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Ψ-asymptotic stability of non-linear matrix Lyapunov systems

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper, first we convert the non-linear matrix Lyapunov system into a Kronecker product matrix system with the help of Kronecker product of matrices. Then, we obtain sufficient conditions for Ψ-asymptotic stability and Ψ-uniform stability of the trivial solutions of the corresponding Kronecker product system.

Keywords: Matrix Lyapunov system, Kronecker product, Fundamental matrix, Ψ-asymptotic stability, Ψ-(uniform) stability.
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1. Introduction

The importance of Matrix Lyapunov systems, which arise in a number of areas of control engineering problems, dynamical systems, and feedback systems are well known. In this paper we focus our attention to the first order non-linear matrix Lyapunov systems of the form

$$X'(t) = A(t)X(t) + X(t)B(t) + F(t, X(t)), \tag{1.1}$$

where $A(t), B(t)$ are square matrices of order n, whose elements a_{ij}, b_{ij}, are real valued continuous functions of t on the interval $R_+ = [0, \infty)$, and $F(t, X(t))$ is a continuous square matrix of order n defined on $(R_+ \times \mathbb{R}^{n\times n})$, such that $F(t, O) = O$, where $\mathbb{R}^{n\times n}$ denote the space of all $n \times n$ real valued matrices.

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Akinyele [1] introduced the notion of Ψ-stability, and this concept was extended to solutions of ordinary differential equations by Constantin [2]. Later Morchalo [6] introduced the concepts of Ψ-(uniform) stability, Ψ-asymptotic stability of trivial solutions of linear and non-linear systems of differential equations. Further, these concepts are extended to non-linear volterra integro-differential equations by Diamandescu [[3], [4]]. Recently, Murty and Suresh Kumar [[7], [8]] extended the concepts of Ψ-boundedness, Ψ-stability and Ψ-instability to matrix Lyapunov systems.

The purpose of our paper is to provide sufficient conditions for Ψ-asymptotic and Ψ-uniform stability of trivial solutions of the Kronecker product system associated with the non-linear matrix Lyapunov system (1.1). Here, we extend the concept of Ψ-stability in [7] to Ψ-asymptotic stability for matrix Lyapunov systems.

The paper is well organized as follows. In section 2 we present some basic definitions and notations relating to Ψ-(uniform) stability, Ψ-asymptotic stability and Kronecker products. First, we convert the non-linear matrix Lyapunov system (1.1) into an equivalent Kronecker product system and obtain its general solution. In section 3 we obtain sufficient conditions for Ψ- asymptotic stability of trivial solutions of the corresponding linear Kronecker product system. In section 4 we study Ψ-asymptotic stability and Ψ-uniform stability of trivial solutions of non-linear Kronecker product system. The main results of this paper are illustrated with suitable examples.

This paper extends some of the results of Ψ-asymptotic stability of trivial solutions of linear equations (Theorem 1 and Theorem 2)in Diamandescu [4] to matrix Lyapunov systems.

2. Preliminaries

In this section we present some basic definitions and results which are useful for later discussion.

Let $\mathbb{R}^n$ be the Euclidean n-dimensional space. Elements in this space are column vectors, denoted by $u = (u_1, u_2, \dots u_n)^T$(^{T} denotes transpose) and their norm defined by

$$\|u\| = \max\{|u_1|, |u_2|, \dots |u_n|\}.$$

For a $n \times n$ real matrix, we define the norm

$$|A| = \sup_{\|x\| \leq 1} \|Ax\|.$$

Let $\Psi_k : R_+ \to (0, \infty)$, $k = 1, 2, \dots n, \dots n^2$,be continuous functions, and let

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_{n^2}].$$

Then the matrix $\Psi(t)$ is an invertible square matrix of order n^2 ,for each $t \geq 0$.

Definition 2.1. [5] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ then the Kronecker product of A and B written $A \otimes B$ is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & a_{2n}B \\ . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & a_{mn}B \end{bmatrix}$$

is an $mp \times nq$ matrix and is in $\mathbb{R}^{mp\times nq}$.

Definition 2.2. [5] Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, we denote

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ . \\ . \\ A_{.n} \end{bmatrix}, \text{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix} (1 \leq\ j \leq\ n).$$

Regarding properties and rules for Kronecker product of matrices we refer to Graham [5].

Now by applying the Vec operator to the non-linear matrix Lyapunov system (1.1) and using the above properties, we have

$$\hat{X}'(t) = H(t)\hat{X}(t) + G(t, \hat{X}(t)), \tag{2.1}$$

where $H(t) = (B^T \otimes I_n) + (I_n \otimes A)$ is a $n^2 \times n^2$ matrix and $G(t, \hat{X}(t)) = VecF(t, X(t))$ is a column matrix of order n^2.

The corresponding linear homogeneous system of (2.1) is

$$\hat{X}'(t) = H(t)\hat{X}(t). \tag{2.2}$$

Definition 2.3. The trivial solution of (2.1) is said to be Ψ-stable on R_+ if for every $\varepsilon > 0$ and every t_0 in R_+, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.1) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta$, also satisfies the inequality $\|\Psi(t)\hat{X}(t)\| < \varepsilon$ for all $t \geq t_0$.

Definition 2.4. The trivial solution of (2.1) is said to be Ψ-uniformly stable on R_+, if $\delta(\varepsilon, t_0)$ in Definition 2.3 can be chosen independent of t_0.

Definition 2.5. The trivial solution of (2.1) is said to be Ψ-asymptotically stable on R_+, if it is Ψ-stable on R_+ and in addition, for any $t_0 \in R_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.1) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$, satisfies the condition $\lim\limits_{t\to\infty} \Psi(t)\hat{X}(t) = 0$.

The following example illustrates the difference between the Ψ-stability and Ψ-asymptotic stability.

Example 2.1. Consider the non-linear matrix Lyapunov system (1.1) with

$$A(t) = \begin{bmatrix} \frac{t}{t^2-1} & 0 \\ 0 & 2t \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 & e^t \\ \frac{-t}{t^2-1} & 0 \end{bmatrix} \text{ and}$$

$$F(t, X(t)) = \begin{bmatrix} \frac{1+t(x_2-3x_1)}{t^2-1} & -e^t x_1 - \frac{tx_2}{t^2-1} + x_2 \\ \frac{tx_4}{t^2-1} - 2tx_3 - x_3 & x_4^2 \sec t - x_4 \tan t - 2tx_4 - e^t x_3 \end{bmatrix}.$$

Then the solution of (2.1) is

$$\hat{X}(t) = \begin{bmatrix} \frac{1}{(t+1)\sqrt{t^2-1}} \\ e^{-t} \\ e^t \\ \frac{-\cos t}{t} \end{bmatrix}.$$

Consider

$$\Psi(t) = \begin{bmatrix} t+1 & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-t} & 0 \\ 0 & 0 & 0 & t \end{bmatrix}$$

for all $t \geq 0$, we have

$$\Psi(t)\hat{X}(t) = \begin{bmatrix} \frac{1}{\sqrt{t^2-1}} \\ 1 \\ 1 \\ -\cos t \end{bmatrix}.$$

It is easily seen from the Definitions 2.3 and 2.5, the trivial solution of the system (2.1) is Ψ-stable on R_+, but, it is not Ψ-asymptotically stable on R_+.

Lemma 2.1. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems*

$$X'(t) = A(t)X(t), \tag{2.3}$$

and

$$[X^T(t)]' = B^T(t)X^T(t) \tag{2.4}$$

respectively. Then the matrix $Z(t) \otimes Y(t)$ is a fundamental matrix of (2.2) and every solution of (2.2) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c$, where c is a n^2-column vector.

Proof. For proof, we refer to Lemma 1 of [7]. □

Theorem 2.1. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the systems (2.3) and (2.4), then any solution of (2.1), satisfying the initial condition $\hat{X}(t_0) = \hat{X}_0$, is given by*

$$\hat{X}(t) = (Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}_0 + \int_{t_0}^{t} (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds. \tag{2.5}$$

Proof. First, we show that any solution of (2.1) is of the form $\hat{X}(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$, where $\widetilde{X}(t)$ is a particular solution of (2.1) and is given by

$$\widetilde{X}(t) = \int_{t_0}^{t} (Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds.$$

Here we observe that, $\hat{X}(t_0) = (Z(t_0) \otimes Y(t_0))c = \hat{X}_0$, $c = (Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}_0$. Let $u(t)$ be any other solution of (2.1), write $w(t) = u(t) - \widetilde{X}(t)$, then w satisfies (2.2), hence $w = (Z(t) \otimes Y(t))c$, $u(t) = (Z(t) \otimes Y(t))c + \widetilde{X}(t)$.

Next, we consider the vector $\widetilde{X}(t) = (Z(t) \otimes Y(t))v(t)$, where $v(t)$ is an arbitrary vector to be determined, so as to satisfy equation (2.1). Consider

$$\begin{aligned}
&\widetilde{X}'(t) = (Z(t) \otimes Y(t))'v(t) + (Z(t) \otimes Y(t))v'(t) \\
&\Rightarrow H(t)\widetilde{X}(t) + G(t, \hat{X}(t)) = H(t)(Z(t) \otimes Y(t))v(t) + (Z(t) \otimes Y(t))v'(t) \\
&\Rightarrow (Z(t) \otimes Y(t))v'(t) = G(t, \hat{X}(t)) \\
&\Rightarrow v'(t) = (Z^{-1}(t) \otimes Y^{-1}(t))G(t, \hat{X}(t)) \\
&\Rightarrow v(t) = \int_{t_0}^{t} (Z^{-1}(s) \otimes Y^{-1}(s))G(s, \hat{X}(s))ds.
\end{aligned}$$

Hence the desired expression follows immediately. □

3. Ψ-asymptotic stability of linear systems

In this section we study the Ψ-asymptotic stability of trivial solutions of linear system (2.2).

Theorem 3.1. *Let $Y(t)$ and $Z(t)$ be the fundamental matrices of (2.3) and (2.4). Then the trivial solution of (2.2) is Ψ-asymptotically stable on R_+ if and only if $\lim_{t\to\infty} \Psi(t)(Z(t) \otimes Y(t)) = 0$.*

Proof. The solution of (2.2) with the initial point at $t_0 \geq 0$ is

$$\hat{X}(t) = (Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}(t_0), \quad \text{for } t \geq 0.$$

First, we suppose that the trivial solution of (2.2) is Ψ-asymptotically stable on R_+. Then, the trivial solution of (2.2) is Ψ-stable on R_+ and for any $t_0 \in R_+$, there exists a $\delta_0 = \delta(t_0) > 0$ such that any solution $\hat{X}(t)$ of (2.2) which satisfies the inequality $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$, and satisfies the condition $\lim_{t\to\infty} \Psi(t)\hat{X}(t) = 0$.

Therefore, for any $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta_0 > 0$ such that $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$ and also satisfies

$$\|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\Psi^{-1}(t_0)\Psi(t_0)\hat{X}(t_0)\| < \epsilon \quad \text{for all } t \geq t_{\epsilon,t_0}.$$

Let $v \in \mathbb{R}^{n^2}$ be such that $\|v\| \leq 1$. For $\hat{X}(t_0) = \frac{\delta_0}{2}\Psi^{-1}(t_0)v$, we have $\|\Psi(t_0)\hat{X}(t_0)\| < \delta_0$ and hence,

$$\|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0)\frac{\delta_0}{2}\Psi^{-1}(t_0)v\| < \epsilon$$

$$\Rightarrow \|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\Psi^{-1}(t_0)\| < \frac{2\epsilon}{\delta_0}$$

$$\Rightarrow |\Psi(t)(Z(t) \otimes Y(t))| \leq \frac{2\epsilon}{\delta_0|(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|},$$

for $t \geq t_{\epsilon,t_0}$. Therfore, $\lim_{t\to\infty} \Psi(t)(Z(t) \otimes Y(t)) = 0$.

Conversely, suppose that $\lim_{t\to\infty} \Psi(t)(Z(t) \otimes Y(t)) = 0$. Then, there exists $M > 0$ such that $|\Psi(t)(Z(t) \otimes Y(t))| \leq M$ for $t \geq 0$. From (i) of Theorem 3 [7], it follows that the trivial solution of (2.2) is Ψ-stable on R_+. For any $\hat{X}(t_0) \in \mathbb{R}^{n^2}$, we have

$$\lim_{t\to\infty} \Psi(t)\hat{X}(t) = \lim_{t\to\infty} \Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\hat{X}(t_0) = 0.$$

Thus, the trivial solution of (2.2) is Ψ-asymptotically stable on R_+. □

The above Theorem 3.1 is illustrated by the following example.

Example 3.1. Consider the linear homogeneous matrix Lyapunov system corresponding to (1.1) with

$$A(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \frac{-1}{t+1} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Then the fundamental matrices of (2.3), (2.4) are

$$Y(t) = \begin{bmatrix} t+1 & 0 \\ 0 & \frac{1}{t+1} \end{bmatrix}, \quad Z(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$

Now the fundamental matrix of (2.2) is

$$Z(t) \otimes Y(t) = \begin{bmatrix} e^t(t+1) & 0 & 0 & 0 \\ 0 & \frac{e^t}{t+1} & 0 & 0 \\ 0 & 0 & (t+1)e^{-2t} & 0 \\ 0 & 0 & 0 & \frac{e^{-2t}}{t+1} \end{bmatrix}.$$

Consider

$$\Psi(t) = \begin{bmatrix} \frac{e^{-2t}}{t+1} & 0 & 0 & 0 \\ 0 & \frac{e^{-t}}{t+1} & 0 & 0 \\ 0 & 0 & \frac{e^{2t}}{(t+1)^2} & 0 \\ 0 & 0 & 0 & \frac{e^{2t}}{\sqrt{t+1}} \end{bmatrix}$$

for all $t \geq 0$, we have

$$\Psi(t)(Z(t) \otimes Y(t)) = \begin{bmatrix} e^{-t} & 0 & 0 & 0 \\ 0 & \frac{1}{(t+1)^2} & 0 & 0 \\ 0 & 0 & \frac{1}{t+1} & 0 \\ 0 & 0 & 0 & \frac{1}{(t+1)^{\frac{3}{2}}} \end{bmatrix}$$

It is easily seen from Theorem 3.1 , the system (2.2) is Ψ-asymptotically stable on R_+.

Remark 3.1. Ψ-asymptotic stability need not imply classical asymptotic stability.

The Remark 3.1 is illustrated by the following example.

Example 3.2. Consider the linear homogeneous matrix Lyapunov system corresponding to (1.1) with

$$A(t) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then the fundamental matrices of (2.3), (2.4) are

$$Y(t) = \begin{bmatrix} e^t \sin t & e^t \cos t \\ -e^t \cos t & e^t \sin t \end{bmatrix}, \quad Z(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Now the fundamental matrix of (2.2) is

$$Z(t) \otimes Y(t) = \begin{bmatrix} \sin t & \cos t & 0 & 0 \\ -\cos t & \sin t & 0 & 0 \\ 0 & 0 & \sin t & \cos t \\ 0 & 0 & -\cos t & \sin t \end{bmatrix}.$$

Clearly the system (2.2) is stable, but it is not asymptotically stable on R_+. Consider

$$\Psi(t) = \begin{bmatrix} \frac{1}{t+1} & 0 & 0 & 0 \\ 0 & \frac{1}{t+1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{t+1}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t+1}} \end{bmatrix}$$

for all $t \geq 0$, we have

$$\Psi(t)(Z(t) \otimes Y(t)) = \begin{bmatrix} \frac{\sin t}{t+1} & \frac{\cos t}{t+1} & 0 & 0 \\ -\frac{\cos t}{t+1} & \frac{\sin t}{t+1} & 0 & 0 \\ 0 & 0 & \frac{\sin t}{\sqrt{t+1}} & \frac{\cos t}{\sqrt{t+1}} \\ 0 & 0 & -\frac{\cos t}{\sqrt{t+1}} & \frac{\sin t}{\sqrt{t+1}} \end{bmatrix}.$$

Thus, from Theorem 3.1 the system (2.2) is Ψ-asymptotically stable on R_+.

Theorem 3.2. *Let $Y(t), Z(t)$ be the fundamental matrices of* (2.2), (2.4). *If there exists a continuous function $\phi : R_+ \to (0, \infty)$ such that $\int_0^\infty \phi(s)ds = \infty$, and a positive constant N satisfying*

$$\int_0^t \phi(s)|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq N, \quad \textit{for all} \quad t \geq 0$$

then, the linear system (2.2) *is Ψ-asymptotically stable on R_+.*

Proof. Let $b(t) = |\Psi(t)(Z(t) \otimes Y(t))|^{-1}$ for $t \geq 0$. From the identity

$$\Big(\int_0^t \phi(s)b(s)\,ds\Big)\Psi(t)(Z(t)\otimes Y(t))$$
$$= \int_0^t \phi(s)\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)\Psi(s)(Z(s)\otimes Y(s))b(s)\,ds,$$

it follows that

$$\Big(\int_0^t \phi(s)b(s)\,ds\Big)|\Psi(t)(Z(t)\otimes Y(t))|$$
$$\leq \int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)||\Psi(s)(Z(s)\otimes Y(s))|b(s)\,ds.$$

Thus, the scalar function $a(t) = \int_0^t \phi(s)b(s)ds$ satisfies the inequality

$$a(t)b^{-1}(t) \leq N, \quad \text{for} \quad t \geq 0.$$

We have $a'(t) = \phi(t)b(t) \geq N^{-1}\phi(t)a(t)$ for $t \geq 0$. It follows that

$$a(t) \geq a(t_1)e^{N^{-1}\int_{t_1}^t \phi(s)\,ds}, \quad \text{for } t \geq t_1 > 0$$

and hence

$$|\Psi(t)(Z(t)\otimes Y(t))| = b^{-1}(t) \leq Na^{-1}(t_1)e^{-N^{-1}\int_{t_1}^t \phi(s)\,ds}, \quad \text{for } t \geq t_1 > 0\,.$$

Since $|\Psi(t)(Z(t)\otimes Y(t))|$ is a continuous function on the compact interval $[0, t_1]$, there exists a positive constant M such that $|\Psi(t)(Z(t)\otimes Y(t))| \leq M$ for $t \geq 0$. Therefore, the trivial solution of (2.2) is Ψ-stable on R_+, and also from

$$\int_0^\infty \phi(s)ds = \infty, \text{ it follws that } \lim_{t\to\infty} \Psi(t)(Z(t)\otimes Y(t)) = 0.$$

Hence by using Theorem 3.1, system (2.2) is Ψ-asymptotically stable. □

4. Ψ-asymptotic stability of non-linear systems

In this section we obtain sufficient conditions for Ψ-asymptotic stability and Ψ-uniform stability of trivial solutions of non-linear system (2.1).

Theorem 4.1. *Suppose that*

(i) The fundamental matrices $Y(t)$ and $Z(t)$ of (2.3), (2.4) are satisfying the condition

$$\int_0^t \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|ds \leq N, \text{ for all } t \geq 0$$

where N is a positive constant and ϕ is a continuous positive function on R_+ such that $\int_0^\infty \phi(s)ds = \infty$.

(ii) The function G satisfies the condition

$$\|\Psi(t)G(t, \hat{X}(t))\| \leq \alpha(t)\|\Psi(t)\hat{X}(t)\|$$

for every vector valued continuous function $\hat{X} : R_+ \to \mathbb{R}^{n^2}$, where α is a continuous non-negative function on R_+ such that

$$q = \sup_{t\geq 0}\frac{\alpha(t)}{\phi(t)} < \frac{1}{N}.$$

Then, the trivial solution of equation (2.1) *is* Ψ*-asymptotically stable on* R_+.

Proof. From the first assumption of the theorem, Theorems 3.1 and 3.2, we have

$$\lim_{t\to\infty} |\Psi(t)(Z(t)\otimes Y(t))| = 0,$$

hence there exists a positive constant M such that

$$|\Psi(t)(Z(t)\otimes Y(t))| \le M, \text{ for all } t \ge 0.$$

From the second assumption of the theorem, we have

$$\frac{\alpha(t)}{\phi(t)} \le \sup_{t\ge 0}\frac{\alpha(t)}{\phi(t)} = q < \frac{1}{N}.$$

For a given $\epsilon > 0$ and $t_0 \ge 0$, we choose $\delta = \min\{\epsilon, \frac{(1-qN)\epsilon}{M|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|}\}$. Let $\hat{X}_0 \in R^{n^2}$ such that $||\Psi(t_0)\hat{X}_0|| < \delta$.

For $\tau > t_0$ and $t \in [t_0, \tau]$. Consider

$$\begin{aligned}\|\Psi(t)\hat{X}(t)\| &\le \|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)\Psi(t_0)\hat{X}(t_0)\| \\ &\quad + \int_{t_0}^{t} |\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))|\|\Psi(s)G(s,\hat{X}(s))\|ds \\ &\le |\Psi(t)(Z(t)\otimes Y(t))||(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\|\Psi(t_0)\hat{X}_0\| \\ &\quad + \int_{t_0}^{t} \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\frac{\alpha(s)}{\phi(s)}\|\Psi(s)\hat{X}(s)\|ds \\ &< M|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\delta + Nq\sup_{t_0\le t\le\tau}\|\Psi(t)\hat{X}(t)\|.\end{aligned}$$

Therefore,

$$\sup_{t_0\le t\le\tau}\|\Psi(t)\hat{X}(t)\| \le (1-Nq)^{-1}M|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\delta < \epsilon.$$

It follows that the trivial solution of equation (2.1) is Ψ-stable on R_+. To prove, the trivial solution of (2.1) is Ψ-asymptotically stable, we must show further that $\lim_{t\to\infty}\|\Psi(t)\hat{X}(t))\| = 0$.

Suppose that $\lim_{t\to\infty}\sup\|\Psi(t)\hat{X}(t))\| = \lambda > 0$. Let θ be such that $qN < \theta < 1$, then there exists $t_1 \ge t_0$ such that $||\Psi(t)\hat{X}(t)|| < \frac{\lambda}{\theta}$ for all $t \ge t_1$. Thus for $t > t_1$, we have

$$\begin{aligned}\|\Psi(t)\hat{X}(t)\| &\le |\Psi(t)(Z(t)\otimes Y(t))||(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\|\Psi(t_0)\hat{X}(t_0)\| \\ &\quad + \int_{t_0}^{t} |\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\|\Psi(s)G(s,\hat{X}(s))\|ds \\ &< |\Psi(t)(Z(t)\otimes Y(t))||(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\delta \\ &\quad + \int_{t_0}^{t_1} |\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\alpha(s)\|\Psi(s)\hat{X}(s)\|ds \\ &\quad + \int_{t_1}^{t} \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\frac{\alpha(s)}{\phi(s)}\|\Psi(s)\hat{X}(s)\|ds \\ &< |\Psi(t)(Z(t)\otimes Y(t))||(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\delta\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_0}^{t_1} |\Psi(t)(Z(t)\otimes Y(t))||(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\alpha(s)\|\Psi(s)\hat{X}(s)\|ds \\
&+\int_{t_1}^{t} \phi(s)|\Psi(t)(Z(t)\otimes Y(t))(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\frac{q\lambda}{\theta}ds \\
\le\ & |\Psi(t)(Z(t)\otimes Y(t))|\ \{|(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\delta \\
&+\int_{t_0}^{t_1} |(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\alpha(s)\|\Psi(s)\hat{X}(s)\|ds\} + \frac{Mq\lambda}{\theta}.
\end{aligned}
$$

From $\lim\limits_{t\to\infty} |\Psi(t)(Z(t)\otimes Y(t))| = 0$, it follows that there exists $T > 0$, sufficiently large, such that

$$|\Psi(t)(Z(t)\otimes Y(t))| < \frac{\lambda - \frac{Mq\lambda}{\theta}}{2Q} \quad \text{for all} \quad t \ge T,$$

where

$$
\begin{aligned}
Q = & |(Z^{-1}(t_0)\otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\delta \\
&+\int_{t_0}^{t_1} |(Z^{-1}(s)\otimes Y^{-1}(s))\Psi^{-1}(s)|\alpha(s)\|\Psi(s)\hat{X}(s)\|ds.
\end{aligned}
$$

Thus, for $t \ge T$ we have

$$
\begin{aligned}
\|\Psi(t)\hat{X}(t)\| &< \frac{\lambda - \frac{Mq\lambda}{\theta}}{2} + \frac{Mq\lambda}{\theta} \\
&< \frac{\lambda + \frac{Mq\lambda}{\theta}}{2}.
\end{aligned}
$$

It follows from the definition of θ

$$\lambda \le \frac{\lambda + \frac{Mq\lambda}{\theta}}{2} < \lambda$$

which is a contradiction. Therefore

$$\lim_{t\to\infty} \|\Psi(t)\hat{X}(t)\| = 0.$$

Thus, the trivial solution of (2.1) is Ψ-asymptotically stable on R_+. □

Example 4.1. Consider the non-linear matrix Lyapunov system (1.1) with

$$A(t) = \begin{bmatrix} \frac{1}{t+1} & 0 \\ 0 & \frac{-1}{t+1} \end{bmatrix}, \ B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \text{and} \ F(t, X(t)) = \begin{bmatrix} \frac{\sin(x_1)}{4(t+1)} & \frac{x_3}{8(t+1)} \\ \frac{x_2}{2(t+1)} & \frac{\sin(x_4)}{6(t+1)} \end{bmatrix}.$$

The fundamental matrices of (2.3), (2.4) are

$$Y(t) = \begin{bmatrix} t+1 & 0 \\ 0 & \frac{1}{t+1} \end{bmatrix}, \ Z(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}.$$

Therefore, the fundamental matrix of (2.2) is

$$Z(t)\otimes Y(t) = \begin{bmatrix} e^t(t+1) & 0 & 0 & 0 \\ 0 & \frac{e^t}{t+1} & 0 & 0 \\ 0 & 0 & (t+1)e^t & 0 \\ 0 & 0 & 0 & \frac{e^t}{t+1} \end{bmatrix}.$$

Consider

$$\Psi(t) = \begin{bmatrix} \frac{e^{-t}}{(t+1)^2} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & \frac{e^{-t}}{(t+1)^2} & 0 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

for all $t \geq 0$, then we have

$$\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s) = \left(\frac{s+1}{t+1}\right) I_4.$$

Taking $\phi(t) = \frac{1}{t+1}$, for all $t \geq 0$. Clearly $\phi(t)$ is continuous on R_+ and $\int\limits_0^\infty \phi(s)ds = \infty$. Also

$$\int_0^t \phi(s)|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|ds = \frac{t}{t+1} \leq 1, \text{ for all } t \geq 0.$$

Further, the matrix G satisfies condition (ii), with $\alpha(t) = \frac{1}{2(t+1)}$, $\alpha(t)$ is a continuous non-negative function on R_+ and satisfies

$$q = \sup_{t \geq 0} \frac{\alpha(t)}{\phi(t)} = \frac{1}{2} < \frac{1}{N} = 1.$$

Thus, from Theorem 4.1, the trivial solution of non-linear system (2.1) is Ψ-asymptotically stable on R_+.

Theorem 4.2. *Let $Y(t)$, $Z(t)$ be the fundamental matrices of* (2.3), (2.4) *respectively satisfying the condition*

$$|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)| \leq L,$$

for all $0 \leq s \leq t < \infty$, where L is a positive number. Assume that the function G satisfies

$$\|\Psi(t)G(t, \hat{X}(t))\| \leq \alpha(t)\|\Psi(t)\hat{X}(t))\|, \ 0 \leq t < \infty$$

and for every $\hat{X} \in \mathbb{R}^{n^2}$, where $\alpha(t)$ is a continuous non-negative function such that $\beta = \int_0^\infty \alpha(s)ds < \infty$. Then, the trivial solution of (2.1) *is Ψ-uniformly stable on R_+.*

Proof. Let $\epsilon > 0$ and $\delta(\epsilon) = \frac{\epsilon}{2L}e^{-L\beta}$. For $t_0 \geq 0$ and $\hat{X}_0 \in \mathbb{R}^{n^2}$ be such that $\|\Psi(t_0)\hat{X}_0\| < \delta(\epsilon)$, we have

$$\begin{aligned} \|\Psi(t)\hat{X}(t)\| &\leq \|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\Psi^{-1}(t_0)\Psi(t_0)\hat{X}(t_0)\| \\ &\quad + \int_{t_0}^t \|\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)\Psi(s)G(s, \hat{X}(s))\|ds \\ &\leq |\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(t_0) \otimes Y^{-1}(t_0))\Psi^{-1}(t_0)|\|\Psi(t_0)\hat{X}_0)\| \\ &\quad + \int_{t_0}^t |\Psi(t)(Z(t) \otimes Y(t))(Z^{-1}(s) \otimes Y^{-1}(s))\Psi^{-1}(s)|\|\Psi(s)G(s, \hat{X}(s))\|ds \\ &\leq L\|\Psi(t_0)\hat{X}_0\| + L\int_{t_0}^t \alpha(s)\|\Psi(s)\hat{X}(s)\|ds. \end{aligned}$$

By Gronwall's inequality

$$\begin{aligned} \|\Psi(t)\hat{X}(t)\| &\leq L\|\Psi(t_0)\hat{X}_0\|e^{L\int_{t_0}^t \alpha(s)ds} \\ &\leq L\delta(\epsilon)e^{L\beta} < \epsilon, \end{aligned}$$

for all $t \geq t_0$. This proves that the trivial solution of (2.1) is Ψ-uniformly stable on R_+. $\square$

Example 4.2. In Example 4.1, taking

$$F(t, X(t)) = \begin{bmatrix} \frac{x_1}{(t+1)^2} & \frac{\sin(x_3)}{(t+1)^2} \\ \frac{\sin(x_2)}{(t+1)^2} & \frac{x_4}{(t+1)^2} \end{bmatrix}.$$

Then the conditions of Theorem 4.2 are satisfied with $L = 1$ and $\alpha(t) = \frac{1}{(t+1)^2}$. Clearly, $\alpha(t)$ is continuous non-negative function and $\int\limits_0^\infty \alpha(s)ds = 1$. Therefore, from Theorem 4.2 the trivial solution of (2.1) is Ψ-uniformly stable on R_+.

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