Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 5 (2012), 44–55
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Existence of positive solutions of singular p-Laplacian equations in a ball

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This paper is dedicated to Professor Ljubomir Ćirić

Communicated by Professor V. Berinde

Abstract

In this paper, we investigate singular p-Laplacian equations of the form $\triangle_p u + f(x, \nabla u)u^{-\lambda} = 0$ with zero Dirichlet boundary condition in a ball $B \subset \mathbf{R}^N$, where $p > 1, \lambda > 0$, and give a sufficient condition for the equation to have a positive solution, by means of a supersolution and a subsolution.

Keywords: Positive solution; singular equation; supersolution and subsolution.
2010 MSC: Primary 58J10; Secondary 35N20.

1. Introduction

We shall establish the results on the existence of positive solutions of singular p-Laplacian equations

$$\triangle_p u + f(x, \nabla u)u^{-\lambda} = 0, \quad x \in B \subset \mathbf{R}^N, \tag{1.1}$$

$$u = 0, \quad x \in \partial B, \tag{1.2}$$

where $p > 1$,$\triangle_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $x = (x_1, x_2, \cdots, x_N)$, ∇ is the gradient operator, B is an open ball centered at the origin of $\mathbf{R}^N$, ∂B is the boundary of B, $\lambda > 0$ is a constant, and $f(x, u)$ is locally *Hölder* continuous with exponent $\theta \in (0, 1)$.

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Received 2011-2-5

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction-diffusion theory [19], non-Newtonian fluid theory [2, 25], non-Newtonian filtration [18] and the turbulent flow of a gas in porous medium [8]. In the non-Newtonian fluid theory, the quantity p is characteristic of the medium.Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids.

During the past three decades, singular elliptic equations have been paid much attention by many mathematicians. In particular, the existence and the uniqueness of positive solutions of the following singular elliptic boundary value problems

$$\begin{cases} -\Delta u = \eta(x)u^{-\lambda}, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\eta(x) \geq 0$, in Ω ,$\lambda > 0$, have been studied widely, see for instance [9, 21, 17] and references therein.

In [4], the authors studied general singular elliptic equation of the following

$$\begin{cases} -\Delta u + h(u)\frac{|\nabla u|^2}{u^\gamma} = f, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbf{R}^N$, $\gamma > 0$ and f is a function which is strictly positive on every compactly contained subset of Ω. They prove that the condition $\gamma < 2$ is necessary and sufficient for the existence of solutions in $H_0^1(\Omega)$ for every sufficiently regular f as above.

Recently, Ahmed Mohammed [1] studied of the existence of the positive solution of the equation

$$\begin{cases} -\Delta_p u = \lambda f(x, u), & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a $C^{1,\varpi}$ bounded domain, for some $0 < \varpi < 1$, $f : \Omega \times (0, \infty) \to [0, \infty)$ is a suitable function and allowed to be singular, $\lambda > 0$.

Yao and Zhou [20] shows the existence of positive solutions for the one-dimensional singular p-Laplacian

$$\begin{cases} [\Phi_p(\psi')]' - \lambda\frac{|\psi'|^p}{\psi} + f(t) = 0, & 0 < t < 1, \\ \psi(1) = \psi(0) = \psi'(1) = \psi'(0) = 0 \end{cases}$$

where $\Phi_p(s) = |s|^{p-2}s, p \geq 2, \lambda > 0, f(t) \in C[0, 1]$ and $f(t) > 0$ on [0,1].

For the other results of singular elliptic equations, see [13, 14, 16, 22, 23, 24] and the references therein.

Motivated by the results of the above cited papers, we shall attempt to treat such equation (1.1)-(1.2), the results of the semilinear equations are extending the quasilinear ones. We can find the related results for $p = 2$ in [10]. The main differences between $p = 2$ and $p \neq 2$ are known in [5, 6]. When $p \neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p = 2$ [3] seem to be lost or at least difficult to verify. The main differences between $p = 2$ and $p \neq 2$ can be found in [5, 6, 26].

This work is organized as follows: In Section 2, we give Several results and lemmas. In Section 3,we give our main results and its proof.

2. Several Results and Lemmas

Before we prove the main results, we need the following lemmas. For (1.1)-(1.2), the following hypotheses on f are adopted.

(B_1).$f : B \times \mathbf{R}^N \to (0,\infty)$ is locally *Hölder* continuous with exponent $\theta \in (0,1)$, and $f(x,p)$ is continuously differentiable in p. For every compact region $\Omega \subset B$, there exists an ordinary number ρ_Ω, such that $|f(x,t)| \le \rho_\Omega(1+|t|^p),\ x \in \Omega,\ t \in \mathbf{R}^N$.

Lemma 2.1. *[11] Suppose that a function f satisfies (B_1) , and that there exist a supersolution v and subsolution w of Eq. (1.1)-(1.2) such that $0 < w(x) < v(x); x \in B$; then Eq. (1.1)-(1.2) has a solution u and $w(x) < u(x) < v(x), x \in B$.*

To establish the supersolution and subsolution, we firstly consider that the function f is radially symmetric, that is $f = F(|x|, |\nabla u|)$. Thus, we introduce the following radial problem

$$(r^{N-1}|w'|^{p-2}w')' + r^{N-1}F(r,|w'|)w^{-\lambda} = 0, \quad 0 < r < 1, \tag{2.1}$$

$$w'(0) = 0, \qquad w(1) = 0. \tag{2.2}$$

We assume that the function F satisfies the following hypotheses.

(A_1).$F : [0,1) \times [0,\infty) \to (0,\infty)$ is continuous, $F(t,z)$ is continuously differentiable in $z \ge 0$. For each fixed $t \in [0,1)$,$F(t,z)$ is strictly increasing in $z \ge 0$;

(A_2). There exists a positive constant $M > 0$ satisfying

$$\int_0^1 (\int_0^s (\frac{t}{s})^{N-1}F(t,d)dt)^{\frac{1}{p-1}}ds \le M,$$

which holds uniformly for every $d \ge 0$.

(A_3). For all $\epsilon > 0$, there exists $\delta > 0$, such that $1-\delta < r < 1$, the following inequality

$$\int_r^1 (\int_0^s (\frac{t}{s})^{N-1}F(t,d)dt)^{\frac{1}{p-1}}ds \le \epsilon,$$

holds uniformly for every $d \ge 0$.

We study the existence of positive solutions of boundary value problems(2.1)-(2.2) by the shooting method. We consider that the unique positive solution of the initial value problem

$$(r^{N-1}|w'|^{p-2}w')' + r^{N-1}F(r,|w'|)w^{-\lambda} = 0, \tag{2.3}$$

$$w(0) = \alpha, \quad w'(0) = 0, \tag{2.4}$$

where $\alpha > 0$ is a parameter. Condition (A_1) implies that problems (2.3)-(2.4) have a unique positive solution $w_\alpha(r) \in C^1[0,T_\alpha) \cap C[0,T_\alpha], r \in [0,T_\alpha)$, where $[0,T_\alpha)$ is the maximal existence interval in $w_\alpha(r)$ (see [15]). Clearly, the value of T_α lies in $0 < T_\alpha \le 1$. If $T_\alpha < 1$, then $w_\alpha(r) > 0, 0 \le r < T_\alpha; w_\alpha(T_\alpha) = 0$. Therefore, $w_\alpha(r)$ depends continuously on its initial value α.

Lemma 2.2. *Suppose that F satisfies $(A_1)-(A_3)$, let α and β be positive numbers satisfying $\alpha > \beta$. If $w_\beta(r)$ exists on $[0,T), (0 \le T < 1)$, then $w_\alpha(r)$ also exists on [0,T) and satisfies*

$$w_\alpha(r) > w_\beta(r), \quad r \in [0,T), \tag{2.5}$$

$$w'_\alpha(r) > w'_\beta(r), \quad r \in [0,T). \tag{2.6}$$

Proof. We prove that lemma 2.2 in three steps.

Step 1. Assume that $w_\alpha(r)$ and $w_\beta(r)$ are defined on [0,T). Then

$$w'_\alpha(r) > w'_\beta(r), \quad r \in [0, T).$$

By (2.3)-(2.4) and (A_1), we obtain

$$|w'_\alpha(r)|^{p-2} w'_\alpha(r) = -\int_0^r (\frac{s}{r})^{N-1} F(s, |w'_\alpha(s)|) w_\alpha^{-\lambda}(s) ds < 0.$$

Therefore $w_\alpha(r)$ is strictly decreasing and $|w'_\alpha(r)| = -w'_\alpha(r)$. Hence,

$$\begin{aligned} &|w'_\alpha(r)|^{p-1} - |w'_\beta(r)|^{p-1} = \\ &\textstyle\int_0^r (\frac{s}{r})^{N-1} [F(s, |w'_\alpha(s)|) w_\alpha^{-\lambda}(s) - F(s, |w'_\beta(s)|) w_\beta^{-\lambda}(s)] ds, \quad r \in [0, T). \end{aligned} \tag{2.7}$$

Choose a positive number γ such that $\beta < \gamma < \alpha$. Since $w_\alpha(r)$ is continuous and $w_\alpha(0) = \alpha,\ r \in [0, T)$, there exists $r_0 > 0 (0 < r_0 < T)$ satisfying $w_\alpha(r) > \gamma,\ 0 \le r \le r_0$,and

$$\begin{aligned} &F(s, |w'_\alpha(s)|) w_\alpha^{-\lambda}(s) - F(s, |w'_\beta(s)|) w_\beta^{-\lambda}(s) \\ &< F(s, |w'_\alpha(s)|) \gamma^{-\lambda} - F(s, 0) \beta^{-\lambda}, \quad 0 \le r \le r_0 \end{aligned} \tag{2.8}$$

Let $F(0,0)\beta^{-\lambda} - F(0,0)\gamma^{-\lambda} = a > 0$. Since $[F(s, |w'_\alpha(s)|) w_\alpha^{-\lambda}(s)$ is continuous and $w'_\alpha(0) = w'_\beta(0) = 0$, there exists $\delta \in [0, r_0)$ such that

$$F(s, |w'_\alpha(r)|) \gamma^{-\lambda} < F(0, 0) \gamma^{-\lambda} + \frac{a}{2},$$

$$F(r, 0) \beta^{-\lambda} > F(0, 0) \beta^{-\lambda} - \frac{a}{2},$$

where $0 < r < \delta$. From (2.8) and the above equalities, we get

$$F(s, |w'_\alpha(s)|) w_\alpha^{-\lambda}(s) - F(s, |w'_\beta(s)|) w_\beta^{-\lambda}(s) < 0, \quad 0 < r < \delta \le T. \tag{2.9}$$

It also follows (2.7) and (2.9) that $|w'_\alpha(r)|^{p-1} < |w'_\beta(r)|^{p-1}, \quad 0 < r < \delta \le T$. It implies that $w'_\alpha(r) > w'_\beta(r), \ \ 0 < r < \delta \le T$. Now we prove that $\delta = T$.

If $\delta < T$, we find that $\delta_1 : \delta < \delta_1 < T$, such that

$$w'_\alpha(r) > w'_\beta(r), \ \ 0 < r < \delta_1, \ \ w'_\alpha(\delta_1) = w'_\beta(\delta_1).$$

Note that

$$w_\alpha(0) - w_\beta(0) = \alpha - \beta, \ w'_\alpha(r) \le 0, \ w'_\beta(r) \le 0, \ 0 < r < \delta_1.$$

Thus,

$$w_\alpha(r) > w_\beta(r), \quad |w'_\alpha(r)| < |w'_\beta(r)|, \ \ 0 < r < \delta_1.$$

Hence, we have

$$F(r, |w'_\alpha(r)|) w_\alpha^{-\lambda}(r) < F(r, |w'_\beta(r)|) w_\beta^{-\lambda}(r), \ \ 0 < r < \delta_1,$$

and

$$\begin{aligned} &0 = |w'_\alpha(\delta_1)|^{p-1} - |w'_\beta(\delta_1)|^{p-1} = \\ &\textstyle\int_0^{\delta_1} (\frac{s}{\delta_1})^{N-1} [F(s, |w'_\alpha(s)|) w_\alpha^{-\lambda}(s) - F(s, |w'_\beta(s)|) w_\beta^{-\lambda}(s)] ds < 0. \end{aligned}$$

This contradiction proves that (2.6) holds.

Step 2. If $w_\beta(r)$ exists on $[0,T)$, then $w_\alpha(r)$ also exists on [0,T).

In fact, we assume that the existence interval of $w_\alpha(r)$ is less than [0,T). Since $w_\alpha(r) > w_\beta(r)$ near the origin, the curve $w_\alpha(r)$ is sure to intersect the curve $w_\beta(r)$. Suppose that the first intersection point is $t = \tau < T$. Then we have

$$w_\alpha(r) > w_\beta(r),\ \ 0 \le r < \tau; \quad w_\alpha(\tau) = w_\beta(\tau).$$

Using (2.3),(2.4),condition(A_1) and the conclusion of step1, we get

$$\begin{aligned}&w_\alpha(\tau) - w_\beta(\tau) + \beta - \alpha \\ &= -\int_0^\tau(\int_0^s(\tfrac{t}{s})^{N-1}F(t,|w'_\alpha(t)|)w_\alpha^{-\lambda}(t)dt)^{\frac{1}{p-1}}ds \\ &-\int_0^\tau(\int_0^s(\tfrac{t}{s})^{N-1}F(t,|w'_\beta(t)|)w_\beta^{-\lambda}(t)dt)^{\frac{1}{p-1}}ds \ge 0.\end{aligned}$$

Thus, $w_\alpha(\tau) - w_\beta(\tau) \ge \alpha - \beta > 0$. This contradiction proves that $w_\alpha(r)$ also exists on [0,T).

Step 3. We prove (2.5) holds. In fact, making use of $w_\alpha(0) - w_\beta(0) = \alpha - \beta > 0$ and the above conclusions, we can prove immediately that (2.5)holds. Lemma 2.2 is proved. □

Lemma 2.3. *Under the assumptions* $(A_1) - (A_3)$*, the boundary value problems (2.1)-(2.2) have a unique positive solution* $w \in C^2([0,1)) \cap C((0,1])$.

Proof. Define the subsets $\bar{S}, \underline{S} \subset (0,\infty)$, respectively, by

$\bar{S}$=$\{\alpha > 0 | w_\alpha(r)$ exists on $[0,1)$ and satisfies $w_\alpha(1) > 0\}$;

$\underline{S}$=$\{\alpha > 0 | w_\alpha(r)$ vanishes before $r = 1\}$.

It follows from Lemma 2.2 that for all $\alpha \in \bar{S}$ and for all $\beta \in \underline{S}$, $\alpha > \beta$. Thus, $\bar{S} \cap \underline{S} = \oslash$. The following results (i)-(v) are valid.

(i). $\bar{S}$ is not empty.

Choosing arbitrarily a positive number α_1 such that $\frac{\alpha_1}{2} > M > 1$. Thus, by condition $(A_2), \alpha_1$ satisfies

$$\begin{aligned}&\int_0^1(\int_0^s(\tfrac{t}{s})^{N-1}F(t,d)(\tfrac{\alpha_1}{2})^{-\lambda}dt)^{\frac{1}{p-1}}ds \\ &< \int_0^1(\int_0^s(\tfrac{t}{s})^{N-1}F(t,d)dt)^{\frac{1}{p-1}}ds < M < \tfrac{\alpha_1}{2},\end{aligned} \tag{2.10}$$

which holds uniformly for every $d \ge 0$. We claim that $w_{\alpha_1}(r) > \frac{\alpha_1}{2}$, for $r \in [0,1)$. In fact, if this is not true, then there exists $r_1 \in (0,1)$ such that

$$w_{\alpha_1}(r) > \frac{\alpha_1}{2},\ \ r \in [0,r_1); \quad w_{\alpha_1}(r_1) = \frac{\alpha_1}{2}. \tag{2.11}$$

Making use of (2.3), (2.4), we get

$$w_{\alpha_1}(r_1) - \alpha_1 + \int_0^{r_1}(\int_0^s(\frac{t}{s})^{N-1}F(t,|w'_{\alpha_1}(t)|)w_{\alpha_1}^{-\lambda}(t)dt)^{\frac{1}{p-1}}ds = 0.$$

Let $d_1 = \max_{0\le t\le r_1}|w'_{\alpha_1}(t)|$. Eqs. (2.10),(2.11) and condition (A_1) can be applied to get

$$\begin{aligned}&\tfrac{\alpha_1}{2} = \int_0^{r_1}(\int_0^s(\tfrac{t}{s})^{N-1}F(t,|w'_{\alpha_1}(t)|)w_{\alpha_1}^{-\lambda}(t)dt)^{\frac{1}{p-1}}ds \\ &\le \int_0^{r_1}(\int_0^s(\tfrac{t}{s})^{N-1}F(t,d_1)(\tfrac{\alpha_1}{2})^{-\lambda}dt)^{\frac{1}{p-1}}ds \\ &\le \int_0^1(\int_0^s(\tfrac{t}{s})^{N-1}F(t,d_1|)(\tfrac{\alpha_1}{2})^{-\lambda}dt)^{\frac{1}{p-1}}ds < \tfrac{\alpha_1}{2}.\end{aligned}$$

This contradiction implies $w_{\alpha_1}(r) > \frac{\alpha_1}{2}, r \in [0,1)$. Thus $\alpha_1 \in \bar{S}$,i.e.$\bar{S}$ is not empty.

(ii). $\underline{S}$ is not empty.

Let

$$\int_0^{\frac{1}{2}} (\int_0^s (\frac{t}{s})^{N-1} F(t,0)dt)^{\frac{1}{p-1}} ds = k.$$

Choose arbitrarily a positive number α' such that $\alpha' <\min\{k, 1\}$, we have

$$\int_0^{\frac{1}{2}} (\int_0^s (\frac{t}{s})^{N-1} F(t,0)(\alpha')^{-\lambda} dt)^{\frac{1}{p-1}} ds > k > \alpha'. \tag{2.12}$$

Then for each α',$w_{\alpha'}(r)$ must vanish before $r = \frac{1}{2}$. In fact, if $w_{\alpha'}(r)$ can be prolonged to $r = \frac{1}{2}$ and $w_{\alpha'}(\frac{1}{2}) > 0$.Since$w_{\alpha'}(r) \le \alpha'$, $r \in [0, \frac{1}{2}]$, by (2.3),(2.4) and (2.12), we obtain

$$\begin{aligned}
&w_{\alpha'}(\tfrac{1}{2}) = \alpha' - \textstyle\int_0^{\frac{1}{2}} (\int_0^s (\frac{t}{s})^{N-1} F(t, |w'_{\alpha'}(t)|) w_{\alpha'}^{-\lambda}(t)dt)^{\frac{1}{p-1}} ds \\
&\le \alpha' - \textstyle\int_0^{\frac{1}{2}} (\int_0^s (\frac{t}{s})^{N-1} F(t,0)(\alpha')^{-\lambda} dt)^{\frac{1}{p-1}} ds \\
&< \alpha' - \alpha' = 0.
\end{aligned}$$

This contradiction implies $\alpha' \in \underline{S}$.

(iii). inf$\bar{S}$ does not belong to $\bar{S}$.

Put $\alpha_* = \inf\bar{S} \in \bar{S}$, it is clear that $\alpha_* \in (0,\infty)$. Suppose that $\alpha_* \in \bar{S}$, then $w_{\alpha_*}(1) = l > 0$. Using condition (A_3), for $(\frac{l}{2})^{\frac{p+\lambda-1}{p-1}} > 0$, there exists $\delta_1 > 0$ and choosing $r_1 \in (0,1)$ sufficiently close to 1, satisfying $1 - \delta_1 < r_1 < 1$, so that

$$\int_{r_1}^1 (\int_0^s (\frac{t}{s})^{N-1} F(t,d)(\frac{l}{2})^{-\lambda}) dt)^{\frac{1}{p-1}} ds < (\frac{l}{2})^{\frac{p+\lambda-1}{p-1}} \cdot (\frac{l}{2})^{\frac{-\lambda}{p-1}} = \frac{l}{2} \tag{2.13}$$

which hold uniformly for every $d \ge 0$. Since $w'_{\alpha_*}(r) < 0$, $(0 < r < 1)$, we get that $w_{\alpha_*}(r_1) > l$. Noting the continuous dependence of solutions of (2.3),(2.4) on initial data, for all $\alpha_0 \in (0,\alpha_*)$ sufficiently close to α_*, $w_{\alpha_0}(r)$ are define on $[0,r_1]$ and satisfy

$$w_{\alpha_0}(r_1) > l. \tag{2.14}$$

Now we claim that such a $w_{\alpha_0}(r)$ satisfies $w_{\alpha_0}(r) > \frac{l}{2}$ on its interval of existence and, consequently, can be extended to [0,1). In fact if this is not true, then, there is $r_2 \in (r_1, 1)$ such that

$$\begin{aligned}
&\tfrac{l}{2} = w_{\alpha_0}(r_2) = w_{\alpha_0}(r_1) - \textstyle\int_{r_1}^{r_2} (\int_0^s (\frac{t}{s})^{N-1} F(t, |w'_{\alpha_0}(t)|) w_{\alpha_0}^{-\lambda}(t)dt)^{\frac{1}{p-1}} ds \\
&\ge l - \textstyle\int_{r_1}^{r_2} (\int_0^s (\frac{t}{s})^{N-1} F(t,d_0)(\frac{l}{2})^{-\lambda} dt)^{\frac{1}{p-1}} ds \\
&> l - \textstyle\int_{r_1}^{1} (\int_0^s (\frac{t}{s})^{N-1} F(t,d_0)(\frac{l}{2})^{-\lambda} dt)^{\frac{1}{p-1}} ds \\
&> l - \tfrac{l}{2} = \tfrac{l}{2},
\end{aligned}$$

where $d_0 = \max_{r_1 \le t \le r_2} |w'_{\alpha_0}(t)|$. Therefore, $\alpha_0 \in \bar{S}$ and $\alpha_0 < \alpha_*$. This contradicts the definition $\alpha_* = \inf\bar{S}$. Thus, inf$\bar{S}$ does not belong to $\bar{S}$.

(iv). sup$\underline{S}$ does not belong to $\underline{S}$.

Suppose that $\alpha^* = \sup\underline{S} \in \underline{S}$. Let r_1 be a point in $(0,1)$ such that $w_{\alpha^*}(r_1) = 0$.Choose $T \in (r_1, 1)$ arbitrarily and let it be fixed. Note that

$$\int_{r_1}^T (\int_0^s (\frac{t}{s})^{N-1} F(t,0)dt)^{\frac{1}{p-1}} ds > 0,$$

there exists $\epsilon > 0$ sufficiently small such that

$$\int_{r_1}^{T}(\int_0^s(\frac{t}{s})^{N-1}F(t,0)(\epsilon)^{-\lambda}dt)^{\frac{1}{p-1}}ds > \epsilon. \tag{2.15}$$

By using Lemma 2.2 and the continuous dependence of solutions on initial data, we find that $w_\beta(r)$ exists on $[0, r_1]$and satisfies $0 < w_\beta(r_1) < \epsilon$ for all $\beta > \alpha^*$ sufficiently close to α^*. Now, we assert that such a $w_\beta(r)$ vanishes before $t = T$. Assume on the contrary that $w_\beta(r)$ exists on $[0, T]$ and remains positive. Then we obtain that $0 < w_\beta(r) < \epsilon$, $r_1 \le r \le T$, and integrating (2.3) twice and using (2.15), (A_1), we obtain

$$\begin{aligned}&w_\beta(T) = w_\beta(r_1) - \textstyle\int_{r_1}^{T}(\int_0^s(\frac{t}{s})^{N-1}F(t,|w'_\beta(t)|)w_\beta^{-\lambda}(t)dt)^{\frac{1}{p-1}}ds\\ &\le \epsilon - \textstyle\int_{r_1}^{T}(\int_0^s(\frac{t}{s})^{N-1}F(t,0)\epsilon^{-\lambda}dt)^{\frac{1}{p-1}}ds\\ &< \epsilon - \epsilon = 0.\end{aligned}$$

This contradiction shows that a β is contained in $\underline{S}$. However, this contradicts the definition of $\alpha^* = \sup\underline{S}$. Thus, $\sup\underline{S}$ does not belong to $\underline{S}$.

(v). $\alpha_0 = \inf\bar{S} = \sup\underline{S}$.

It is obvious that for all $\alpha \in \bar{S}$ and for all $\beta \in \underline{S}$, then $\beta < \alpha$. Thus, $\inf\bar{S} \ge \sup\underline{S}$. Now, we claim that $\inf\bar{S} > \sup\underline{S}$ does not hold. In fact, if this is no true, denote $\alpha = \inf\bar{S}, \beta = \sup\underline{S}$, then $\alpha > \beta$. We see clearly that α, β belong neither to $\bar{S}$ or $\underline{S}$. We find that $w_\alpha(r), w_\beta(r)$ exist in $[0, 1)$,$w_\alpha(1) = w_\beta(1) = 0$ from the definition of $\bar{S}$ and $\underline{S}$. Since $\alpha > \beta$, by Lemma 2.2 we have

$$w'_\alpha(r) > w'_\beta(r), \quad r \in (0, 1).$$

Thus, $w_\alpha(r) - w_\beta(r)$ is strictly increasing in $[0, 1)$ and we have

$$w_\alpha(1) - w_\beta(1) > w_\alpha(0) - w_\beta(0) = \alpha - \beta > 0.$$

This contradiction proves that $\alpha \ge \beta$ does not hold. Thus, $\alpha_0 = \inf\bar{S} = \sup\underline{S}$. It follows that $w_{\alpha_0}(r)$ is unique positive solution of class $C^1[0, 1) \cap C[0, 1]$ of problems (2.1)-(2.2). This completes the proof of Lemma 2.3. □

3. The Main Result

We consider the singular elliptic boundary value problems (1.1)-(1.2)under the following conditions:

(B_2). There exists functions $f^*, f_* : [0, 1) \times [0, \infty) \to (0, \infty), f^*, f_* \in C^\theta_{loc}([0, 1) \times ([0, \infty))$. Both $f^*(t, z), f_*(t, z)$ are continuously differentiable in z, strictly increasing in $z \ge 0$ for every fixed $t \in [0, 1)$ and satisfy

$$0 < f_*(|t|, |z|) \le f(x, p) \le f^*(|t|, |z|), \quad (x, p) \in B \times R^N. \tag{3.1}$$

From [21, 23], we get the following comparison principle which plays an important role in the proof of Theorem 3.2.

Lemma 3.1. *(Weak comparison principle) Let Ω be a bounded domain in $\mathbf{R}^N (N \ge 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0, \infty) \to (0, \infty)$ is continuous and nondecreasing. Let $u_1,\ u_2 \in W^{1,p}(\Omega)$ satisfy*

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla\psi dx + \int_\Omega \theta(u_1)\psi dx \le \int_\Omega |\nabla u_2|^{p-2}\nabla u_2 \nabla\psi dx + \int_\Omega \theta(u_2)\psi dx,$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$ *satisfy*

$$u_1 \leq u_2 \quad on \quad \partial\Omega,$$

implies that

$$u_1 \leq u_2 \quad in \quad \Omega.$$

The main results of this paper are as follows:

Theorem 3.2. *Suppose that conditions* (B_1) *and* (B_2) *hold,* f^* *and* f_* *satisfy the conditions* (A_2) *and* (A_3). *Then, there exists a positive solution* u *of class* $C^1[0,1) \cap C[0,1]$ *for singular elliptic boundary value problems (1.1)-(1.2).*

Proof. We consider the following boundary value problems:

$$\triangle_p u + f^*(|x|, |\nabla u|)u^{-\lambda} = 0, \quad x \in B, \tag{3.2}$$

$$u = 0, \quad x \in \partial B, \tag{3.3}$$

$$\triangle_p u + f_*(|x|, 0)u^{-\lambda} = 0, \quad x \in B, \tag{3.4}$$

$$u = 0, \quad x \in \partial B. \tag{3.5}$$

Applying Lemma 2.3 to these problems, we see that problems (3.2)-(3.3) and (3.4)-(3.5), respectively, have positive radial solutions $\bar{u}(|x|)$ and $\underline{u}(|x|)$ of class $C^2_{loc}(B) \cap C(\bar{B})$. Note that$f^*, f_* \in C^2_{loc}$. The regular theorem implies that $\bar{u}, \underline{u} \in C^{2+\theta}_{loc}(B) \cap C(\bar{B})$. It is obvious that $\bar{u}, \underline{u}$ are a supersolution and a subsolution respectively of the boundary value problems (1.1)-(1.2). We next prove that $\bar{u}(|x|) \geq \underline{u}(|x|)$, $x \in B$. Since $\bar{u} - \underline{u}$ satisfies

$$\triangle_p \bar{u} - \triangle_p \underline{u} + f^*(|x|, |\nabla \bar{u}|)(\bar{u})^{-\lambda} - f_*(|x|, 0)(\underline{u})^{-\lambda} = 0, \;\; x \in B, \tag{3.6}$$

$$\bar{u} - \underline{u} = 0, \quad x \in \partial B, \tag{3.7}$$

We can change (3.6) as follows:

$$\begin{aligned} &\triangle_p \bar{u} - \triangle_p \underline{u} + f^*(|x|, |\nabla \bar{u}|)(\bar{u})^{-\lambda} - \\ &f_*(|x|, 0)(\underline{u})^{-\lambda} + f_*(|x|, 0)(\bar{u})^{-\lambda} - f_*(|x|, 0)(\bar{u})^{-\lambda} = 0, \;\; x \in B. \end{aligned} \tag{3.8}$$

Condition (B2)and (3.8) can be applied to obtain

$$\triangle_p \bar{u} - \triangle_p \underline{u} + f_*(|x|, 0)(\bar{u})^{-\lambda} - f_*(|x|, 0)(\underline{u})^{-\lambda} \leq 0, \;\; x \in B, \tag{3.9}$$

Thus, we obtain

$$\triangle_p \bar{u} - \triangle_p \underline{u} + c(x)(\bar{u} - \underline{u}) \leq 0, \;\; x \in B, \tag{3.10}$$

$$c(x) = -\lambda f_*(|x|, 0) \int_0^1 (t\bar{u} + (1-t)\underline{u})^{-(\lambda+1)} dt \leq 0. \tag{3.11}$$

Making use of (3.10), (3.7) and Lemma 3.1, we have

$$\bar{u}(|x|) \geq \underline{u}(|x|), \;\; x \in B. \tag{3.12}$$

Let $B_n = \{x \in R^N | |x| < 1 - \frac{1}{n}\}$ for $n = 2, 3, \ldots$, and h be a function of class $C^{2+\theta}_{loc}(B) \cap C(\bar{B})$ satisfying $\underline{u}(|x|) \leq h(x) \leq \bar{u}(|x|)$ in B. Since $\bar{u}(|x|)$ and $\underline{u}(|x|)$ are a supersolution and a subsolution respectively, of boundary value problems (1.1)-(1.2), we see clearly that $\bar{u}, \underline{u}$ are also a supersolution and a subsolution of the following boundary value problems

$$\triangle_p u + f(x, \nabla u))u^{-\lambda} = 0, \quad x \in B_n, \tag{3.13}$$

$$u = h(x), \quad x \in \partial B_n, \tag{3.14}$$

for each $n \geq 2$, and satisfy

$$\bar{u}(|x|) \geq \underline{u}(|x|), \ \ x \in \bar{B}_n.$$

Using condition (B_1) and Lemma 2.1, we find that there exists a positive $u_n \in C^{1+\theta}_{loc}(\bar{B}_n)(n \geq 2)$ for boundary value problems (3.13)-(3.14) and satisfy

$$\bar{u}(|x|) \geq u_n(x) \geq \underline{u}(|x|), \ \ x \in \bar{B}_n.$$

Now, we want to apply elliptic interior estimates together with a diagonal process to conclude: $\{u_k : k \geq 1\}$ has a subsequence $\{u_{k_i} : k_i \uparrow \infty\}$ such that $\{u_{k_i}\}$ converges to a function u in B (pointwise) and this convergence is in C^1 on every compact set in B. (Therefore, $u \in C^1$ and $\triangle_p u + f(x, \nabla u))u^{-\lambda} = 0$ in B with $u = 0$ on ∂B, and this concludes the proof.)

Step 1. On $B_2, \{u_k : k \geq 2\}$ is uniformly bounded by $\underline{u}(x)$ and $\overline{u}(x)$. Since both $\underline{u}(x)$ and $\overline{u}(x)$ are bounded functions on B_2, there exists $M > 0$ such that

$$\|u_k(x)\|_{L^\infty(B_2)} \leq M,$$

for all $k \geq 2$.

From (1.1), u_k satisfies

$$\int_{B_2} |\nabla u_k|^p \leq -\int_{B_2} f(x, \nabla u_k)(u_k)^{-\lambda} u_k.$$

Therefore,

$$\int_{B_2} |\nabla u_k|^p \leq M(\text{meas}B_2)^{1/q'} C_1 \|\nabla u_k\|_p.$$

Here$1/q' + 1/p = 1$, and C_1 is the Sobolev embedding constant. So, $\|u_k\|_{1,p} \leq C_2$. When $1 < m < N$, the embedding of $W^{1,p}_0(B_2)$ in $L^{Np/(N-p)}(B_2)$ implies that $u_k \in L^{Np/(N-p)}(B_2)$. Applying Theorem 7.1 in [26], Page 286-287, we obtain the estimate

$$\sup\{|u_k|; x \in B_2\} \leq C_3, \tag{3.15}$$

here $C_3 = C_3(\|\psi\|_0)$. If $p \geq N$, we get (3.15) from the Sobolev embedding theorem. Using Theorem 1.1 in[26], Page 251, we see that u_k belongs to $C^\alpha(\overline{B}_2)$ for some $0 < \alpha < 1$, and

$$\|u_k\|_{C^\alpha} \leq C_4,$$

here C_4 is determined by C_3. By Proposition 3.7 in [27], Page 806, we also know that u_k belongs to $C^{2,\alpha}(\overline{B_2})$ and

$$\|u_k\|_{C^{1,\alpha}} \leq C_5.$$

Here C_5 is determined by C_4.

From the arguments above we see that there exists $C > 0$ such that

$$\|u_k\|_{C^{1+\alpha}(B_1)} \leq C, \ \ \text{for all} \ \ k \geq 2.$$

Since the embedding $C^{1+\alpha}(B_1) \to C^1(B_1)$ is compact, there exists a sequence denoted by $\{u_{k_{1j}}\}_{j=1,2...}$ (where $k_{1j} \uparrow \infty$), which converges in $C^1(B_1)$. Let $u_1(x) = \lim_{j\to\infty} u_{k_{1j}}(x)$, for $x \in B_1$; then u_1 is a solution of (1.1) with $\underline{u}(x) \leq u_1 \leq \overline{u}(x)$.

Step 2. Repeat Step 1 up to the existence of the sequence $\{u_{k_{1j}}\}_{j=1,2...}$ to get a subsequence $\{u_{k_{2i}}\}_{i=1,2...}$ converging in $C^1(B_2)$ to a limit u_2. Then likewise u_2 is a solution of (2.4), (2.5) and $u_2|_{B_1} = u_1$. Repeat Step 1 again on B_3, ..., etc. In this way, we obtain a sequence $\{u_{k_{nj}}\}_{j=1,2...}$ which converges in $C^1(B_k)$ and is a subsequence of $\{u_{k_{(n-1)j}}\}_{j=1,2...}$. Let $u_k = \lim_{j\to\infty} u_{k_{nj}}$, then, u_k is a solution of (3.13), (3.14) in B_k and $u_k|_{B_{k-1}} = u_{k-1}$.

Step 3. By a diagonal process, $\{u_{k_{nn}}\}_{n=1,2...}$ is a subsequence of $\{u_{k_{nj}}\}_{j=1,2...}$ for every n. Thus, on B_k for each k we have

$$\lim_{n\to\infty} u_{k_{nn}} = u_k.$$

So, if we define $u(x) = \lim_{n\to\infty} u_{k_{nn}}(x)$, then $u(x)$ satisfies

$$\triangle_p u + f(x, \nabla u))u^{-\lambda} = 0,$$

and $\underline{u} \le u(x) \le \overline{u}$ (since $\underline{u} \le u_k(x) \le \overline{u}$) for every k. This complete the proof of Theorem 3.1. □

Now, we give an example below to show the application of Theorem 3.2. The principal part of the equation below is p-Laplacian and the nonlinear function $f(x, z)$ has singularity at the boundary of the unit N-ball and isn't increasing in $z \ge 0$.

Example 3.3. Consider the singular boundary value problem

$$\begin{cases} (r^{N-1}|u'(r)|^{p-2}u'(r))' + \frac{r^{N-1}(2-\frac{3}{2}e^{-|u'(r)|^2}+e^{-2|u'(r)|^2})}{u(r)(1-r^2)^{\frac{1}{2}}} = 0, \quad r \in [0,1), \\ u(r) > 0, r \in [0,1), \\ u'(0) = 0, u(1) = 0, \end{cases} \tag{3.16}$$

where $p \ge 2$.
Here

$$f(r, z) = \frac{2 - \frac{3}{2}e^{-z^2} + e^{-2z^2}}{(1-r^2)^{\frac{1}{2}}} : [0, 1) \times [0, \infty) \to (0, \infty),$$

where $z = |u'(r)|$. We take

$$f_*(r, z) = \frac{2 - \frac{3}{2}e^{-z^2}}{(1-r^2)^{\frac{1}{2}}}, f^*(r, z) = \frac{2 - \frac{1}{2}e^{-z^2}}{(1-r^2)^{\frac{1}{2}}}.$$

It is easy to check that $f(r, z)$ isn't increasing in $z \ge 0$,condition(B_2) is satisfied and f^*, f_* satisfy the condition (A_1). Now,we check f^*, f_* also satisfy conditions (A_2) and (A_3).

(A_2)

$$\begin{aligned} &\int_0^1 (\int_0^s (\tfrac{t}{s})^{N-1} f_*(t, d)dt)^{\frac{1}{p-1}} ds \\ &= \int_0^1 (\int_0^s (\tfrac{t}{s})^{N-1} \tfrac{(2-\frac{3}{2}e^{-d^2})}{(1-t^2)^{\frac{1}{2}}} dt)^{\frac{1}{p-1}} ds \\ &\le \int_0^1 (\int_0^s \tfrac{2}{(1-t^2)^{\frac{1}{2}}} dt)^{\frac{1}{p-1}} ds \\ &\le (\int_0^1 \int_0^s \tfrac{2}{(1-t^2)^{\frac{1}{2}}} dtds)^{\frac{1}{1/(p-1)}} \\ &\le (\int_0^1 \tfrac{2s}{(1-s^2)^{\frac{1}{2}}} ds)^{\frac{1}{p-1}} = 2^{1/(p-1)}. \end{aligned}$$

From the above compute, we know that f_* satisfy the condition (A_2).By the same way,we can check that f^* satisfy the conditions (A_2).

(A_3) When $0 < r < 1$,similar to the above computation, we have

$$\begin{aligned}&\int_r^1(\int_0^s(\tfrac{t}{s})^{N-1}f_*(t,d)dt)^{\frac{1}{p-1}}ds\\&=\int_r^1(\int_0^s(\tfrac{t}{s})^{N-1}\tfrac{(2-\frac{3}{2}e^{-d^2})}{(1-t^2)^{\frac{1}{2}}}dt)^{\frac{1}{p-1}}ds\\&\le(\int_r^1\tfrac{2s}{(1-s^2)^{\frac{1}{2}}}ds)^{\frac{1}{p-1}}=2^{1/p-1}(1-r^2)^{\frac{1}{p-1}}\le\epsilon,\end{aligned}$$

when $r\ge(1-\frac{\epsilon^{p-1}}{2})^{1/2}$.Choosing $\delta=1-(1-\frac{\epsilon^{p-1}}{2})^{1/2}$,$f_*$ satisfy the condition (A_3).The same result can be attained by f^*.Therefore, by Theorem 3.2, problem (3.16) has a positive solution in the unit N-ball.

Acknowledgements:

This project was supported by the National Natural Science Foundation of China(No.11171092); and by the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No.08KJB110005).

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