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Fixed point theorems for A-contraction mappings of integral type

Mantu Saha[a], Debashis Dey[b,]

[a] Department of Mathematics, The University of Burdwan, Burdwan-713104, West Bengal, India.
[b]Koshigram Union Institution, Koshigram-713150, Burdwan, West Bengal, India.

This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In the present paper, we prove analogues of some fixed point results for A-contraction type mappings in integral setting.

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1. Introduction and Preliminaries

Fixed point theory plays a crucial part in nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations. First important result on fixed points for contractive type mapping was given by S. Banach [3] in 1922. In the general setting of complete metric space, this theorem runs as follows (see Theorem 2.1,[8] or, Theorem 1.2.2,[17])).

Theorem 1.1. (Banach contraction principle) *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(fx, fy) \leq cd(x, y) \tag{1.1}$$

then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

*Corresponding author
Email addresses: mantusaha@yahoo.com (Mantu Saha), debashisdey@yahoo.com (Debashis Dey)

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An elementary account of the Banach contraction principle and some applications, including its role in solving nonlinear ordinary differential equations, is in [6]. The contraction mapping theorem is used to prove the inverse function theorem in [([15], pp. 221-223). A beautiful application of contraction mappings to the construction of fractals (interpreted as fixed points in a metric space whose 'points' are compact subsets of the plane) is in ([16], Chap. 5).
After the classical result by Banach, Kannan [9] gave a substantially new contractive mapping to prove the fixed point theorem. Since then there have been many theorems emerged as generalizations under various contractive conditions. Such conditions involve linear and nonlinear expressions (rational, irrational, and general type). The intrested reader who wants to know more about this matter is recommended to go deep into the survey articles by Rhoades ([12], [13], [14]) and Bianchini [4], and into the references therein.

1.1. A-contraction

On the otherhand, Akram et al.[2] introduced a new class of contraction maps, called A-contraction, which is a proper superclass of Kannan's[9], Bianchini's[4] and Reich's[10] type contractions. Akram et al.[2] defined A-contractions as follows:

Let a non-empty set A consisting of all functions $\alpha : R_+^3 \to R_+$ satisfying:

(A_1): α is continuous on the set R_+^3 of all triplets of non-negative reals(with respect to the Euclidean metric on R^3).
(A_2): $a \le kb$ for some $k \in [0, 1)$ whenever $a \le \alpha(a, b, b)$ or $a \le \alpha(b, a, b)$ or $a \le \alpha(b, b, a)$ for all a, b.

Definition 1.2. A self-map T on a metric space X is said to be A-contraction, if it satisfies the condition

$$d(Tx, Ty) \le \alpha(d(x, y), d(x, Tx), d(y, Ty))$$

for all $x, y \in X$ and some $\alpha \in A$.

Example 1.3. Let a self-map T on a metric space (X, d) satisfying

$$d(Tx, Ty) \le \beta \max\{d(Tx, x) + d(Ty, y), d(Ty, y) + d(x, y), d(Tx, x) + d(x, y)\}$$

for all $x, y \in X$ and some $\beta \in \left[0, \frac{1}{2}\right)$, is an A-contraction. (see [2] for detail and comparison with other contraction maps.)

In 2002, A.Branciari[5] analyzed the existence of fixed point for mapping T defined on a complete metric space (X, d) satisfying a general contractive condition of integral type in the following theorem.

Theorem 1.4. (Branciari) *Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $T : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt \tag{1.2}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, non-negative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$, then T has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades[11] extending the result of Branciari by replacing the condition (1.2) by the following

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{[d(x,Ty)+d(y,Tx)]}{2}\}} \varphi(t)dt \tag{1.3}$$

for all $x, y \in X$ with some $c \in [0, 1)$. In a very recent paper, Dey et al.[7] proved some fixed point theorems for mixed type of contraction mappings of integral type in complete metric space.

Motivated and inspired by these consequent works, we introduce the analogues of some fixed point results for A-contraction mappings in integral setting which in turn generalize several known results. Also we have analyzed the existence of fixed point of mapping over two related metrics due to Theorem 4 of [1] in integral setting. Our results substantially extend, improve, and generalize comparable results in the literature.

2. Main results

Theorem 2.1. *Let T be a self-mapping of a complete metric space (X, d) satisfying the following condition:*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha\left(\int_0^{d(x,y)} \varphi(t)dt, \int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(y,Ty)} \varphi(t)dt\right) \tag{2.1}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0, +\infty)$, non-negative, and such that

$$\text{for each } \epsilon > 0, \quad \int_0^{\epsilon} \varphi(t)dt > 0 \tag{2.2}$$

Then T has a unique fixed point $z \in X$ and for each $x \in X$, $\lim_n T^n x = z$.

Proof. Let $x_0 \in X$ be arbitrary and, for brevity, define $x_{n+1} = Tx_n$. For each integer $n \geq 1$, from (2.1) we get,

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt &= \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)dt \\
&\leq \alpha\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_{n-1},Tx_{n-1})} \varphi(t)dt, \int_0^{d(x_n,Tx_n)} \varphi(t)dt\right) \\
&\leq \alpha\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_{n-1},x_n)} \varphi(t)dt, \int_0^{d(x_n,x_{n+1})} \varphi(t)dt\right)
\end{aligned}$$

Then by the axiom A_2 of function α,

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \tag{2.3}$$

for some $k \in [0, 1)$ as $\alpha \in A$.

In this fashion, one can obtain

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt &\leq k \int_0^{d(x_{n-1},x_n)} \varphi(t)dt \\
&\leq k^2 \int_0^{d(x_{n-2},x_{n-1})} \varphi(t)dt \\
&\leq \dots \\
&\leq k^n \int_0^{d(x_0,x_1)} \varphi(t)dt
\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\lim_n \int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0 \qquad \text{as } k \in [0, 1)$$

which, from(2.2) implies that

$$\lim_n d(x_n, x_{n+1}) = 0 \tag{2.4}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there exists an $\epsilon > 0$ and subsequences $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \epsilon, \qquad d(x_{m(p)}, x_{n(p)-1}) < \epsilon \tag{2.5}$$

Now

$$\begin{aligned} d(x_{m(p)-1}, x_{n(p)-1}) &\leq d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) \\ &< d(x_{m(p)-1}, x_{m(p)}) + \epsilon \end{aligned} \tag{2.6}$$

So by (2.4) and (2.6) we get

$$\lim_p \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \varphi(t)dt \leq \int_0^{\epsilon} \varphi(t)dt \tag{2.7}$$

Using (2.3), (2.5) and (2.7) we get

$$\int_0^{\epsilon} \varphi(t)dt \leq \int_0^{d(x_{m(p)}, x_{n(p)})} \varphi(t)dt \leq k \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \varphi(t)dt \leq k \int_0^{\epsilon} \varphi(t)dt$$

which is a contradiction, since $k \in [0, 1)$. Therefore, $\{x_n\}$ is Cauchy, hence convergent. Call the limit z.
From (2.1) we get

$$\begin{aligned} \int_0^{d(Tz, x_{n+1})} \varphi(t)dt &= \int_0^{d(Tz, Tx_n)} \varphi(t)dt \\ &\leq \alpha\left(\int_0^{d(z, x_n)} \varphi(t)dt, \int_0^{d(z, Tz)} \varphi(t)dt, \int_0^{d(x_n, x_{n+1})} \varphi(t)dt \right) \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\int_0^{d(Tz, z)} \varphi(t)dt \leq \alpha\left(o, \int_0^{d(z, Tz)} \varphi(t)dt, 0 \right)$$

So by the axiom A_2 of function α,

$$\int_0^{d(Tz, z)} \varphi(t)dt = k.0 = 0$$

which, from (2.2), implies that $d(Tz, z) = 0$ or, $Tz = z$.
Next suppose that $w(\neq z)$ be another fixed point of T. Then from (2.1) we have

$$\begin{aligned} \int_0^{d(z, w)} \varphi(t)dt &= \int_0^{d(Tz, Tw)} \varphi(t)dt \\ &\leq \alpha\left(\int_0^{d(z, w)} \varphi(t)dt, \int_0^{d(z, Tz)} \varphi(t)dt, \int_0^{d(w, Tw)} \varphi(t)dt \right) \\ &= \alpha\left(\int_0^{d(z, w)} \varphi(t)dt, \int_0^{d(z, z)} \varphi(t)dt, \int_0^{d(w, w)} \varphi(t)dt \right) \\ &= \alpha\left(\int_0^{d(z, w)} \varphi(t)dt, 0, 0 \right) \end{aligned}$$

So by the axiom A_2 of function α,

$$\int_0^{d(z,w)} \varphi(t)dt = 0$$

which, from (2.2), implies that $d(z,w) = 0$ or, $z = w$ and so the fixed point is unique. □

Next theorem describes common fixed point of two self-maps on X having two related metrics in integral setting. This result generalizes Theorem 4 of [1] in integral seting.

Theorem 2.2. *Let X be a set with two metrics d and δ satisfying the following conditions:*
(i) $\int_0^{d(x,y)} \varphi(t)dt \le \int_0^{\delta(x,y)} \varphi(t)dt$ *for all* $x, y \in X$*;*
(ii) X *is complete with respect to* d*;*
(iii) S, T *are self-maps on* X *such that* T *is continuous with respect to* d *and*

$$\int_0^{\delta(Tx,Sy)} \varphi(t)dt \le \alpha\left(\int_0^{\delta(x,y)} \varphi(t)dt, \int_0^{\delta(x,Tx)} \varphi(t)dt, \int_0^{\delta(y,Sy)} \varphi(t)dt\right) \tag{2.8}$$

for each $x, y \in X$ *with some* $\alpha \in A$*, where* $\varphi : [0,+\infty) \to [0,+\infty)$ *is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of* $[0,+\infty)$*, non-negative, and such that*

$$\text{for each } \epsilon > 0, \quad \int_0^{\epsilon} \varphi(t)dt > 0 \tag{2.9}$$

Then S *and* T *have a unique common fixed point* $z \in X$*.*

Proof. For each integer $n \ge 0$, we define

$$\begin{aligned} x_{2n+1} &= Tx_{2n} \\ x_{2n+2} &= Sx_{2n+1} \end{aligned}$$

Then from (2.8) we get,

$$\begin{aligned} \int_0^{\delta(x_1,x_2)} \varphi(t)dt &= \int_0^{\delta(Tx_0,Sx_1)} \varphi(t)dt \\ &\le \alpha\left(\int_0^{\delta(x_0,x_1)} \varphi(t)dt, \int_0^{\delta(x_0,Tx_0)} \varphi(t)dt, \int_0^{\delta(x_1,Sx_1)} \varphi(t)dt\right) \\ &\le \alpha\left(\int_0^{\delta(x_0,x_1)} \varphi(t)dt, \int_0^{\delta(x_0,x_1)} \varphi(t)dt, \int_0^{\delta(x_1,x_2)} \varphi(t)dt\right) \end{aligned}$$

Then by the axiom A_2 of function α,

$$\int_0^{\delta(x_1,x_2)} \varphi(t)dt \le k \int_0^{\delta(x_0,x_1)} \varphi(t)dt \tag{2.10}$$

for some $k \in [0,1)$. Similarly one can show that

$$\int_0^{\delta(x_2,x_3)} \varphi(t)dt \le k \int_0^{\delta(x_1,x_2)} \varphi(t)dt \tag{2.11}$$

for some $k \in [0,1)$. In general, for any $r \in N$ odd or even,

$$\int_0^{\delta(x_r,x_{r+1})} \varphi(t)dt \le k \int_0^{\delta(x_{r-1},x_r)} \varphi(t)dt \tag{2.12}$$

and so for any $n \in N$ odd or even, one can easily obtain that

$$\int_0^{\delta(x_n,x_{n+1})} \varphi(t)dt \le k^n \int_0^{\delta(x_0,x_1)} \varphi(t)dt \tag{2.13}$$

Then by the condition (i) of the theorem one obtains

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \le \int_0^{\delta(x_n,x_{n+1})}\varphi(t)dt \le k^n\int_0^{\delta(x_0,x_1)}\varphi(t)dt$$

Taking limit as $n\to\infty$, we get

$$\lim_n \int_0^{d(x_n,x_{n+1})}\varphi(t)dt = 0 \qquad \text{as } k\in[0,1)$$

which, from(2.9) implies that

$$\lim_n d(x_n,x_{n+1}) = 0 \tag{2.14}$$

We now show that $\{x_n\}$ is a Cauchy sequence with respect to (X,d). So for any integer $p>0$,

$$\begin{aligned}\int_0^{d(x_n,x_{n+p})}\varphi(t)dt &\le \int_0^{\delta(x_n,x_{n+p})}\varphi(t)dt\\ &\le \int_0^{\delta(x_n,x_{n+1})}\varphi(t)dt + \int_0^{\delta(x_{n+1},x_{n+2})}\varphi(t)dt\\ &\quad +\dots \int_0^{\delta(x_{n+p-1},x_{n+p})}\varphi(t)dt\\ &\le k^n\int_0^{\delta(x_0,x_1)}\varphi(t)dt + k^{n+1}\int_0^{\delta(x_0,x_1)}\varphi(t)dt\\ &\quad +\dots k^{n+p-1}\int_0^{\delta(x_0,x_1)}\varphi(t)dt\\ &\le \frac{k^n}{1-k}\int_0^{\delta(x_0,x_1)}\varphi(t)dt \to 0 \text{ as } n\to\infty\end{aligned}$$

since $k\in[0,1)$. Therefore, $\{x_n\}$ is Cauchy. Hence by completeness of X, $\{x_n\}$ converges to some $z\in X$, i.e. $d(x_n,z)\to 0$ as $n\to\infty$ for some $z\in X$. Since T is given to be continuous with the respect to d we have

$$0=\lim_n\int_0^{d(x_{2n+1},z)}\varphi(t)dt = \lim_n\int_0^{d(Tx_{2n},z)}\varphi(t)dt = \lim_n\int_0^{d(Tz,z)}\varphi(t)dt$$

So by (2.9) $d(Tz,z)=0$ i.e. $Tz=z$.
Now by (2.8)

$$\begin{aligned}\int_0^{\delta(z,Sz)}\varphi(t)dt &= \int_0^{\delta(Tz,Sz)}\varphi(t)dt\\ &\le \alpha\Big(\int_0^{\delta(z,z)}\varphi(t)dt, \int_0^{\delta(z,Tz)}\varphi(t)dt, \int_0^{\delta(z,Sz)}\varphi(t)dt\Big)\\ &\le \alpha\Big(0,0,\int_0^{\delta(z,Sz)}\varphi(t)dt\Big)\end{aligned}$$

Then by the axiom A_2 of function α,

$$\int_0^{\delta(z,Sz)}\varphi(t)dt \le k.0 = 0 \tag{2.15}$$

and so by (2.9) $Sz = z$. Thus z is a common fixed point of S and T. For the uniqueness, let $w(\neq z)$ be another common fixed point of S and T in X. Then by (2.8)

$$\begin{aligned}\int_0^{\delta(z,w)}\varphi(t)dt &= \int_0^{\delta(Tz,Sw)}\varphi(t)dt\\ &\leq \alpha(\int_0^{\delta(z,w)}\varphi(t)dt, \int_0^{\delta(z,Tz)}\varphi(t)dt, \int_0^{\delta(w,Sw)}\varphi(t)dt)\\ &\leq \alpha(\int_0^{\delta(z,w)}\varphi(t)dt, 0, 0)\\ &\leq k.0 = 0 \text{ as } \alpha \in A\end{aligned}$$

Then by (2.9) we have $\delta(z, w) = 0$ and so $z = w$. □

If $S = T$, then the Theorem 2.2 gives as follows.

Corollary 2.3. *Let X be a set with two metrics d and δ satisfying the following conditions:*
(i) $\int_0^{d(x,y)}\varphi(t)dt \leq \int_0^{\delta(x,y)}\varphi(t)dt$ *for all $x, y \in X$;*
(ii) *X is complete with respect to d;*
(iii) *T is a self-map on X such that T is continuous with respect to d and*

$$\int_0^{\delta(Tx,Ty)}\varphi(t)dt \leq \alpha\left(\int_0^{\delta(x,y)}\varphi(t)dt, \int_0^{\delta(x,Tx)}\varphi(t)dt, \int_0^{\delta(y,Ty)}\varphi(t)dt\right) \tag{2.16}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative, and such that

$$\text{for each } \epsilon > 0, \quad \int_0^{\epsilon}\varphi(t)dt > 0 \tag{2.17}$$

Then T has a unique fixed point $z \in X$.

We have another similar result if we omit the condition (ii) of corollary 2.3 and the continuity of T with respect to d is replaced by assuming the continuity at a point. Then we get the same conclusion under much less restricted condition.

Theorem 2.4. *Let X be a set with two metrics d and δ satisfying the following conditions:*
(i) $\int_0^{d(x,y)}\varphi(t)dt \leq \int_0^{\delta(x,y)}\varphi(t)dt$ *for all $x, y \in X$;*
(ii) *T is a self-map on X such that T is continuous at $z \in X$ with respect to d and*

$$\int_0^{\delta(Tx,Ty)}\varphi(t)dt \leq \alpha\left(\int_0^{\delta(x,y)}\varphi(t)dt, \int_0^{\delta(x,Tx)}\varphi(t)dt, \int_0^{\delta(y,Ty)}\varphi(t)dt\right) \tag{2.18}$$

for each $x, y \in X$ with some $\alpha \in A$, where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,+\infty)$, non-negative, and such that

$$\text{for each } \epsilon > 0, \quad \int_0^{\epsilon}\varphi(t)dt > 0 \tag{2.19}$$

(iii) *There exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n x_0\}$ has a subsequence $\{T^{n_i} x_0\}$ converging to z in (X, d).*
Then T has a unique fixed point $z \in X$.

Proof. Considering the sequence $\{x_n\}$ as defined by $x_{n+1} = Tx_n$ for $n \geq 0$ i.e. $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$,..., $x_n = T^n x_0$ and proceeding as in the proof of theorem 2.2 we can easily arrive at a conclusion that the sequence is Cauchy with respect to d. Since the subsequence $\{x_{n_i}\}$ of the Cauchy sequence $\{x_n\}$ converges to z, therefore $\{x_n\}$ converges to z in X with respect to d i.e. $\lim\limits_{n\to\infty} x_n = z$. Since T is given to be continuous at z with the respect to d we have

$$0 = \lim_n \int_0^{d(x_{n+1},z)} \varphi(t)dt = \lim_n \int_0^{d(Tx_n,z)} \varphi(t)dt = \lim_n \int_0^{d(Tz,z)} \varphi(t)dt$$

So by (2.9) $d(Tz, z) = 0$ i.e. $Tz = z$. Thus T has a fixed point. Uniqueness of z is also very clear. □

Remark 2.5. On setting $\varphi(t) = 1$ over $[0, +\infty)$ in each results, the contractive condition of integral type transforms into a general contractive condition not involving integrals.

3. Example and application

Let $X = \{0, 1, 2, 3, 4\}$ and d be the usual metric of reals. Let $T : X \to X$ be given by

$$\begin{aligned} Tx &= 2, \text{if } x = 0 \\ &= 1, \text{otherwise} \end{aligned}$$

Again let $\varphi : R_+ \to R_+$ be given by $\varphi(t) = 1$ for all $t \in R_+$.
Then $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lesbesgue-integrable mapping which is summable on each compact subset of $[0, +\infty)$, non-negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$.
Now as we know from Example 1.3, a self-map T satisfying

$$d(Tx, Ty) \leq \beta \max \{d(Tx, x) + d(Ty, y), d(Ty, y) + d(x, y), d(Tx, x) + d(x, y)\}$$

for all $x, y \in X$ and some $\beta \in \left[0, \frac{1}{2}\right)$, is an A-contraction; we have

$$\begin{aligned} \int_0^{d(Tx,Ty)} \varphi(t)dt &\leq \alpha \left(\int_0^{d(x,y)} \varphi(t)dt, \int_0^{d(x,Tx)} \varphi(t)dt, \int_0^{d(y,Ty)} \varphi(t)dt \right) \\ &= \beta \max \left\{ \int_0^{d(Tx,x)+d(x,y)} \varphi(t)dt, \int_0^{d(Tx,x)+d(Ty,y)} \varphi(t)dt, \right. \\ &\quad \left. \int_0^{d(Ty,y)+d(x,y)} \varphi(t)dt \right\} \end{aligned}$$

which is satisfied for all $x, y \in X$ and some $\beta \in [0, \frac{1}{2})$ (see Theorem 2, Akram et al.[2]).
So all the axioms of Theorem 2.1 are satisfied and 1, is of course a unique fixed point of T.
We also can show the clear distinction between our result and that of Branciari (contractive condition 1.2) and that of Rhoades (contractive condition 1.3)
Let us take $x = 0$, $y = 1$. Then from condition 1.2, we have

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt \quad \text{implies } c \geq 1$$

which is not true. So T does not satisfy the condition 1.2 of Branciari.
Again for same $x, y \in X$,

$$\begin{aligned} 1 = \int_0^{d(Tx,Ty)} \varphi(t)dt &\leq c \int_0^{\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{[d(x,Ty)+d(y,Tx)]}{2}\}} \varphi(t)dt \\ &= c \max\{1, 2, 0, 1\} \end{aligned}$$

which implies $c \geq \frac{1}{2}$. Now if we take $0 < c < \frac{1}{2}$, the condition 1.3 of Rhoades does not satisfy.

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