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Existence of nonoscillatory solutions to second-order nonlinear neutral difference equations

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Abstract

We study a class of second-order neutral delay difference equations with positive and negative coefficients

$$\Delta(r_n(\Delta(x_n + px_{n-m}))) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0, \quad n = n_0, n_0 + 1, \ldots,$$

where $p \in R$, $m, k, l, n_0 \in N$, $p_n, q_n, r_n \in R^+$, $f, g \in C(R, R)$ with $xf(x) > 0$ and $xg(x) > 0$ $(x \neq 0)$. Some sufficient conditions for the existence of a nonoscillatory solution of the studied equation expressed in terms of $\sum^\infty R_n p_n < \infty$ and $\sum^\infty R_n q_n < \infty$ are obtained, where $R_n = \sum_{s=n_0}^{n} \frac{1}{r_s}, n \geq n_0$.

Keywords: Nonoscillatory solution, neutral delay difference equation, second-order, positive and negative coefficients.
2010 MSC: 34B15, 34B25.

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1. Introduction

This paper is concerned with a second-order neutral delay difference equation with positive and negative coefficients

$$\Delta(r_n(\Delta(x_n + px_{n-m}))) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0, \quad n = n_0, n_0 + 1, \ldots, \tag{1.1}$$

where Δ stands for the forward difference operator, $\Delta x_n = x_{n+1} - x_n$, $p \in R$, $m, k, l, n_0 \in N$, $p_n, q_n, r_n \in R^+$, $f, g \in C(R, R)$, $xf(x) > 0$, and $xg(x) > 0$ for all $x \neq 0$. Throughout, we suppose that the following assumptions are satisfied.

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(H_1) f and g satisfy local Lipchitz conditions, Lipchitz constants are denoted by $L_f(A)$ and $L_g(A)$, where A is the domain that f and g are defined;

(H_2) $R_n = \sum_{s=n_0}^{n} \frac{1}{r_s}, n \geq n_0$, $\sum^{\infty} R_s p_s < \infty$, and $\sum^{\infty} R_s q_s < \infty$.

In recent years, there has been an increasing interest in studying the oscillatory and nonoscillatory behavior of various classes of differential, difference, and dynamic equations; see, for instance, the monographs [1, 2], papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], and the references cited therein. Candan [3] investigated a higher-order nonlinear neutral differential equation

$$\left[r(t)(x(t)+P(t)x(t-\tau))^{(n-1)}\right]' + (-1)^n \left[Q_1(t)g_1(x(t-\sigma_1)) - Q_2(t)g_2(x(t-\sigma_2)) - f(t)\right] = 0, \tag{1.2}$$

where $t \geq t_0$, $n \geq 2$ is an integer, $r \in C([t_0,\infty), R^+)$, $P, f \in C([t_0,\infty), R)$, $Q_i \in C([t_0,\infty), R^+)$, $i = 1,2$, and $g_i \in C(R,R)$, $i = 1,2$, satisfy the local Lipschitz condition with $xg_i(x) > 0$, $i = 1,2$ for $x \neq 0$. Using the Banach contraction principle, the author obtained some sufficient conditions for the existence of nonoscillatory solutions to (1.2). Cheng [6] studied the existence of nonoscillatory solution of a second-order linear neutral difference equation

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n = n_0, n_0+1, \ldots, \tag{1.3}$$

where $p \in R$, $m,k,l,n_0 \in N$, $p_n, q_n \in R^+$, and some other special cases of equation (1.3) were considered by Li et al. [9] and Zhang and Zhou [13]. In particular, Cheng [6] established the following result.

Theorem 1.1 (See [6, Theorem 1]). *Suppose that* $p \neq -1$, $\sum^{\infty} sp_s < \infty$, *and* $\sum^{\infty} sq_s < \infty$. *Then equation* (1.3) *has a nonoscillatory solution.*

To the best of our knowledge, there are few results for second-order nonlinear difference equations with positive and negative coefficients. Motivated by the ideas exploited in [3, 6], we obtain the global results (with respect to p), which are some sufficient conditions for the existence of a nonoscillatory solution of (1.1) for $p \neq -1$. The results obtained extend those reported in [6]. An example is considered to illustrate the possible applications.

2. Main results

Theorem 2.1. *Assume that* $p \neq -1$ *and conditions* (H_1) *and* (H_2) *are satisfied. Then* (1.1) *has a bounded nonoscillatory solution.*

Proof. The proof of Theorem 2.1 will be divided into five cases, depending on the five different ranges of the parameter p. Let $l_{n_0}^{\infty}$ be the Banach space which is composed of all bounded real sequences $x = \{x_n\}_{n=n_0}^{\infty}$ with the norm $||x|| = \sup_{n \geq n_0} |x_n|$.

Case 1. $p = 1$. By (H_1) and (H_2), one can choose an $n_* \geq n_0 + \max\{m,k,l\}$ sufficiently large such that, for all $n \geq n_*$,

$$\begin{aligned} &\sum_{u=n}^{\infty}(R_u - R_{n-1})p_u \leq \frac{1}{\alpha}, \\ &\sum_{u=n}^{\infty}(R_u - R_{n-1})q_u \leq \frac{1}{\beta}, \\ &\sum_{u=n}^{\infty}(R_u - R_{n-1})(p_u + q_u) < \min\left\{\frac{1}{L}, \frac{1}{\alpha} + \frac{1}{\beta}\right\}, \end{aligned} \tag{2.1}$$

where $\alpha = \max_{1 \leq x \leq 3}\{f(x)\}$, $\beta = \max_{1 \leq x \leq 3}\{g(x)\}$, and $L = \max\{L_f([1,3]), L_g([1,3])\}$.

We define a bounded, closed, and convex subset S in $l_{n_0}^{\infty}$ by

$$S = \{x = \{x_n\} \in l_{n_0}^{\infty} : 1 \leq x_n \leq 3, n \geq n_0\}.$$

Consider the operator $T : S \to l^\infty_{n_0}$ defined by

$$(Tx)_n = \begin{cases} 2 - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)m}^{n+2jm} \left(\frac{1}{r_s} \sum_{u=s}^{\infty} (p_u f(x_{u-k}) - q_u g(x_{u-l})) \right), & n \geq n_*, \\ (Tx)_{n_*}, \quad n_0 \leq n \leq n_*. \end{cases}$$

Clearly, Tx_n is a real sequence. It is not difficult to show that T is a continuous mapping on S. For every $x = \{x_n\} \in S$ and $n \geq n_*$, we obtain

$$\begin{aligned} (Tx)_n &\leq 2 + \sum_{j=1}^{\infty} \left[\sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{r_s} \sum_{u=s}^{\infty} q_u g(x_{u-l}) + \sum_{s=n+(2j-2)m}^{n+(2j-1)m} \frac{1}{r_s} \sum_{u=s}^{\infty} q_u g(x_{u-l}) \right] \\ &= 2 + \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{u=s}^{\infty} q_u g(x_{u-l}) = 2 + \sum_{u=n}^{\infty} \sum_{s=n}^{u} \frac{1}{r_s} q_u g(x_{u-l}) \\ &= 2 + \sum_{u=n}^{\infty} (R_u - R_{n-1}) q_u g(x_{u-l}) \leq 2 + \beta \sum_{u=n}^{\infty} (R_u - R_{n-1}) q_u \leq 3. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (Tx)_n &\geq 2 - \sum_{j=1}^{\infty} \left[\sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{r_s} \sum_{u=s}^{\infty} p_u f(x_{u-k}) + \sum_{s=n+(2j-2)m}^{n+(2j-1)m} \frac{1}{r_s} \sum_{u=s}^{\infty} p_u f(x_{u-k}) \right] \\ &= 2 - \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{u=s}^{\infty} p_u f(x_{u-k}) = 2 - \sum_{u=n}^{\infty} \sum_{s=n}^{u} \frac{1}{r_s} p_u f(x_{u-k}) \\ &= 2 - \sum_{u=n}^{\infty} (R_u - R_{n-1}) p_u f(x_{u-l}) \geq 2 - \alpha \sum_{u=n}^{\infty} (R_u - R_{n-1}) p_u \geq 1. \end{aligned}$$

Thus, we conclude that $TS \subseteq S$.

Next, we prove that T is a contraction mapping on S. As a matter of fact, for every $x, y \in S$ and $n \geq n_*$, we get

$$\begin{aligned} |Tx_n - Ty_n| &\leq \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)m}^{n+2jm} \left(\frac{1}{r_s} \sum_{u=s}^{\infty} (p_u |f(x_{u-k}) - f(y_{u-k})| + q_u |g(x_{u-l}) - g(y_{u-l})|) \right) \\ &\leq L||x-y|| \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{u=s}^{\infty} (p_u + q_u) = L||x-y|| \sum_{u=n}^{\infty} \sum_{s=n}^{u} \frac{1}{r_s} (p_u + q_u) \\ &= L||x-y|| \sum_{u=n}^{\infty} (R_u - R_{n-1})(p_u + q_u) = p_0 ||x-y||, \end{aligned}$$

which implies that

$$||Tx - Ty|| \leq p_0 ||x-y||,$$

where $p_0 = L \sum_{u=n}^{\infty} (R_u - R_{n-1})(p_u + q_u)$. Using (2.1), we have $p_0 < 1$, and thus T is a contraction mapping. Consequently, T has a unique fixed x such that $(Tx)_n = x_n$, that is,

$$x_n = \begin{cases} 2 - \sum_{j=1}^{\infty} \sum_{s=n+(2j-1)m}^{n+2jm} \frac{1}{r_s} \sum_{u=s}^{\infty} [p_u f(x_{u-k}) - q_u g(x_{u-l})], & n \geq n_*, \\ (Tx)_{n_*}, \quad n_0 \leq n \leq n_*. \end{cases}$$

Furthermore, we have

$$
\begin{aligned}
x_n + x_{n-m} &= 4 - \sum_{j=1}^{\infty} \left[\sum_{s=n+(2j-1)m}^{n+2jm} + \sum_{s=n+(2j-2)m}^{n+(2j-1)m} \right] \frac{1}{r_s} \sum_{u=s}^{\infty} (p_u f(x_{u-k}) - q_u g(x_{u-l})) \\
&= 4 - \sum_{s=n}^{\infty} \frac{1}{r_s} \sum_{u=s}^{\infty} (p_u f(x_{u-k}) - q_u g(x_{u-l})).
\end{aligned}
$$

Therefore,

$$
\Delta(r_n(\Delta(x_n + x_{n-m}))) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0,
$$

and x_n is obviously a positive solution of (1.1). This completes the proof of Case 1.

Case 2. $p \in (0, 1)$. By virtue of conditions (H_1) and (H_2), we can choose an $n_1 \geq n_0 + \max\{m, k, l\}$ sufficiently large such that

$$
\begin{aligned}
&\sum_{s=n}^{\infty} R_s p_s \leq \frac{p - (1 - N_1)}{\alpha_1}, \\
&\sum_{s=n}^{\infty} R_s q_s \leq \frac{1 - p - pN_1 - M_1}{\beta_1}, \\
&\sum_{s=n}^{\infty} R_s (p_s + q_s) < \frac{1 - p}{L_1}
\end{aligned} \tag{2.2}
$$

hold for all $n \geq n_1$, where $N_1 \geq M_1 > 0$, $1 - N_1 < p < (1 - M_1)/(1 + N_1)$, $\alpha_1 = \max_{M_1 \leq x \leq N_1} \{f(x)\}$, $\beta_1 = \max_{M_1 \leq x \leq N_1} \{g(x)\}$, and $L_1 = \max\{L_f([M_1, N_1]), L_g([M_1, N_1])\}$. Set

$$
A_1 = \{x = \{x_n\} \in l_{n_0}^{\infty} : M_1 \leq x_n \leq N_1, n \geq n_0\}.
$$

Define an operator $T : A_1 \to l_{n_0}^{\infty}$ by

$$
(Tx)_n = \begin{cases}
1 - p - px_{n-m} + R_{n-1} \displaystyle\sum_{s=n-1}^{\infty} (p_s f(x_{s-k}) - q_s g(x_{s-l})) \\
\quad + \displaystyle\sum_{s=n_1}^{n-2} R_s (p_s f(x_{s-k}) - q_s g(x_{s-l})), \quad n \geq n_1, \\
(Tx)_{n_1}, \quad n_0 \leq n \leq n_1.
\end{cases}
$$

For every $x \in A_1$ and $n \geq n_1$, we have

$$
\begin{aligned}
(Tx)_n &\leq 1 - p + \alpha_1 R_{n-1} \sum_{s=n-1}^{\infty} p_s + \alpha_1 \sum_{s=n_1}^{n-2} R_s p_s \\
&\leq 1 - p + \alpha_1 \sum_{s=n_1}^{\infty} R_s p_s \leq N_1.
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
(Tx)_n &\geq 1 - p - pN_1 - R_{n-1} \sum_{s=n-1}^{\infty} q_s g(x_{s-l}) - \sum_{s=n_1}^{n-2} R_s q_s g(x_{s-l}) \\
&\geq 1 - p - pN_1 - \beta_1 \sum_{s=n_1}^{\infty} R_s q_s \geq M_1,
\end{aligned}
$$

and hence $TA_1 \subseteq A_1$.

Now, for $x, y \in A_1$ and $n \geq n_1$, we obtain

$$\begin{aligned}|Tx_n - Ty_n| &\leq p|x_{n-m} - y_{n-m}| + R_{n-1}\sum_{s=n-1}^{\infty} p_s|f(x_{s-k}) - f(y_{s-k})| \\ &\quad + R_{n-1}\sum_{s=n-1}^{\infty} q_s|g(x_{s-l}) - g(y_{s-l})| + \sum_{s=n_1}^{n-2} R_s p_s|f(x_{s-k}) - f(y_{s-k})| \\ &\quad + \sum_{s=n_1}^{n-2} R_s q_s|g(x_{s-l}) - g(y_{s-l})| \\ &\leq p||x-y|| + L_1||x-y||\sum_{s=n_1}^{\infty} R_s(p_s+q_s) \\ &= \hat{q}_1||x-y||,\end{aligned}$$

where $\hat{q}_1 = p + L_1\sum_{s=n_1}^{\infty} R_s(p_s+q_s) < 1$ due to (2.2). This immediately yields

$$||Tx - Ty|| \leq \hat{q}_1||x-y||,$$

and so T is a contraction mapping. Consequently, T has a unique fixed x, which is obviously a positive solution of (1.1). This completes the proof of Case 2.

Case 3. $p \in (1, \infty)$. From (H_1) and (H_2), one can choose an $n_2 \geq n_0 + \max\{m, k, l\}$ sufficiently large such that

$$\begin{aligned}&\sum_{s=n}^{\infty} R_s p_s \leq \frac{1 - p(1-N_2)}{\alpha_2}, \\ &\sum_{s=n}^{\infty} R_s q_s \leq \frac{(1-M_2)p - (1+N_2)}{\beta_2}, \\ &\sum_{s=n}^{\infty} R_s(p_s+q_s) < \frac{p-1}{L_2}\end{aligned} \tag{2.3}$$

hold for all $n \geq n_2$, where $N_2 \geq M_2 > 0$, $(1-M_2)p > 1 + N_2$, $p(1-N_2) < 1$, $\alpha_2 = \max_{M_2 \leq x \leq N_2}\{f(x)\}$, $\beta_2 = \max_{M_2 \leq x \leq N_2}\{g(x)\}$, and $L_2 = \max\{L_f([M_2, N_2]), L_g([M_2, N_2])\}$. Set

$$A_2 = \{x = \{x_n\} \in l_{n_0}^{\infty} : M_2 \leq x_n \leq N_2, n \geq n_0\}.$$

Define an operator $T : A_2 \to l_{n_0}^{\infty}$ as

$$(Tx)_n = \begin{cases} 1 - \dfrac{1}{p} - \dfrac{1}{p}x_{n+m} + \dfrac{1}{p}R_{n+m-1}\displaystyle\sum_{s=n+m-1}^{\infty}(p_s f(x_{s-k}) - q_s g(x_{s-l})) \\ \quad + \dfrac{1}{p}\displaystyle\sum_{s=n_2}^{n+m-2} R_s(p_s f(x_{s-k}) - q_s g(x_{s-l})), \quad n \geq n_2, \\ (Tx)_{n_2}, \quad n_0 \leq n \leq n_2. \end{cases}$$

For every $x \in A_2$ and $n \geq n_2$, we get

$$\begin{aligned}(Tx)_n &\leq 1 - \frac{1}{p} + \frac{1}{p}\alpha_2 R_{n+m-1}\sum_{s=n+m-1}^{\infty} p_s + \frac{1}{p}\alpha_2\sum_{s=n_2}^{n+m-2} R_s p_s \\ &\leq 1 - \frac{1}{p} + \frac{1}{p}\alpha_2\sum_{s=n_2}^{\infty} R_s p_s \leq N_2.\end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
(Tx)_n &\geq 1-\frac{1}{p}-\frac{1}{p}N_2-\frac{1}{p}\beta_2 R_{n+m-1}\sum_{s=n+m-1}^{\infty} q_s-\frac{1}{p}\beta_2\sum_{s=n_2}^{n+m-2} R_s q_s\\
&\geq 1-\frac{1}{p}-\frac{1}{p}N_2-\frac{1}{p}\beta_2\sum_{s=n_2}^{\infty} R_s q_s \geq M_2,
\end{aligned}
$$

and thus $TA_2 \subseteq A_2$. Since A_2 is a bounded, closed, and convex subset of $l^\infty_{n_0}$, we have to prove that T is a contraction mapping on A_2 to apply the contraction principle.

Now, for $x, y \in A_2$ and $n \geq n_2$, we obtain

$$
\begin{aligned}
|Tx_n - Ty_n| &\leq \frac{1}{p}|x_{n+m}-y_{n+m}| + \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty} p_s|f(x_{s-k})-f(y_{s-k})|\\
&\quad + \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty} q_s|g(x_{s-l})-g(y_{s-l})| + \frac{1}{p}\sum_{s=n_2}^{n+m-2} R_s p_s|f(x_{s-k})-f(y_{s-k})|\\
&\quad + \frac{1}{p}\sum_{s=n_2}^{n+m-2} R_s q_s|g(x_{s-l})-g(y_{s-l})|\\
&\leq \frac{1}{p}||x-y|| + \frac{1}{p}L_2||x-y||\sum_{s=n_2}^{\infty} R_s(p_s+q_s)\\
&= \hat{q}_2||x-y||,
\end{aligned}
$$

which yields

$$
||Tx - Ty|| \leq \hat{q}_2||x-y||.
$$

From (2.3), we have $\hat{q}_2 = 1/p(1 + L_2\Sigma_{s=n_2}^{\infty} R_s(p_s + q_s)) < 1$. Therefore, T is a contraction mapping. Consequently, T has a unique fixed x, which is obviously a positive solution of (1.1). The proof of Case 3 is complete.

Case 4. $p \in (-1, 0)$. Combining (H_1) and (H_2), we can choose an $n_3 \geq n_0 + \max\{m, k, l\}$ sufficiently large such that

$$
\begin{aligned}
&\sum_{s=n}^{\infty} R_s p_s \leq \frac{(1+p)N_3-(1+p)}{\alpha_3},\\
&\sum_{s=n}^{\infty} R_s q_s \leq \frac{1+p-M_3(1+p)}{\beta_3},\\
&\sum_{s=n}^{\infty} R_s(p_s+q_s) < \frac{1+p}{L_3}
\end{aligned}
\tag{2.4}
$$

hold for all $n \geq n_3$, where M_3 and N_3 are positive constants satisfying $0 < M_3 < 1 < N_3$, $\alpha_3 = \max_{M_3\leq x\leq N_3}\{f(x)\}$, $\beta_3 = \max_{M_3\leq x\leq N_3}\{g(x)\}$, and $L_3 = \max\{L_f([M_3, N_3]), L_g([M_3, N_3])\}$. Set

$$
A_3 = \{x = \{x_n\} \in l^\infty_{n_0} : M_3 \leq x_n \leq N_3,\ n \geq n_0\}.
$$

Define an operator $T : A_3 \to l^\infty_{n_0}$ by

$$
(Tx)_n = \begin{cases}
1+p-px_{n-m}+R_{n-1}\displaystyle\sum_{s=n-1}^{\infty}(p_s f(x_{s-k})-q_s g(x_{s-l}))\\
\quad +\displaystyle\sum_{s=n_3}^{n-2} R_s(p_s f(x_{s-k})-q_s g(x_{s-l})), \quad n \geq n_3,\\
(Tx)_{n_3}, \quad n_0 \leq n \leq n_3.
\end{cases}
$$

For every $x \in A$ and $n \geq n_3$, we have

$$\begin{aligned}(Tx)_n &\leq 1 + p - pN_3 + \alpha_3 R_{n-1} \sum_{s=n-1}^{\infty} p_s + \alpha_3 \sum_{s=n_3}^{n-2} R_s p_s \\ &\leq 1 + p - pN_3 + \alpha_3 \sum_{s=n_3}^{\infty} R_s p_s \leq N_3.\end{aligned}$$

Furthermore, we conclude that

$$\begin{aligned}(Tx)_n &\geq 1 + p - pM_3 - R_{n-1} \sum_{s=n-1}^{\infty} q_s g(x_{s-l}) - \sum_{s=n_3}^{n-2} R_s q_s g(x_{s-l}) \\ &\geq 1 + p - pM_3 - \beta_3 \sum_{s=n_3}^{\infty} R_s q_s \geq M_3,\end{aligned}$$

and thus $TA_3 \subseteq A_3$.

Next, we prove that T is a contraction mapping on A_3. In fact, for every $x, y \in A_3$ and $n \geq n_3$, we have

$$\begin{aligned}|Tx_n - Ty_n| \leq& -p|x_{n-m} - y_{n-m}| + R_{n-1} \sum_{s=n-1}^{\infty} p_s |f(x_{s-k}) - f(y_{s-k})| \\ &+ R_{n-1} \sum_{s=n-1}^{\infty} q_s |g(x_{s-l}) - g(y_{s-l})| + \sum_{s=n_3}^{n-2} R_s p_s |f(x_{s-k}) - f(y_{s-k})| \\ &+ \sum_{s=n_3}^{n-2} R_s q_s |g(x_{s-l}) - g(y_{s-l})| \\ \leq& -p||x-y|| + L_3 ||x-y|| \sum_{s=n_3}^{\infty} R_s (p_s + q_s) \\ =& \hat{q}_3 ||x-y||.\end{aligned}$$

This immediately yields

$$||Tx - Ty|| \leq \hat{q}_3 ||x - y||,$$

where $\hat{q}_3 = -p + L_3 \sum_{s=n_3}^{\infty} R_s (p_s + q_s) < 1$ due to (2.4), which implies that T is a contraction mapping. Consequently, T has a unique fixed x, which is obviously a positive solution of (1.1). This completes the proof of Case 4.

Case 5. $p \in (-\infty, -1)$. From (H_1) and (H_2), one can choose an $n_4 \geq n_0 + \max\{m, k, l\}$ sufficiently large such that

$$\begin{aligned}&\sum_{s=n}^{\infty} R_s p_s \leq \frac{-(p+1)(N_4 - 1)}{\beta_4}, \\ &\sum_{s=n}^{\infty} R_s q_s \leq \frac{-(1+p)(1 - M_4)}{\alpha_4}, \\ &\sum_{s=n}^{\infty} R_s (p_s + q_s) < \frac{-(p+1)}{L_4}\end{aligned} \tag{2.5}$$

hold for all $n \geq n_4$, where M_4 and N_4 are positive constants satisfying $0 < M_4 < 1 < N_4$, $\alpha_4 = \max_{M_4 \leq x \leq N_4}\{f(x)\}$, $\beta_4 = \max_{M_4 \leq x \leq N_4}\{g(x)\}$, and $L_4 = \max\{L_f([M_4, N_4]), L_g([M_4, N_4])\}$. Set

$$A_4 = \{x = \{x_n\} \in l_{n_0}^{\infty} : M_4 \leq x_n \leq N_4,\ n \geq n_0\}.$$

Define an operator $T : A_4 \to l^\infty_{n_0}$ as

$$(Tx)_n = \begin{cases} 1 + \frac{1}{p} - \frac{1}{p}x_{n+m} + \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty}(p_s f(x_{s-k}) - q_s g(x_{s-l})) \\ \quad + \frac{1}{p}\sum_{s=n_4}^{n+m-2} R_s(q_s f(x_{s-k}) - q_s g(x_{s-l})), \quad n \geq n_4, \\ (Tx)_{n_4}, \quad n_0 \leq n \leq n_4. \end{cases}$$

For every $x \in A_4$ and $n \geq n_4$, we get

$$\begin{aligned}(Tx)_n &\leq 1 + \frac{1}{p} - \frac{1}{p}N_4 - \frac{1}{p}\beta_4 R_{n+m-1}\sum_{s=n+m-1}^{\infty} q_s - \frac{1}{p}\beta_4\sum_{s=n_4}^{n+m-2} R_s q_s \\ &\leq 1 + \frac{1}{p} - \frac{1}{p}N_4 - \frac{1}{p}\beta_4\sum_{s=n_4}^{\infty} R_s q_s \leq N_4.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}(Tx)_n &\geq 1 + \frac{1}{p} - \frac{1}{p}M_4 + \frac{1}{p}\alpha_4 R_{n+m-1}\sum_{s=n+m-1}^{\infty} p_s + \frac{1}{p}\alpha_4\sum_{s=n_4}^{n+m-2} R_s p_s \\ &\geq 1 + \frac{1}{p} - \frac{1}{p}M_4 + \frac{1}{p}\alpha_4\sum_{s=n_4}^{\infty} R_s q_s \geq M_4,\end{aligned}$$

and so $TA_4 \subseteq A_4$. Since A_4 is a bounded, closed, and convex subset of $l^\infty_{n_0}$, we have to prove that T is a contraction mapping on A_4 to apply the contraction principle.

Now, for $x, y \in A_4$ and $n \geq n_4$, we have

$$\begin{aligned}|Tx_n - Ty_n| &\leq -\frac{1}{p}|x_{n+m} - y_{n+m}| - \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty} p_s|f(x_{s-k}) - f(y_{s-k})| \\ &\quad - \frac{1}{p}R_{n+m-1}\sum_{s=n+m-1}^{\infty} q_s|g(x_{s-l}) - g(y_{s-l})| - \frac{1}{p}\sum_{s=n_4}^{n+m-2} R_s p_s|f(x_{s-k}) - f(y_{s-k})| \\ &\quad - \frac{1}{p}\sum_{s=n_4}^{n+m-2} R_s q_s|g(x_{s-l}) - g(y_{s-l})| \\ &\leq -\frac{1}{p}||x-y|| - \frac{1}{p}L_4||x-y||\sum_{s=n_4}^{\infty} R_s(p_s + q_s) \\ &= \hat{q}_4||x-y||.\end{aligned}$$

This immediately implies that

$$||Tx - Ty|| \leq \hat{q}_4||x-y||.$$

By virtue of (2.5), we get $\hat{q}_4 = 1/p(-1 - L_4\Sigma_{s=n_4}^{\infty} R_s(p_s + q_s)) < 1$, which proves that T is a contraction mapping. Consequently, T has a unique fixed x, which is obviously a positive solution of (1.1). This completes the proof of Case 5. Therefore, the proof of Theorem 2.1 is complete. □

Remark 2.2. One can easily see that Theorem 2.1 includes Theorem 1.1 when $r_n = 1$ and $f(u) = g(u) = u$.

3. Applications

Example 3.1. Consider a second-order difference equation

$$\Delta(r_n\Delta(x_n + x_{n-1})) + p_n x_{n-2} - q_n x_{n-2}^3 = 0, \quad n = 2, 3, \ldots, \tag{3.1}$$

where $p = 1$, $r_n = 1/n$, $f(x) = x$, $g(x) = x^3$,

$$p_n = \frac{2(n-2)}{(n+2)(n+1)n(n-1)(2n-3)},$$

and

$$q_n = \frac{8(n-2)^3}{(n+2)(n+1)n(n-1)(2n-3)^3}.$$

It is easy to verify that

$$R_n = \sum_{s=2}^{n} \frac{1}{r_s} = \sum_{s=2}^{n} s = \frac{1}{2}(n+2)(n-1),$$

$$\sum_{n=2}^{\infty} R_n p_n < \infty, \quad \text{and} \quad \sum_{n=2}^{\infty} R_n q_n < \infty.$$

Therefore, conditions (H_1) and (H_2) are satisfied. By Theorem 2.1, equation (3.1) has a bounded nonoscillatory solution. As a matter of fact, the sequence $\{x_n\} = \{2 + 1/n\}$ is a nonoscillatory solution of (3.1).

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