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Positive solutions for Sturm-Liouville eigenvalue problems

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Abstract

By means of the lower and upper solutions argument and fixed index theorem in the frame of the ODE technique, we consider the existence and nonexistence of multiple positive solutions for fourth-order eigenvalue Sturm-Liouville boundary value problem. Our results significantly extend and improve many known results including singular and nonsingular cases.

Keywords: Fourth-order singular differential equation, lower and upper solutions, positive solutions.
2010 MSC: 34B16.

1. Introduction

In this paper, we will study the existence and nonexistence of positive solutions for the following fourth-order nonlinear Sturm-Liouville boundary value problem (BVP)

$$\begin{cases} \dfrac{1}{p(t)}(p(t)u'''(t))' - \lambda g(t) f(u(t)) = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \ \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 \lim\limits_{t\to 0+} p(t)u'''(t) = 0, \\ \gamma_2 u''(1) + \delta_2 \lim\limits_{t\to 1-} p(t)u'''(t) = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is positive real parameters, $\alpha_i, \beta_i, \delta_i, \gamma_i \geq 0$ $(i = 1, 2)$ are constants, and $p \in C^1((0,1),(0,+\infty))$. Moreover g, p may be singular at $t = 0$ and/or 1.

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The boundary value problems for ordinary differential equations play a very important role in both theory and application. They are used to describe a large number of physical, biological and chemical phenomena. BVP (1.1) is often referred to as the deformation of an elastic beam under a variety of boundary conditions(see [3, 6, 10, 11, 12, 13]. For example, as $\lambda = 1$, BVP (1.1) subject to Lidstone boundary value conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ are used to model such phenomena as the deflection of elastic beam simply supported at the endpoints, see [1, 2, 4, 6, 7, 8, 9, 10, 13]. Particularly, when $g(t) = 1,\ \beta_1 = \delta_1 = 0$, Zhang [11] only discussed existence of positive solutions for BVP (1.1).

The aim of this paper is to consider the existence of positive solutions for the more general Sturm-Liouville boundary value problem by using the lower and upper solutions argument and fixed index theorem in the frame of the ODE technique. Here we allow p, g have singularity at $t = 0,\ 1$, as far as we know, there were fewer works to be done. This paper attempts to fill part of this gap in the literature.

This paper is organized as follows. In Section 2, we firstly present some properties of Green's functions that are used to define a positive operator. Next we approximate the singular fourth-order boundary value problem to singular second-order boundary value problem by constructing an integral operator. In Section 3, A sufficient condition for the existence of multiple positive solutions and no positive solutions of BVP (1.1) will be established. In section 4, we give one example as the application.

2. Preliminaries and Lemmas

In this paper, we all suppose $J = [0, 1]$, R is a real number space, $R^+ = [0, +\infty)$ and let

$$C(J, R) = \{u : J \to R \mid u(t) \ \text{ continuous}\},$$
$$C^i(J, R) = \{u : J \to R \mid u(t) \text{ is } i\text{th-order continuously differentiable}\},\ i = 1, 2, \cdots .$$

For $u = u(t) \in C(J, R)$, let $\|u\| = \max_{t\in J} |u(t)|$, then $E = C(J, R)$ is a Banach space with the norm $\|\cdot\|$.

Definition 2.1. A function $u(t)$ is said to be a positive solution of the boundary value problem (1.1) if $u \in C^2([0, 1], R) \bigcap C^3((0, 1), R)$ satisfies $u > 0$, $pu''' \in C^1((0, 1), R^+)$ and the BVP (1.1).

Definition 2.2. $\alpha \in C^2([0, 1], R) \bigcap C^3((0, 1), R),\ p(t)\alpha'''(t) \in C^1((0, 1), R^+)$ is called a lower solution of BVP (1.1) if

$$\begin{cases} \dfrac{1}{p(t)}(p(t)\alpha'''(t))' - \lambda g(t)f(\alpha(t)) \ge 0, \quad 0 < t < 1,\\ \alpha_1\alpha(0) - \beta_1\alpha'(0) \le 0,\ \gamma_1\alpha(1) + \delta_1\alpha'(1) \le 0,\\ \alpha_2\alpha''(0) - \beta_2 \lim\limits_{t\to 0+} p(t)\alpha'''(t) \ge 0,\\ \gamma_2\alpha''(1) + \delta_2 \lim\limits_{t\to 1-} p(t)\alpha'''(t) \ge 0. \end{cases}$$

Definition 2.3. $\beta \in C^2([0, 1], R) \bigcap C^3((0, 1), R),\ p(t)\beta'''(t) \in C^1((0, 1), R^+)$ is called a upper solution of BVP (1.1) if

$$\begin{cases} \dfrac{1}{p(t)}(p(t)\beta'''(t))' - \lambda g(t)f(\beta(t)) \le 0, \quad 0 < t < 1,\\ \alpha_1\beta(0) - \beta_1\beta'(0) \ge 0,\ \gamma_1\beta(1) + \delta_1\beta'(1) \ge 0,\\ \alpha_2\beta''(0) - \beta_2 \lim\limits_{t\to 0+} p(t)\beta'''(t) \le 0,\\ \gamma_2\beta''(1) + \delta_2 \lim\limits_{t\to 1-} p(t)\beta'''(t) \le 0. \end{cases}$$

We notice that if $u(t)$ is a positive solution of the BVP (1.1) and $p \in C^1((0, 1), (0, +\infty))$, then $u(t) \in C^4(0, 1)$.

Now we denote that $H(t,s)$ and $G(t,s)$ are Green's functions for the following boundary value problem

$$\begin{cases} -u'' = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \ \gamma_1 u(1) + \delta_1 u'(1) = 0 \end{cases}$$

and

$$\begin{cases} \dfrac{1}{p(t)}(p(t)v'(t))' = 0, \quad 0 < t < 1, \\ \alpha_2 v(0) - \lim\limits_{t\to 0+} \beta_2 p(t)v'(t) = 0, \\ \gamma_2 v(1) + \lim\limits_{t\to 1-} \delta_2 p(t)u'(t) = 0, \end{cases}$$

respectively. It is well known that $H(t,s)$ and $G(t,s)$ can be written by

$$H(t,s) = \frac{1}{\rho_1}\begin{cases} (\beta_1 + \alpha_1 s)(\delta_1 + \gamma_1(1-t)), & 0 \le s \le t \le 1, \\ (\beta_1 + \alpha_1 t)(\delta_1 + \gamma_1(1-s)), & 0 \le t \le s \le 1 \end{cases} \tag{2.1}$$

and

$$G(t,s) = \frac{1}{\rho_2}\begin{cases} (\beta_2 + \alpha_2 B(0,s))(\delta_2 + \gamma_2 B(t,1)), & 0 \le s \le t \le 1, \\ (\beta_2 + \alpha_2 B(0,t))(\delta_2 + \gamma_2 B(s,1)), & 0 \le t \le s \le 1, \end{cases} \tag{2.2}$$

where $\rho_1 = \gamma_1\beta_1 + \alpha_1\gamma_1 + \alpha_1\delta_1 > 0$, $B(t,s) = \displaystyle\int_t^s \frac{\mathrm{d}\tau}{p(\tau)}$, $\rho_2 = \alpha_2\delta_2 + \alpha_2\gamma_2 B(0,1) + \beta_2\gamma_2 > 0$.

It is easy to verify the following properties of $H(t,s)$ and $G(t,s)$

(I) $G(t,s) \le G(s,s) < +\infty$, $H(t,s) \le H(s,s) < +\infty$;

(II) $G(t,s) \ge \rho G(s,s)$, $H(t,s) \ge \xi H(s,s)$, for any $t \in [a,b] \subset (0,1)$, $s \in [0,1]$, where

$$\rho = \min\left\{\frac{\delta_2 + \gamma_2 B(b,1)}{\delta_2 + \gamma_2 B(0,1)}, \frac{\beta_2 + \alpha_2 B(0,a)}{\beta_2 + \alpha_2 B(0,1)}\right\}, \tag{2.3}$$

$$\xi = \min\left\{\frac{\delta_1 + \gamma_1(1-b)}{\delta_1 + \gamma_1}, \frac{\beta_1 + \alpha_1 a}{\beta_1 + \alpha_1}\right\}.$$

For $[a,b] \subset (0,1)$ which is given by (II), we set

$$\theta_1 = \int_a^b G(s,s)p(s)g(s)\mathrm{d}s, \quad \theta_2 = \int_a^b H(\tau,\tau)\mathrm{d}\tau.$$

Throughout this paper, we adopt the following assumptions

(H_1) $p \in C^1((0,1),(0,+\infty))$, $g \in C((0,1),[0,+\infty))$ and satisfies

$$0 < \int_0^1 \frac{\mathrm{d}s}{p(s)} < +\infty, \ 0 < e = \int_0^1 G(s,s)p(s)g(s)\mathrm{d}s < +\infty.$$

(H_2) $f(u) \in C([0,+\infty),[0,+\infty))$ and f is nondecreasing on $[0,+\infty)$.

(H_3)

$$f(0) > 0, \ f_\infty = \lim_{u\to+\infty}\frac{f(u)}{u} = +\infty.$$

Now we define an integral operator $S : C[0,1] \to C[0,1]$ by

$$Sv(t) = \int_0^1 H(t,\tau)v(\tau)\mathrm{d}\tau. \tag{2.4}$$

Then S is linear nondecreasing continuous operator and by the expressed of $H(t,s)$, we have

$$\begin{cases} (Sv)''(t) = -v(t),\ 0 < t < 1, \\ \alpha_1 (Sv)(0) - \beta_1 (Sv)'(0) = 0, \\ \gamma_1 (Sv)(1) + \delta_1 (Sv)'(1) = 0. \end{cases} \tag{2.5}$$

The following Lemmas play an important role in this paper.

Lemma 2.4. *The Sturm-Liouville BVP* (1.1) *has a positive solution v if and only if v is a positive solution of the following integral-differential boundary value problem*

$$\begin{cases} \dfrac{1}{p(t)}(p(t)v'(t))' + \lambda g(t) f(Sv(t)) = 0, \quad 0 < t < 1, \\ \alpha_2 v(0) - \lim\limits_{t\to 0+} \beta_2 p(t) v'(t) = 0, \\ \gamma_2 v(1) + \lim\limits_{t\to 1-} \delta_2 p(t) v'(t) = 0, \end{cases} \tag{2.6}$$

where S is given in (2.4).

Proof. In fact, if u is a positive solution of (1.1), let $u = Sv$, then $v = -u''$. This implies $u'' = -v$ is a solution of (2.6). Conversely, if v is a positive solution of (2.6). Let $u = Sv$, by (2.5), $u'' = (Sv)'' = -v$. Thus $u = Sv$ is a positive solution of (1.1). This completes the proof of Lemma 2.4.

Now, for the given $[a,b] \subset (0,1)$, ρ as above in (II), we introduce

$$K = \{u \in C[0,1] : u(t) \geq \rho u(s),\ t \in [a,b],\ s \in [0,1]\}.$$

It is easy to check that K is a cone in $C[0,1]$ and for $u(t) \in K, t \in [a,b]$, we have $u(t) \geq \rho \|u\|$.

Next, for any $\lambda \in (0,+\infty)$, we define an operator A, T_λ given by

$$Av(t) = \int_0^1 G(t,s)p(s)g(s)f(Sv(s))\mathrm{d}s,\ t \in [0,1], \tag{2.7}$$

$$T_\lambda v(t) = \lambda A v(t), \forall\ v \in K. \tag{2.8}$$

Clearly, v is a solution of the BVP (2.6) if and only if $u = Sv$ is a fixed point of the operator T_λ, therefor, by Lemma 2.4, $u = Sv$ is also a solution of the BVP (1.1).

Through direct calculation, by (II) and for $v \in K,\ t \in [a,b],\ s \in J$, we have

$$\begin{aligned} T_\lambda v(t) &= \lambda \int_0^1 G(t,s)p(s)g(s)f(Sv(s))\mathrm{d}s \\ &\geq \lambda\rho \int_0^1 G(s,s)p(s)g(s)f(Sv(s))\mathrm{d}s \\ &= \rho T_\lambda v(s). \end{aligned}$$

So, this implies that $T_\lambda K \subset K$. □

Lemma 2.5. *Assume that (H_1), (H_2), (H_3) hold. Then $T_\lambda : K \to K$ is completely continuous.*

Proof. Firstly, The continuity of T_λ is easily obtained. In fact, if $v_n,\ v \in K$ and $v_n \to v$ in the sup norm as $n \to \infty$, then for any $t \in J$, we get

$$|T_\lambda v_n(t) - T_\lambda v(t)| \leq \lambda \max_{s\in J} |f(Sv_n(t)) - f(Sv(t))| \int_0^1 G(s,s)p(s)g(s)\mathrm{d}s,$$

so, by the continuity of f, S, we have

$$\|T_\lambda v_n - T_\lambda v\| = \sup_{t\in J} |T_\lambda v_n(t) - T_\lambda v(t)| \to 0, \quad \text{as } n \to \infty.$$

This implies that $T_\lambda v_n \to T_\lambda v$ in the sup norm as $n \to \infty$, i.e., T_λ is continuous.

Now, let $B \subset K$ is a bounded set. It follows from condition (H_2) and the continuity of S that there exists a positive number L such that $\|f(Sv)\| \le L$ for any $v \in B$. Then, we can get

$$\|T_\lambda v(t)\| \le \lambda L e < \infty, \ \forall\, t \in J, \ v \in B.$$

So, $T_\lambda(B) \subset K$ is a bounded set in K.

For any $\varepsilon > 0$, by (H_1), there exists a $\delta' > 0$ such that

$$\int_0^{\delta'} G(s,s)p(s)g(s)\mathrm{d}s \le \frac{\varepsilon}{6\lambda L}, \quad \int_{1-\delta'}^1 G(s,s)p(s)g(s)\mathrm{d}s \le \frac{\varepsilon}{6\lambda L}.$$

Let $G = \max\limits_{t\in[\delta',1-\delta']} g(t)$, $P = \max\limits_{t\in[\delta',1-\delta']} p(t)$. It follows from the uniformly continuity of $G(t,s)$ on $[0,1]\times[0,1]$ that there exists $\delta > 0$ such that

$$|G(t,s) - G(t',s)| \le \frac{\varepsilon}{3\lambda GPL}, \ |t - t'| < \delta, \ t, t' \in [0,1].$$

Consequently, when $|t - t'| < \delta$, $t, t' \in [0,1]$, $v \in B$, we have

$$\begin{aligned}
|T_\lambda v(t) - T_\lambda v(t')| &= \left|\lambda \int_0^1 (G(t,s) - G(t',s))p(s)g(s)f(Sv(s))\mathrm{d}s\right| \\
&\le \lambda \int_0^{\delta'} |G(t,s) - G(t',s)|p(s)g(s)|f(Sv(s))|\mathrm{d}s \\
&\quad + \lambda \int_{\delta'}^{1-\delta'} |G(t,s) - G(t',s)|p(s)g(s)|f(Sv(s))|\mathrm{d}s \\
&\quad + \lambda \int_{1-\delta'}^{1} |G(t,s) - G(t',s)| \cdot |p(s)| \cdot |g(s)| \cdot |f(Sv(s))|\mathrm{d}s \\
&\le 2\lambda L \int_0^{\delta'} G(s,s)p(s)g(s)\mathrm{d}s + 2\lambda L \int_{1-\delta'}^1 G(s,s)p(s)g(s)\mathrm{d}s \\
&\quad + \lambda GPL \int_0^1 |G(t,s) - G(t',s)|\mathrm{d}s \\
&\le \varepsilon.
\end{aligned}$$

This implies that $T_\lambda(B)$ is equicontinuous set on J.

Therefore, it is easy to check by Arzela-ascoli Theorem that $T_\lambda : K \to K$ is completely continuous. The proof is complete. □

Lemma 2.6 ([5]). *Let E be a real Banach space, $K \subset E$ be a cone, and Ω be a boundary open set in E. Suppose $A : \overline{K \bigcap \Omega} \to K$ is a completely continuous operator. If*

$$Au \ne \mu u, \ \forall\, u \in K \cap \partial\Omega, \ \mu \ge 1.$$

Then the fixed point index $i(A, K \bigcap \Omega, K) = 1$.

Lemma 2.7 ([5]). *Let K be a positive cone in real Banach space E. Suppose $A : \Omega \bigcap K \to K$ is a completely continuous operator and satisfies:*

$$\|Au\| \ge \|u\|, \ u \in \partial\Omega \cap K.$$

Then fixed point index $i(A, \Omega \bigcap K, K) = 0$.

3. Main results

In this section, we will give our main results. Let

$$\Sigma = \{(\lambda,\ u) \mid (\lambda,\ u) \in R^+ \times K,\ \lambda > 0,\ u \neq \theta,\ u = T_\lambda u\},$$

$$\Lambda = \{\lambda \mid \lambda \in R^+,\ \lambda > 0,\ \text{there exists }\ u \neq \theta \text{ such that }\ (\lambda,\ u) \in \Sigma\}.$$

Theorem 3.1. *Suppose that conditions $(H_1)-(H_3)$ are satisfied. Then, for λ sufficiently small, BVP* (2.6) *has at least one positive solution; i.e., for λ sufficiently small, BVP* (1.1) *has at least one positive solution. In addition, for λ sufficiently large, BVP* (2.6) *has no positive solution; i.e., for λ sufficiently large, BVP* (1.1) *has no positive solution.*

Proof. For $l > 0$, let $d(l) = \sup\limits_{u\in K, \|u\|=l} f(Su(s))$. Combine (H_3), there exist $r > 0$ such that $d(r) > 0$.

Let $\lambda_1 = \dfrac{r}{d(r)e}$ and set $K_1 = \{u \in K : \|u\| \leq r\}$. Then for $\lambda \in (0, \lambda_1)$ and $u \in \partial K_1$, we have

$$\begin{aligned} T_\lambda u(t) \ &= \lambda \int_0^1 G(t,s)p(s)g(s)f(Su(s))\mathrm{d}s \\ &\leq \lambda \int_0^1 G(s,s)p(s)g(s)d(r)\mathrm{d}s \\ &\leq \lambda_1 d(r) \int_0^1 G(s,s)p(s)g(s)\mathrm{d}s \\ &= \lambda_1 d(r)e = r = \|u\|, \end{aligned}$$

Therefore,

$$\|T_\lambda u\| \leq \|u\|,\ u \in \partial K_1.$$

On the other hand, since (H_3), (2.4) and the property of limits, we can have $\lim\limits_{u\to\infty} \dfrac{f(Su)}{u} = +\infty$. Then, there exists H big enough such that $f(Su) \leq mu,\ t \in [a,b],\ u \geq H$, where $m > 0$ is choose so that

$$\lambda m\rho^2 \int_0^1 G(s,s)p(s)g(s)\mathrm{d}s \geq 1.$$

Let $R \geq r + \dfrac{H}{\rho} \geq H$ and set $K_2 = \{u \in K : \|u\| \leq R\}$. Then for $u \in \partial K_2$, by (II), we have

$$\begin{aligned} T_\lambda u(t) \ &= \lambda \int_0^1 G(t,s)p(s)g(s)f(Su(s))\mathrm{d}s \\ &\geq \lambda \int_a^b G(t,s)p(s)g(s)f(Su(s))\mathrm{d}s \\ &\geq \lambda\rho^2 m \int_0^1 G(s,s)p(s)g(s)\mathrm{d}s\|u\| \\ &\geq \|u\|, \end{aligned}$$

Therefore,

$$\|T_\lambda u\| \geq \|u\|,\ u \in \partial K_2.$$

Then, by the fixed-point theorem of cone expansion and compression, there has at least one positive fixed point $u_* \in K_2 \backslash (\overline{K_1})$ for the BVP (2.6).

To prove the nonexistence part. we note by $\lim_{u\to\infty} \dfrac{f(Su)}{u} = +\infty$, there exist $\varrho > 0$ such that

$$f(Su) \geq \varrho u, \ \forall \ u \geq 0.$$

We suppose that $u \in K$ be a positive solution for BVP (1.1). Now choose λ big enough such that $\lambda\varrho\rho^2\theta_1 > 1$. Then, we have

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)p(s)g(s)f(Su(s))\mathrm{d}s \\ &\geq \lambda\varrho \int_a^b G(t,s)p(s)g(s)u(s)\mathrm{d}s \\ &\geq \lambda\varrho\rho^2 \int_a^b G(s,s)p(s)g(s)\mathrm{d}s\|u\| \\ &\geq \lambda\rho^2\varrho\theta_1\|u\| > \|u\|, \quad \forall \ t \in [a,b]. \end{aligned}$$

It is a contradiction.

Now, by Lemma 2.4 we see that $u_* = Sv_*$ is a position solution of BVP (1.1) for λ sufficiently small, whereas for λ sufficiently large, BVP (1.1) has no positive solution. The proof is completed. □

Furthermore, we can get the next more precise result by using the lower and upper solutions argument and fixed index theorem.

Theorem 3.2. *Suppose that conditions* $(H_1) - (H_3)$ *are satisfied. Then, there exists a* $\lambda^* \in R$ *with* $0 < \lambda^* < +\infty$ *such that*

(1) the BVP (1.1) *has no solution when* $\lambda > \lambda^*$.

(2) the BVP (1.1) *has at least one positive solution when* $\lambda = \lambda^*$.

(3) the BVP (1.1) *has at least two positive solutions when* $0 < \lambda < \lambda^*$.

In order to obtain the proof of Theorem 3.1, we first give the following Lemmas.

Lemma 3.3. *Suppose that conditions* (H_1), (H_2), (H_3) *hold. Then set* Λ *is nonempty.*

Proof. By the definition of operator T_λ and the compact of operator A, we know that for any $r > 0$, we can choose an adequately small positive number $\lambda_0 > 0$ such that

$$\lambda_0 \sup_{u\in\overline{K_r}} \|Au\| < r, \quad \text{where} \quad K_r = \{u \in K : \|u\| < r\}.$$

Therefore,

$$T_{\lambda_0}u \ngeq u, \ u \in \partial K_r. \tag{3.1}$$

On the other hand, by (H_3), we can choose an appropriately big positive number $R > r > 0$ such that

$$f(u) \geq \sigma u, \ \forall \ u \geq \xi\rho\theta_2 R, \tag{3.2}$$

where σ satisfies

$$\lambda_0\sigma\xi\rho^2\theta_1\theta_2 > 1. \tag{3.3}$$

Then, let $K_R = \{u \in K : \|u\| < R\}$ and for any $t \in [a,b], \ u \in \partial K_R$, by (3.2), (3.3), we have

$$\begin{aligned} T_{\lambda_0}u(t) &= \lambda_0 Au(t) = \lambda_0 \int_0^1 G(t,s)p(s)g(s)f(Su(s))\mathrm{d}s \\ &\geq \lambda_0 \int_a^b G(t,s)p(s)g(s)f\left(\int_a^b H(s,\tau)u(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ &\geq \lambda_0\rho \int_a^b G(s,s)p(s)g(s)f\left(\xi\rho\int_a^b H(\tau,\tau)\mathrm{d}\tau\|u\|\right)\mathrm{d}s \\ &\geq \lambda_0\rho^2\sigma\xi\theta_1\theta_2\|u\| \\ &> \|u\|. \end{aligned}$$

Therefore,

$$T_{\lambda_0}u \not\leq u,\ u \in \partial K_R. \tag{3.4}$$

Then, by (3.1), (3.4) and the fixed-point theorem of cone expansion and compression, T_λ has at least one positive fixed point $u_* \in K_R \backslash (\overline{K_r})$. The proof is completed. □

Lemma 3.4. *Suppose that conditions (H_1), (H_2) hold. And also suppose that there is a positive solution at the point λ_1. Then for any $0 < \lambda \leq \lambda_1$, there is a positive solution of BVP (1.1).*

Proof. Suppose that v_1 is the positive solution of BVP (2.6) at the point λ_1. Then, v_1 and θ are the upper solution and lower solution od BVP (2.6) for $0 < \lambda \leq \lambda_1$.

In fact, for $0 < \lambda \leq \lambda_1$, we have

$$\begin{cases} \dfrac{1}{p(t)}(p(t)v_1'''(t))' + \lambda g(t)f(Sv_1(t)) & = -\lambda_1 g(t)(fSv_1(t)) + \lambda g(t)f(Sv_1(t)) \\ & = (\lambda - \lambda_1)g(t)(fSv_1(t)) \leq 0, \quad 0 < t < 1, \\ \alpha_1 v_1(0) - \beta_1 v_1'(0) = 0,\ \gamma_1 v_1(1) + \delta_1 v_1'(1) = 0, & \\ \alpha_2 v_1''(0) - \beta_2 \lim\limits_{t\to 0+} p(t)v_1'''(t) = 0, & \\ \gamma_2 v_1''(1) + \delta_2 \lim\limits_{t\to 1-} p(t)v_1'''(t) = 0, & \end{cases}$$

This means that $v_1(t)$ is an upper solution of BVP (2.6). Obviously, $v_0 \equiv \theta,\ t \in J$ is a lower solution of BVP (2.6).

Because $T_\lambda : K \to K$ is completely continuous, then therefore exist $v_*(t) \in [\theta, v_1]$ such that $v_* = T_\lambda v_*$. Therefore, BVP (2.6) has a positive solution for any $0 < \lambda \leq \lambda_1$.

Now, by Lemma 2.4 we see that $u_* = Sv_*$ is a position solution of BVP (1.1) for any $0 < \lambda \leq \lambda_1$. The proof is completed. □

Lemma 3.5. *Suppose that conditions (H_1), (H_2), (H_3) hold. Then set Λ has upper bound. And if $\lambda^* = \sup \Lambda$, there exist u^* such that $\lambda^* \in \Lambda$ and $(\lambda^*, u^*) \in \Sigma$.*

Proof. For any $\lambda \in \Lambda$, we suppose that u_λ is the positive solution of BVP (1.1) at the point λ. By (H_2), (H_3), there exist $\varrho > 0$ such that

$$f(u) \geq \varrho u,\ \forall\ u \geq 0. \tag{3.5}$$

Then,

$$\begin{aligned} u_\lambda(t) & = \lambda \int_0^1 G(t,s)p(s)g(s)f(Su_\lambda(s))\mathrm{d}s \\ & \geq \lambda \int_a^b G(t,s)p(s)g(s)f\left(\int_a^b H(s,\tau)u(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ & \geq \lambda\rho \int_a^b G(s,s)p(s)g(s)f\left(\xi\rho\int_a^b H(\tau,\tau)\mathrm{d}\tau\|u\|\right)\mathrm{d}s \\ & \geq \lambda\rho^2\varrho\xi\theta_1\theta_2\|u\|, \quad \forall\ t \in [a,b]. \end{aligned}$$

i.e.,

$$\|u\| \geq u_\lambda(t) \geq \lambda\rho^2\varrho\xi\theta_1\theta_2\|u\|,$$

therefore,

$$\lambda \leq \left[\rho^2\varrho\xi\theta_1\theta_2\right]^{-1},$$

So, set Λ has upper bound.

Suppose $\lambda^* = \sup \Lambda$. Next we will show that $\lambda^* \in \Lambda$.

For $\lambda^* = \sup \Lambda$, there exist $\lambda_n \in \Lambda$ such that $\lambda_n \to \lambda$, as $n \to +\infty$. And for $\lambda_n \in \Lambda$, there exist $u_n \in K \backslash \{\theta\}$ such that $(\lambda_n, u_n) \in \Sigma$. Then, $\{u_n\}$ is bounded.

In fact, if not, there exist $\{u_{nk}\} \subset \{u_n\}$ such that $u_{nk} \to +\infty$, as $k \to +\infty$. Then, by (2.4) (II), for any $s \in [a, b],\ u_{nk} \in K$, we have

$$Su_{nk}(s) = \int_0^1 H(s,\tau)u_{nk}(\tau)\mathrm{d}\tau \geq \xi\theta_2\|u_{nk}\|. \tag{3.6}$$

By (H_2), (H_3), we can choose an appropriately big positive number $\varepsilon > 0$ and positive integer $N > 0$ such that for any $s \in [a, b]$,

$$f(Su_{nk}(s)) \geq f\left(\xi\theta_2\|u_{nk}\|\right) \geq \varepsilon\xi\theta_2\|u_{nk}\|, \ \forall\ k \geq N, \tag{3.7}$$

$$\lambda_{nk}\varepsilon\xi\rho^2\theta_1\theta_2 > 1. \tag{3.8}$$

Therefore, by (3.6), (3.7), (3.8), we have

$$\begin{aligned} \|u_{nk}\| &\geq u_{nk}(s) = \lambda_{nk}\int_0^1 G(t,s)p(s)g(s)f(Su_{nk}(s))\mathrm{d}s \\ &\geq \lambda_{nk}\varepsilon\xi\rho^2\theta_1\theta_2\|u_{nk}\| > \|u_{nk}\|, \quad \forall\ k \geq N, \end{aligned}$$

this is a contrafactual. So $\{u_n\}$ is sequence compact set. Therefore, there exist $u^* \in K\backslash\{\theta\}$ and $\{u_{ni}\} \subset \{u_n\}$ such that $\{u_{ni}\} \subset \{u^*\},\ i \to +\infty$, and $u_{ni}(s) = T_{\lambda_{nk}}u_{ni}(s) = \lambda_{nk}Au_{ni}(s)$.

Then, it is easy to see by the completely continuous of $T_{\lambda_{nk}}$ that $u^* = T_{\lambda^*}u^* = \lambda^*Au^*$, i.e., u^* is the fixed point of T_{λ^*}, i.e., $\lambda^* \in \Lambda$ and $(\lambda^*, u^*) \in \Sigma$. The proof is completed. □

Lemma 3.6. *Suppose that conditions (H_1), (H_2), (H_3) hold and $0 < \lambda < \lambda^*$. Then there have at least two positive fixed point of T_λ, i.e., there exist at least two positive solutions of BVP* (1.1).

Proof. By Lemma 3.5, we suppose that v^* is the positive solution of BVP (2.6) at the point λ^*. For any $0 < \lambda < \lambda^*$, we can obtain that there exist $\varepsilon^* > 0$ such that for any $0 < \varepsilon \leq \varepsilon^*$, $u_\varepsilon^*(t) = u^*(t) + \varepsilon,\ t \in [0, 1]$ is a upper solution of BVP (2.6) at the point λ.

In fact, for $v_\varepsilon^*(t) = v^*(t) + \varepsilon,\ t \in [0, 1]$, by (H_2), we have

$$\begin{aligned} &\frac{1}{p(t)}(p(t)(v_\varepsilon^*(t))''')' + \lambda g(t)f(Sv_\varepsilon^*(t)) \\ &\quad = \frac{1}{p(t)}(p(t)(v^*(t))''')' + \lambda g(t)f(Sv^*(t) + \varepsilon) \\ &\quad = -\lambda^* g(t)(fSv^*(t)) + \lambda g(t)f(Sv^*(t) + \varepsilon) \\ &\quad = (\lambda - \lambda^*)g(t)(f(Sv^*(t) + \varepsilon) - f(Sv^*(t))) \leq 0, \end{aligned}$$

and

$$\begin{aligned} &\alpha_1 v_\varepsilon^*(0) - \beta_1(v_\varepsilon^*)'(0) = a\varepsilon \geq 0,\ \gamma_1 v_\varepsilon^*(1) + \delta_1(v_\varepsilon^*)'(1) = c\varepsilon \geq 0, \\ &\alpha_2(v_\varepsilon^*)''(0) - \beta_2 \lim_{t\to 0+} p(t)(v_\varepsilon^*)'''(t) = 0, \\ &\gamma_2(v_\varepsilon^*)''(1) + \delta_2 \lim_{t\to 1-} p(t)(v_\varepsilon^*)'''(t) = 0, \end{aligned}$$

This means that $v_\varepsilon^*(t)$ is an upper solution of BVP (2.6).

Let $\Omega = \{u \in C(J, R)| - \varepsilon < v(t) < v_\varepsilon^*(t),\ t \in [0, 1]\}$, then Ω is bounded open in $C(J, R)$ and $\theta \in \Omega$. Obviously, $T_\lambda : K \cap \overline{\Omega} \to K$ is completely continuous.

Suppose $u \in K \cap \partial\Omega$, then there exist $t_0 \in [0,1]$ such that $u(t_0) = v_\varepsilon^*(t_0)$. So, by (H_2), (2.4) and Lemma 3.5, we have

$$T_\lambda u(t_0) \leq T_\lambda v_\varepsilon^*(t_0) < T_{\lambda^*} v_\varepsilon^*(t_0) \leq v_\varepsilon^*(t_0) = u(t_0) \leq \mu u(t_0),\ \forall\ \mu \geq 1.$$

Therefore, we have

$$T_\lambda u \neq \mu u,\ u \in K \cap \partial\Omega,\ \mu \geq 1,$$

so, by Lemma 2.6, we have

$$i(T_\lambda, K \bigcap \Omega, K) = 1. \tag{3.9}$$

On the other hand, by (H_3), we can choose an appropriately big positive number $R' > 0$ such that

$$f(u) \geq \sigma u,\ \forall\ u \geq \xi\rho\theta_2 R', \tag{3.10}$$

where σ satisfies

$$\lambda\sigma\xi\rho^2\theta_1\theta_2 > 1.$$

Then, let $R = \max\{R'/\rho, 2\|v_\varepsilon^*\|\}$ and $K_R = \{u \in K : \|u\| < R\}$. And for any $t \in [a,b],\ u \in \partial K_R$, by using the same method as in Lemma 3.5, we have

$$u \neq T_\lambda u,\ \forall\ u \in \partial K_R,$$

Furthermore, if $u \in \partial K_R$, then

$$\min_{t\in[a,b]} u(t) \geq \rho\|u\| \geq R'.$$

Thus by (3.10), for any $t \in [a,b]$, we have

$$\begin{aligned}
\|T_\lambda u\| \ &\geq T_\lambda u(t) = \lambda A u(t) = \lambda \int_0^1 G(t,s)p(s)g(s)f(Su(s))\mathrm{d}s \\
&\geq \lambda \int_a^b G(t,s)p(s)g(s)f\left(\int_a^b H(s,\tau)u(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\
&\geq \lambda\rho \int_a^b G(s,s)p(s)g(s)f\left(\xi\rho\int_a^b H(\tau,\tau)\mathrm{d}\tau\|u\|\right)\mathrm{d}s \\
&\geq \lambda\rho^2\sigma\xi \int_a^b H(\tau,\tau)\mathrm{d}\tau \int_a^b G(s,s)p(s)g(s)\mathrm{d}s\|u\| \\
&> \|u\|.
\end{aligned}$$

Therefore, $\|T_\lambda u\| > \|u\|,\ u \in K \cap \partial K_R$ and by Lemma 2.4, we have

$$i(T_\lambda, K_R, K) = 0. \tag{3.11}$$

Consequently, by the additivity of the fixed point index,

$$0 = i(T_\lambda, K_R, K) = i(T_\lambda, K \cap \Omega, K) + i(T_\lambda, K_R \backslash \overline{K \cap \Omega}, K).$$

Therefore, $i(T_\lambda, K \bigcap \Omega, K) = 1,\ i(T_\lambda, K_R \backslash \overline{K \cap \Omega}, K) = -1$, and thus, T_λ have at lease two positive fixed points on $K \bigcap \Omega$ and $K_R \backslash \overline{K \cap \Omega}$ respectively, i.e., by Lemma 2.4, there exist at least two positive solutions of BVP (1.1). The proof is completed. □

Proof of Theorem 3.2. By Lemma 3.3–Lemma 3.6, we can obtain the results of Theorem 3.2. The proof is completed. □

Remark 3.7. In Theorem 3.1, we not only derive an explicit interval of λ such that for any λ in this interval, the existence of at least one positive solution to the boundary value problem is guaranteed, and the non-existence of solutions for λ in an appropriate interval is also discussed which is different from the previous papers. So our conclusion extend and improve the corresponding results of papers.

4. Application

In the section, in order to illustrate our results, we consider the following concrete fourth-order singular boundary value problem

Example 4.1. Consider the following singular boundary value problem (SBVP)

$$\begin{cases} 15\sqrt{t}(1-t)\left(\dfrac{1}{15\sqrt{t}(1-t)}u'''(t)\right)' + t^{\frac{1}{2}}(1-t)\left[(u+1)^2+(u+1)^3\right]=0, \quad 0<t<1, \\ u(0)-3u'(0)=0, \quad u(1)+2u'(1)=0, \\ 3u''(0)-\lim\limits_{t\to 0^+}u'''(t)=0, \quad u''(1)+\lim\limits_{t\to 1^-}u'''(t)=0, \end{cases} \tag{4.1}$$

where

$$\alpha_1=\gamma_1=1,\ \beta_1=3,\ \delta_1=2,\ \beta_2=\gamma_2=\delta_2=1,\ \alpha_2=3,$$

$$p(t)=\frac{1}{15\sqrt{t}(1-t)},\ g(t)=t^{\frac{1}{2}}(1-t),\ f(u)=(u+1)^2+(u+1)^3.$$

Then obviously,

$$\int_0^1 g(t)dt=4/15,\ \int_0^1\frac{1}{p(t)}dt=4,\ f_\infty=+\infty,$$

By computing, we know that the Green's function are

$$H(t,s)=\frac{1}{6}\begin{cases}(3+s)(3-t), & 0\le s\le t\le 1,\\ (3+t)(3-s), & 0\le t\le s\le 1.\end{cases}$$

$$G(t,s)=\frac{1}{16}\begin{cases}\left(1+30s^{\frac{3}{2}}-18s^{\frac{5}{2}}\right)\left(5-10t^{\frac{3}{2}}+6t^{\frac{5}{2}}\right), & 0\le s\le t\le 1,\\ \left(1+30t^{\frac{3}{2}}-18t^{\frac{5}{2}}\right)\left(5-10s^{\frac{3}{2}}+6s^{\frac{5}{2}}\right), & 0\le t\le s\le 1.\end{cases}$$

It is easy to note that $0\le G(s,s)\le 1$ and conditions (H_1), (H_2), (H_3) hold.

Next, by computing, we know that

$$e=\int_0^1 G(s,s)p(s)g(s)\mathrm{d}s=0.08,$$

we choose $r=3$, it follows from a direct calculation that

$$d(3)=\sup_{u\in K,\|u\|=3} f\left(3\int_0^1 H(s,s)\mathrm{d}s\right)=294.$$

Let $\lambda_1=\dfrac{r}{d(r)e}=0.13$. Therefore, from Theorem 3.1, when $\lambda\in(0,0.13)$, BVP (1.1) has at least one positive solution.

Let $[1/4,3/4]\subset(0,1)$, from $\lambda\varrho\rho^2\theta_1>1$, it follows from a direct calculation that $\lambda>277.95$. Then by Theorem 3.1, BVP (4.1) has no positive solutions when $\lambda>277.95$.

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