Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 5 (2012), 27–33
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Solvability of multi-point boundary value problems on the half-line

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This paper is dedicated to Professor Ljubomir Ćirić

Communicated by Professor V. Berinde

Abstract

In this work, using the Leray-Schauder continuation principle, we study the existence of at least one solution to the quasilinear second-order multi-point boundary value problems on the half-line.

Keywords: Solvability, m-point boundary value problem, p-Laplacian, half-line
2010 MSC: Primary 34B10; Secondary 34B15, 34B40.

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1. Introduction

Boundary value problems on the half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and in various applications such as an unsteady flow of gas through a semi-infinite porous media, theory of drain flows and plasma physics. There have been many works concerning the existence of solutions for the boundary value problems on the half-line. We refer the reader to [1, 2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 19, 20, 22, 23] and the references therein.

Recently, Lian and Ge ([11]) studied the second-order three-point boundary value problem

$$\begin{aligned}
&x''(t) + g(t, x(t), x'(t)) = 0, \ \text{a.e. } t \in \mathbb{R}_+,\\
&x(0) = \alpha x(\eta), \ \ \lim_{t\to\infty} x'(t) = 0,
\end{aligned}$$

where $\mathbb{R}_+ = [0, \infty)$, $\alpha \neq 1$ and $\eta > 0$. The authors investigated the existence of at least one solution under the assumption that $g(t, \cdot, \cdot)$ and $tg(t, \cdot, \cdot)$ are Carathéodory with respect to $L^1(\mathbb{R}_+)$.

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Received 2011-4-7

More recently, Kosmatov ([10]) studied the second-order nonlinear differential equation

$$(q(t)y'(t))' = k(t, y(t), y'(t)), \text{ a.e. } t \in \mathbb{R}_+,$$

satisfying two sets of boundary conditions:

$$y'(0) = 0, \ \lim_{t\to\infty} y(t) = 0$$

and

$$y(0) = 0, \ \lim_{t\to\infty} y(t) = 0,$$

where $k : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory with respect to $L^1(\mathbb{R}_+)$, $\mathbb{R} = (-\infty, \infty)$, $q \in C(\mathbb{R}_+) \cap C^1(0, \infty)$, $1/q \in L^1(\mathbb{R}_+)$ and $q(t) > 0$ for all $t \in \mathbb{R}_+$. The author obtained the existence of at least one solution to the above problems using the Leray-Schauder continuation principle. In the end of the paper, the author pointed out that the assumption $q(0) > 0$ could be omitted, in which case one would have to work in a Banach space equipped with a weighted norm after the boundary conditions are adjusted accordingly.

Motivated by the above works ([10, 11]), we study the quasilinear second-order nonlinear differential equation

$$(w(t)\varphi_p(u'(t)))' + f(t, u(t), u'(t)) = 0, \ a.e. \ t \in \mathbb{R}_+, \tag{P}$$

satisfying the following four sets of boundary conditions:

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \ \lim_{t\to\infty} (\varphi_p^{-1}(w)u')(t) = 0, \tag{BC_1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i (\varphi_p^{-1}(w)u')(\xi_i), \ \lim_{t\to\infty} (\varphi_p^{-1}(w)u')(t) = 0, \tag{BC_2}$$

$$\lim_{t\to 0+} (\varphi_p^{-1}(w)u')(t) = 0, \ \lim_{t\to\infty} u(t) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{BC_3}$$

$$\lim_{t\to 0+} (\varphi_p^{-1}(w)u')(t) = 0, \ \lim_{t\to\infty} u(t) = \sum_{i=1}^{m-2} a_i (\varphi_p^{-1}(w)u')(\xi_i), \tag{BC_4}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\xi_i \in \mathbb{R}_+$ with $0 \le \xi_1 < \xi_2 < \cdots < \xi_{m-2}$, $a_i \in \mathbb{R}$ with $\sum_{i=1}^{m-2} a_i \neq 1$, $w \in C(\mathbb{R}_+, \mathbb{R})$ and $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f = f(t, u, v)$ is Lebesgue measurable in t for all $(u, v) \in \mathbb{R} \times \mathbb{R}$ and continuous in (u, v) for almost all $t \in \mathbb{R}_+$. We further assume the following conditions hold.

(F) There exist measurable functions α, β and γ such that

$$\alpha, \ \beta/w, \ \gamma \in L^1(\mathbb{R}_+)$$

and

$$|f(t, u, v)| \le \alpha(t)|u|^{p-1} + \beta(t)|v|^{p-1} + \gamma(t), \ a.e. \ t \in \mathbb{R}_+.$$

(W) $\varphi_p^{-1}(1/w) \in L^1(\mathbb{R}_+)$ and $Z_w = \{t \in \mathbb{R}_+ \mid w(t) = 0\}$ is a finite set.

By a solution to problem $(P), (BC_i)$, we understand a function $u \in C(\mathbb{R}_+) \cap C^1(\mathbb{R}_+ \setminus Z_w)$ with $w\varphi_p(u') \in AC(\mathbb{R}_+)$ satisfying $(P), (BC_i)$ $(i = 1, 2, 3, 4)$.

To the author's knowledge, the multi-point boundary value problems with sign-changing weight w have not been investigated until now. The purpose of this paper is to establish the existence of at least one solution to p-Laplacian boundary value problems $(P), (BC_i)$ $(i = 1, 2, 3, 4)$ with sign-changing weight w.

Since $\mathbb{R}_+$ is not compact, the related compactness principle on a bounded interval $[0,1]$ does not hold. In addition, solutions u of $(P),(BC_i)$ $(i=1,2,3,4)$ may not be in $C^1(\mathbb{R}_+)$ since w may have zeros in $\mathbb{R}_+$. In order to overcome these difficulties, a new Banach space equipped with a weighted norm is introduced, and then we can proceed with the Leray-Schauder continuation principle which was used in many works (see, e.g., [5, 7, 9, 10, 11, 17]) in order to prove the existence of a solution for the problems $(P),(BC_i)$ $(i=1,2,3,4)$.

The rest of this paper is organized as follows. In Section 2, a weighted Banach space and corresponding operators to problems $(P),(BC_i)$ $(i=1,2,3,4)$ are introduced, and lemmas are presented. In Section 3, our main results are given, and also an example to illustrate our results is presented.

2. Preliminaries

Let X be the Banach space

$$X=\{u\in C^1(\mathbb{R}_+\setminus Z_w)\mid u \text{ and } \varphi_p^{-1}(w)u' \text{ are continuous and bounded functions on } \mathbb{R}_+\}$$

equipped with norm

$$\|u\|=\|u\|_\infty+\|\varphi_p^{-1}(w)u'\|_\infty,$$

where $\|v\|_\infty=\sup_{t\in\mathbb{R}_+}|v(t)|$ and let Y be the Banach space $L^1(\mathbb{R}_+)$ equipped with norm

$$\|h\|_1=\int_0^\infty |h(s)|ds.$$

For convenience, we will use the following constants

$A=1-\sum_{i=1}^{m-2}a_i,$

$$B=|A|^{-1}\sum_{i=1}^{m-2}|a_i|\int_0^{\xi_i}\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds+\int_0^\infty\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds,$$

$$C=\sum_{i=1}^{m-2}|a_i|+\int_0^\infty\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds,$$

$$D=|A|^{-1}\sum_{i=1}^{m-2}|a_i|\int_{\xi_i}^\infty\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds+\int_0^\infty\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds.$$

For each $h\in Y$, we define, for $t\in\mathbb{R}_+$,

$$\begin{aligned}(T_1h)(t) &= A^{-1}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_p^{-1}\left(\frac{1}{w(s)}\int_s^\infty h(\tau)d\tau\right)ds\\ &\quad+\int_0^t\varphi_p^{-1}\left(\frac{1}{w(s)}\int_s^\infty h(\tau)d\tau\right)ds,\end{aligned}$$

$$\begin{aligned}(T_2h)(t) &= \sum_{i=1}^{m-2}a_i\varphi_p^{-1}\left(\int_{\xi_i}^\infty h(s)ds\right)\\ &\quad+\int_0^t\varphi_p^{-1}\left(\frac{1}{w(s)}\int_s^\infty h(\tau)d\tau\right)ds,\end{aligned}$$

$$\begin{aligned}(T_3h)(t) &= A^{-1}\sum_{i=1}^{m-2}a_i\int_{\xi_i}^\infty\varphi_p^{-1}\left(\frac{1}{w(s)}\int_0^s h(\tau)d\tau\right)ds\\ &\quad+\int_t^\infty\varphi_p^{-1}\left(\frac{1}{w(s)}\int_0^s h(\tau)d\tau\right)ds\end{aligned}$$

and

$$\begin{aligned}(T_4h)(t) &= -\sum_{i=1}^{m-2} a_i\varphi_p^{-1}\left(\int_0^{\xi_i} h(s)ds\right) \\ &\quad + \int_t^{\infty} \varphi_p^{-1}\left(\frac{1}{w(s)}\int_0^s h(\tau)d\tau\right)ds.\end{aligned}$$

Then $T_i : Y \to X$ is well defined and for each $h \in Y$, T_ih is the unique solution of the differential equation

$$(w(t)\varphi_p(u'(t)))' + h(t) = 0,\ a.e.\ t \in \mathbb{R}_+,$$

subject to the boundary conditions (BC_i) $(i = 1, 2, 3, 4)$.

Lemma 2.1. *Let $h \in Y$. Then T_1h satisfies*

$$\|T_1h\|_\infty \leq B\|h\|_1^{1/(p-1)} \tag{2.1}$$

and

$$\|\varphi_p^{-1}(w)(T_1h)'\|_\infty \leq \|h\|_1^{1/(p-1)}. \tag{2.2}$$

Proof. Let $h \in Y$. Then, for all $t \in \mathbb{R}_+$, one has

$$\begin{aligned}|T_1h(t)| &\leq \left(|A|^{-1}\sum_{i=1}^{m-2}|a_i|\int_0^{\xi_i}\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds\right)\left(\int_0^{\infty}|h(s)|ds\right)^{1/(p-1)} \\ &\quad + \left(\int_0^{t}\varphi_p^{-1}\left(\frac{1}{|w(s)|}\right)ds\right)\left(\int_0^{\infty}|h(s)|ds\right)^{1/(p-1)} \\ &\leq B\|h\|_1^{1/(p-1)}.\end{aligned}$$

Similarly, for all $t \in \mathbb{R}_+$, one has

$$\begin{aligned}|(\varphi_p^{-1}(w)(T_1h)')(t)| &= \left|\varphi_p^{-1}\left(\int_t^{\infty} h(s)ds\right)\right| \\ &\leq \|h\|_1^{1/(p-1)}.\end{aligned}$$

Thus the proof is complete. □

The following lemmas can be proved by the similar manner and so we omit the proofs.

Lemma 2.2. *Let $h \in Y$. Then, for each $i = 2, 4$, T_ih satisfies*

$$\|T_ih\|_\infty \leq C\|h\|_1^{1/(p-1)}$$

and

$$\|\varphi_p^{-1}(w)(T_ih)'\|_\infty \leq \|h\|_1^{1/(p-1)}.$$

Lemma 2.3. *Let $h \in Y$. Then T_3h satisfies*

$$\|T_ih\|_\infty \leq D\|h\|_1^{1/(p-1)}$$

and

$$\|\varphi_p^{-1}(w)(T_ih)'\|_\infty \leq \|h\|_1^{1/(p-1)}.$$

We define the Nemiskii operator $N : X \to Y$ by

$$(Nu)(t) = f(t, u(t), u'(t)),\ t \in \mathbb{R}_+.$$

It follows from (F) that N maps bounded sets of X into bounded sets of Y and is continuous. For each $i \in \{1, 2, 3, 4\}$, define $L_i \triangleq T_i N : X \to X$. Then L_i is well defined and problem $(P), (BC_i)$ has a solution u if and only if L_i has a fixed point u in X.

To show the compactness of the operators L_i $(i = 1, 2, 3, 4)$, we use the following compactness criterion.

Theorem 2.4. *([2]) Let Z be the space of all bounded continuous vector-valued functions on $\mathbb{R}_+$ and $S \subset Z$. Then S is relatively compact in Z if the following conditions hold.*
(i) S is bounded in Z.
(ii) the functions from S are equicontinuous on any compact interval of $\mathbb{R}_+$.
(iii) the functions from S are equiconvergent, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $\|\phi(t) - \phi(\infty)\|_{\mathbb{R}^n} < \epsilon$, for all $t > T$ and all $\phi \in S$.

Lemma 2.5. *For each $i \in \{1, 2, 3, 4\}$, the mapping $L_i : X \to X$ is completely continuous.*

Proof. We only prove that $L_1 : X \to X$ is completely continuous since other cases can be proved by the similar manner.

First, we show that L_1 is compact. Let Σ be bounded in X, i.e., there exists $M > 0$ such that $\|u\| \le M$ for all $u \in \Sigma$. Then there exists $h_M \in Y$ such that $|(Nu)(t)| \le h_M(t)$ for all $t \in \mathbb{R}_+$ and all $u \in \Sigma$. By Lemma 2.1, $L_1(\Sigma)$ is bounded in X.

For $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, one has

$$\begin{aligned} |(L_1u)(t_1) - (L_1u)(t_2)| &= \left| \int_{t_1}^{t_2} \varphi_p^{-1} \left(\frac{1}{w(s)} \int_s^\infty (Nu)(\tau) d\tau \right) ds \right| \\ &\le \|h_M\|_1^{1/(p-1)} \int_{t_1}^{t_2} \varphi_p^{-1} \left(\frac{1}{|w(s)|} \right) ds \end{aligned}$$

and

$$\begin{aligned} &|(\varphi_p^{-1}(w)(L_1u)')(t_1) - (\varphi_p^{-1}(w)(L_1u)')(t_2)| \\ = &\left| \varphi_p^{-1} \left(\int_{t_1}^\infty (Nu)(s) ds \right) - \varphi_p^{-1} \left(\int_{t_2}^\infty (Nu)(s) ds \right) \right|, \end{aligned}$$

which yield that $L_1(\Sigma)$ and $\{\varphi_p^{-1}(w)(L_1u)' \mid u \in \Sigma\}$ are equicontinuous on $\mathbb{R}_+$ by the facts that φ_p^{-1} is uniformly continuous on $[-1, 1]$ and $|(Nu)(t)| \le h_M(t)$ for all $t \in \mathbb{R}_+$.

For $u \in \Sigma$, one has

$$\lim_{t\to\infty} (\varphi_p^{-1}(w)(L_1u)')(t) = \lim_{t\to\infty} \varphi_p^{-1} \left(\int_t^\infty (Nu)(s) ds \right) = 0.$$

Then

$$\begin{aligned} |L_1u(t) - \lim_{t\to\infty} L_1u(t)| &= \left| \int_t^\infty \varphi_p^{-1} \left(\frac{1}{w(s)} \int_s^\infty (Nu)(\tau) d\tau \right) ds \right| \\ &\le \|h_M\|_1^{1/(p-1)} \int_t^\infty \varphi_p^{-1} \left(\frac{1}{|w(s)|} \right) ds \end{aligned}$$

and

$$\begin{aligned} |(\varphi_p^{-1}(w)(L_1u)')(t) - \lim_{t\to\infty} (\varphi_p^{-1}(w)(L_1u)')(t)| &= \left| \varphi_p^{-1} \left(\int_t^\infty (Nu)(s) ds \right) \right| \\ &\le \left(\int_t^\infty |h_M(s)| ds \right)^{1/(p-1)}, \end{aligned}$$

which yield that $L_1(\Sigma)$ and $\{\varphi_p^{-1}(w)(L_1u)' \mid u \in \Sigma\}$ are equiconvergent. By Theorem 2.4, we can conclude that T_1 is compact.

It follows from the Lebesgue dominated convergence theorem that $L_1 : X \to X$ is continuous, and thus the proof is complete.

□

3. Main results

In this section, we give our main results.

Theorem 3.1. *Assume $B^{p-1}\|\alpha\|_1 + \|\beta/w\|_1 < 1$. Then problem $(P), (BC_1)$ has at least one solution for every $\gamma \in Y$.*

Proof. Consider the differential equation, for $\lambda \in [0, 1]$,

$$(w(t)\varphi_p(u'(t)))' + \lambda f(t, u(t), u'(t)) = 0, \ a.e. \ t \in \mathbb{R}_+, \tag{3.1}$$

subject to the boundary condition (BC_1).

Let u be any solution of $(3.1), (BC_1)$. Then, by (F) and Lemma 2.1, one has

$$\begin{aligned}
\|(w\varphi_p(u'))'\|_1 &= \lambda\|Nu\|_1 \\
&\leq \|\alpha\|_1\|u\|_\infty^{p-1} + \|\beta/w\|_1\|\varphi_p^{-1}(w)u'\|_\infty^{p-1} + \|\gamma\|_1 \\
&\leq B^{p-1}\|\alpha\|_1\|(w\varphi_p(u'))'\|_1 + \|\beta/w\|_1\|(w\varphi_p(u'))'\|_1 + \|\gamma\|_1,
\end{aligned}$$

which yields

$$\|(w\varphi_p(u'))'\|_1 \leq \frac{\|\gamma\|_1}{1 - (B^{p-1}\|\alpha\|_1 + \|\beta/w\|_1)}.$$

It follows from Lemma 2.1 that the set of all possible solutions to problem $(3.1), (BC_1)$ is a priori bounded by a constant independent of $\lambda \in [0, 1]$. Thus the proof is complete in view of the Leray-Schauder continuation principle (see, e.g., [18, 21]). □

Similarly, the following results are obtained.

Theorem 3.2. *Assume $C^{p-1}\|\alpha\|_1 + \|\beta/w\|_1 < 1$. Then problems $(P), (BC_i)$ $(i = 2, 4)$ have at least one solution for every $\gamma \in Y$.*

Theorem 3.3. *Assume $D^{p-1}\|\alpha\|_1 + \|\beta/w\|_1 < 1$. Then problem $(P), (BC_3)$ has at least one solution for every $\gamma \in Y$.*

Finally, we give an example to illustrate our results.

Example 3.4. In problems $(P), (BC_i)$ $(i = 1, 2, 3, 4)$, let $p = 3$, $m = 3$, $a_1 = 1/2$, $\xi_1 = 1$, and

$$w(t) = \begin{cases} \varphi_3(-(1-t)^{1/2}), \ 0 \leq t < 1, \\ \varphi_3((t-1)^{1/2}), \ 1 \leq t < 2, \\ \varphi_3(\exp(t-2)), \ t \geq 2. \end{cases}$$

Then $A = 1/2$, $B = 7$, $C = 11/2$ and $D = 8$. For any $\gamma \in Y$, we set

$$f(t, u, v) = \frac{\sin t}{(t+70)^2}\varphi_3(u) + \frac{w(t)}{(t+70)^2}\varphi_3(v) + \gamma(t).$$

Then $\alpha(t) = \beta/w = 1/(t+70)^2$, and $\|\alpha\|_1 = \|\beta/w\|_1 = 1/70$. Thus by Theorems 3.1, 3.2 and 3.3, problems $(P), (BC_i)$ $(i = 1, 2, 3, 4)$ has at least one solution for every $\gamma \in Y$.

Acknowledgements: The author thanks the referee for his or her helpful remarks.

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