difficult to show that there exists a unique solution $w_\varepsilon(x,t)$ for problem (4.1). In addition, from the arguments of Section 2 in [24], it follows that

$$\lim_{\varepsilon\to0^+} w_\varepsilon(x,t) = u(x,t),$$

where $u(x,t)$ is the solution of problem (1.1).

Lemma 4.1. *Suppose that hypotheses $(H_1) - (H_5)$ hold, and assume that $p > 1$, $q = 1$ and $\mathcal{N} \leq 1$. Then $(x^\alpha u_x)_x \leq 0$ holds for $(x,t) \in I_T$.*

Proof. Taking $\eta = (x^\alpha w_{\varepsilon x})_x$, then from (4.1), we have

$$\eta_t = \left\{x^\alpha\left[(x^\alpha w_{\varepsilon x})_x + \int_0^l w_\varepsilon^p\right]_x\right\}_x = (x^\alpha \eta_x)_x \tag{4.2}$$

holds for any $(x,t) \in (\varepsilon, l-\varepsilon) \times (0,T)$. On the other hand, for any $t \in (0,T)$, we have

$$\begin{aligned} \eta(\varepsilon,t) &= \int_\varepsilon^{l-\varepsilon} f(x) w_{\varepsilon t}(x,t)\,dx - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\,dx \\ &= \int_\varepsilon^{l-\varepsilon} f(x)\left((x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\,dx\right)dx - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\,dx \\ &= \int_\varepsilon^{l-\varepsilon} f(x)\eta(x,t)\,dx + \left(\int_\varepsilon^{l-\varepsilon} f(x)\,dx - 1\right)\int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t)\,dx. \end{aligned} \tag{4.3}$$

By the assumption $\mathcal{N} \leq 1$, we can claim from (4.3) that

$$\eta(\varepsilon, t) \leq \int_{\varepsilon}^{l-\varepsilon} f(x)\, \eta(x, t)\, dx, \quad t \in (0, T). \tag{4.4}$$

Applying the analogous arguments, we can also verify that

$$\eta(l-\varepsilon, t) \leq \int_{\varepsilon}^{l-\varepsilon} g(x)\, \eta(x, t)\, dx \tag{4.5}$$

holds for all $t \in (0, T)$.

Moreover, noticing that $\eta(x, 0) = (x^{\alpha} w_{0\varepsilon x})_x \leq 0$ holds for $x \in (\varepsilon, l-\varepsilon)$. Then, maximum principle tells us that $\eta(x, t) = (x^{\alpha} w_{\varepsilon x})_x \leq 0$ holds for all $(x, t) \in (\varepsilon, l-\varepsilon) \times (0, T)$. In addition, by the arbitrariness of ε, we know that $(x^{\alpha} u_x)_x \leq 0$ holds in I_T. The proof of Lemma 4.1 is complete. □

In what follows, for the sake of simplicity, we denote

$$\psi(t) = \int_0^l u^p(x, t)\, dx \text{ and } \Psi(t) = \int_0^t \psi(\tau)\, d\tau.$$

Proof of Theorem 1.5. First, for any given open interval $(l_1, l_2) \subset\subset (0, l)$, let $m = \inf_{x \in (l_1, l_2)} \mu(x)$, where $\mu(x)$ is the unique positive solution of the following boundary value problem

$$\begin{cases} -(x^{\alpha}\mu_x)_x = 1, & 0 < x < l, \\ \mu(0) = \mu(l) = 0. \end{cases} \tag{4.6}$$

In fact, $\mu(x)$ can be expressed in an explicit form that

$$\mu(x) = \frac{1}{2-\alpha} x^{1-\alpha}(l-x), \quad x \in [0, l].$$

Lemma 4.1 leads us to get that

$$\int_0^l u dx = -\int_0^l u (x^{\alpha}\mu_x)_x\, dx = -\int_0^l \mu (x^{\alpha} u_x)_x\, dx \geq -m \int_{l_1}^{l_2} (x^{\alpha} u_x)_x\, dx,$$

and hence, we obtain

$$0 \leq \lim_{t \to T} -m \int_{l_1}^{l_2} \frac{(x^{\alpha} u_x)_x}{\psi(t)} dx \leq \lim_{t \to T} \frac{\int_0^l u dx}{\int_0^l u^p dx} \leq 0,$$

which implies that

$$\lim_{t \to T} \int_{l_1}^{l_2} \frac{(x^{\alpha} u_x)_x}{\psi(t)} dx = 0.$$

From the arbitrariness of (l_1, l_2), one can infer that

$$\lim_{t \to T} \frac{(x^{\alpha} u_x)_x}{\psi(t)} = 0 \text{ a.e. in } (0, l). \tag{4.7}$$

Now, integrating the first equation in problem (1.1) from 0 to t, then for $(x, t) \in I_T$, we get

$$u(x, t) - u_0(x) = \int_0^t \left(x^{\alpha} u_x(x, s)_x\right) ds + \Psi(t). \tag{4.8}$$

Since $\lim_{t \to T} \|u(\cdot, t)\|_{\infty} = \infty$, then from Lemma 4.1 and (4.8), one can immediately deduce that

$$\lim_{t \to T} \Psi(t) = \infty. \tag{4.9}$$

It follows from (4.7) and (4.9) that

$$\lim_{t\to T}\frac{\int_0^t\left(x^{\alpha}u_x\left(x,s\right)_x\right)ds}{\Psi\left(t\right)}=0 \text{ a.e. in } \left(0,l\right). \tag{4.10}$$

Dividing both sides of (4.8) by $\Psi(t)$ and letting $t\to T$, we see that

$$\lim_{t\to T}\frac{u\left(x,t\right)}{\Psi\left(t\right)}=1 \text{ a.e. in } \left(0,l\right), \tag{4.11}$$

which means that the solution $u(x,t)$ of problem (1.1) blows up at almost everywhere in $(0,l)$. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. Equation (4.11) tells us that

$$u\left(x,t\right)\sim\Psi\left(t\right) \text{ a.e. in } \left(0,l\right) \text{ as } t\to T. \tag{4.12}$$

When t is sufficiently closed to T, by employing (4.8) and (4.9), we know that there exists a constant M such that

$$0\leq\frac{u\left(x,t\right)}{\Psi\left(t\right)}\leq M \tag{4.13}$$

holds for all $x\in(0,l)$. Up to now, Lebesgue's dominated convergence theorem can be used to get

$$\Psi'\left(t\right)=\psi\left(t\right)=\int_0^l u^p\left(x,t\right)dx\sim l\Psi^p\left(t\right) \quad \text{as } t\to T. \tag{4.14}$$

Integrating (4.14) over (t,T), one has

$$\Psi\left(t\right)\sim\left[l\left(p-1\right)\left(T-t\right)\right]^{-\frac{1}{p-1}} \quad \text{as } t\to T. \tag{4.15}$$

Combining (4.12) and (4.15), we can easily obtain that

$$u\left(x,t\right)\sim\left[l\left(p-1\right)\left(T-t\right)\right]^{-\frac{1}{p-1}} \quad \text{as } t\to T. \tag{4.16}$$

The proof of Theorem 1.6 is complete. □

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