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Coupled common fixed point results in ordered G-metric spaces

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`This paper is dedicated to Professor Ljubomir Ćirić`

`Communicated by Professor V. Berinde`

Abstract

In the present paper, we prove coupled common fixed point theorems in the setting of a partially ordered G-metric space in the sense of Z. Mustafa and B. Sims. Examples are given to support the usability of our results and to distinguish them from the existing ones.©2012 NGA. All rights reserved.

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1. INTRODUCTION

The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and sciences. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors.

In 1963, S. Gähler introduced the notion of 2-metric spaces but different authors proved that there is no relation between these two functions and there is no easy relationship between results obtained in the two settings. Because of that, Dhage [18] introduced a new concept of the measure of nearness between three or more objects. But topological structure of so called D-metric spaces was incorrect. In 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G-metric space [25].

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In fact, Mustafa et al. studied many fixed point results for a self-mapping in G-metric space under certain conditions; see [24]–[30]. This is a generalization of metric spaces in which a non-negative real number is assigned to every triplet of elements. Analysis of the structure of these spaces was done in details in [26]. Fixed point theory in these spacee was initiated in [27]. Particularly, Banach contraction mapping principle was established in this work. After that several fixed point results were proved in these spaces. Some of these works are noted in [1, 2, 28, 29, 30, 42, 43].

In present era, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. Fixed point problems have also been considered in partially ordered probabilistic metric spaces [17], partially ordered G-metric spaces [8, 39], partially ordered cone metric spaces [13, 23, 43], partially ordered fuzzy metric spaces [40] and partially ordered non-Archimedean fuzzy metric spaces [4, 5].

Mixed monotone operators were introduce by Guo and Lakshmikantham in [19]. Their study has not only important theoretical meaning but also wide applications in engineering, nuclear physics, biological chemistry technology, etc. Particularly, a coupled fixed point result in partially ordered metric spaces was established by Bhaskar and Lakshmikantham [10]. After the publication of this work, several coupled fixed point and coincidence point results have appeared in the recent literature. Works noted in [3, 6, 7, 9, 11, 16, 17, 20, 21, 31, 32, 33, 34, 35, 36, 37, 41, 44] are some relevant examples.

In [10], Bhaskar and Lakshmikantham introduced the notions of a mixed monotone mapping and a coupled fixed point. Lakshmikantham et al. [15] introduced the concept of a coupled coincidence point of a mapping F from $X \times X$ into X and a mapping g from X into X and studied fixed point theorems in partially ordered metric spaces. Recently, Choudhury and Maity [13] also established coupled fixed point theorems in a partially ordered G-metric space.

The aim of this paper is to prove a coupled coincidence and common fixed point theorems for commuting mappings with mixed g-monotone property in partially ordered G-metric spaces. In this paper, we extend the results of Choudhury and Maity [13] for a pair of commutative maps. Examples are given to support the usability of our results and to distinguish them from the existing ones.

2. PRELIMINARIES

Throughout this paper $(X, \preceq)$ denotes a partially ordered set with the partial order $\preceq$. By '$x \succeq y$ holds', we mean that '$y \preceq x$ holds' and by '$x \prec y$ holds' we mean that '$x \preceq y$ holds and $x \neq y$'.

Definition 2.1. (G-Metric Space [26]). Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$, if $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2.2. ([26]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and one says that the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$.

It was shown in [26] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

Definition 2.3. ([26]). Let (X, G) be a G-metric space. A sequence $\{x_n\}$ in X is called G-Cauchy if for every $\epsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Lemma 2.4. *([26]). If (X, G) is a G-metric space, then the following are equivalent:*

(1) $\{x_n\}$ is G-convergent to x;
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$;
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$;
(4) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Lemma 2.5. *([25]). If (X, G) is a G-metric space, then the following are equivalent:*

(1) The sequence $\{x_n\}$ is G-Cauchy.
(2) For every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.

Lemma 2.6. *([13, 26]). If (X, G) is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.*

Combining Lemmas 2.5 and 2.6 we have the following result.

Lemma 2.7. *([13]) If (X, G) is a G-metric space then $\{x_n\}$ is a G-Cauchy sequence if and only if for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$, for all $m > n \geq N$.*

Definition 2.8. ([26]). Let (X, G), (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Definition 2.9. ([26]). A G-metric space (X, G) is called symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.10. ([26]). A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X, G) is convergent in X.

Definition 2.11. Let X be a nonempty set. Then $(X, G, \preceq)$ is called an ordered G-metric space if:

(i) (X, G) is a metric space,

(ii) $(X, \preceq)$ is a partially ordered set.

Definition 2.12. Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 2.13. ([10]) . Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, &\quad x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y), \\ y_1, y_2 \in X, &\quad y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned}$$

This definition coincides with the notion of a mixed monotone function on $\mathbb{R}^2$ when $\leq$ represents the usual total order on $\mathbb{R}$.

Definition 2.14. ([10]). An element $(x, y) \in X \times X$ is a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

The concept of the mixed monotone property is generalized in [15].

Definition 2.15. ([15]). Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y) \tag{2.1}$$

and

$$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2). \tag{2.2}$$

Clearly, if g is the identity mapping, then Definition 2.15 reduces to Definition 2.13.

Definition 2.16. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x) \quad \text{and} \quad F(y, x) = g(y),$$

It is a common coupled fixed point of F and g if

$$F(x, y) = g(x) = x, \text{ and } F(y, x) = g(y) = y.$$

Definition 2.17. Let (X, d) be a metric space and $F : X \times X \to X$ and $g : X \to X$ be mappings. We say that F and g are commutative if

$$F(g(x), g(y)) = g(F(x, y))$$

for all $x, y \in X$.

Definition 2.18. Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two sequences $\{x_n\}$ and $\{y_n\}$ G-converging to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

3. MAIN RESULTS

Our first result is the following.

Theorem 3.1. *Let $(X, G, \preceq)$ be a partially ordered G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists a $k \in [0, \frac{1}{2})$ such that for $x, y, z, u, v, w \in X$, the following holds:*

$$G(F(x, y), F(u, v), F(z, w)) \leq k[G(gx, gu, gz) + G(gy, gv, gw)] \tag{3.1}$$

for all $gx \succeq gu \succeq gz$ and $gy \preceq gv \preceq gw$ where either $gu \neq gz$ or $gv \neq gw$. We assume the following hypotheses:

(i) $F(X \times X) \subseteq g(X)$,

(ii) $g(X)$ is G-complete,

(iii) g is G-continuous and commutes with F.

Then F and g have a coupled coincidence point. If $gu = gz$ and $gv = gw$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x) = x.$$

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$.

Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \ \forall \ n \geq 0. \tag{3.2}$$

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{3.3}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{3.4}$$

We shall use the mathematical induction. Let $n = 0$. Since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \preceq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) \succeq F(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (3.3) and (3.4) hold for $n = 0$. We presume that (3.3) and (3.4) hold for some $n > 0$. As F has the mixed g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (3.2), we get

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n) \tag{3.5}$$

and

$$F(y_{n+1}, x_n) \preceq F(y_n, x_n) = g(y_{n+1}). \tag{3.6}$$

Also for the same reason we have

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n) \text{ and } F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2}).$$

Then from (3.2) and (3.3), we obtain

$$g(x_{n+1}) \preceq g(x_{n+2}) \text{ and } g(y_{n+1}) \succeq g(y_{n+2}).$$

Thus by the mathematical induction, we conclude that (3.3) and (3.4) hold for all $n \geq 0$.

Continuing this process, one can easily verify that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_{n+1}) \preceq \cdots$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_{n+1}) \succeq \cdots .$$

If $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then F and g have a coupled coincidence point. So we assume $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$, that is, we assume that either $g(x_{n+1}) = F(x_n, y_n) \neq g(x_n)$ or $g(y_{n+1}) = F(y_n, x_n) \neq g(y_n)$.

Next, we claim that, for all $n \geq 0$,

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{1}{2}(2k)^n [G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)]. \tag{3.7}$$

For $n = 1$, we have

$$\begin{aligned} G(gx_1, gx_2, gx_2) &= G(F(x_0, y_0), F(x_1, y_1), F(x_1, y_1)) \\ &\leq k[G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)] \\ &= \frac{1}{2}(2k)^1 [G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)]. \end{aligned}$$

Thus (3.7) holds for $n = 1$. Therefore, we presume that (3.7) holds $n > 0$. Since $g(x_{n+1}) \succeq g(x_n)$ and $g(y_{n+1}) \preceq g(y_n)$, from (3.1) and (3.2), we have

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n)) \quad (3.8) \\ &\leq k[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)]. \quad (3.9) \end{aligned}$$

From

$$\begin{aligned} G(gx_{n-1}, gx_n, gx_n) &= G(F(x_{n-2}, y_{n-2}), F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1})) \\ &\leq k[G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})], \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} G(gy_{n-1}, gy_n, gy_n) &= G(F(y_{n-2}, x_{n-2}), F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1})] \\ &\leq k[G(gy_{n-2}, gy_{n-1}, gy_{n-1}) + G(gx_{n-2}, gx_{n-1}, gx_{n-1})]. \end{aligned} \tag{3.11}$$

By combining (3.10) and (3.11), we have

$$G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n) \leq 2k[G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})]$$

holds for all $n \in N$. Thus, from (3.8)

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &\leq k[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)) \\ &\leq 2k^2[G(gx_{n-2}, gx_{n-1}, gx_{n-1}) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})] \\ &\vdots \\ &\leq \frac{1}{2}(2k)^n[G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)]. \end{aligned}$$

Thus for each $n \in N$, we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{1}{2}(2k)^n[G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)]. \tag{3.12}$$

Let $m, n \in N$ with $m > n$. By Axiom (G5) of definition of G-metric spaces, we have

$$G(gx_n, gx_m, gx_m) \leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) + \cdots + G(gx_{m-1}, gx_m, gx_m).$$

Since $2k < 1$, by (3.12) we get

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq \frac{1}{2}\sum_{i=n}^{m-1}(2k)^i[G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)] \\ &\leq \frac{(2k)^n}{2(1-2k)}[G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)]. \end{aligned}$$

Letting $n, m \to +\infty$, we have

$$\lim_{n,m\to+\infty} G(g(x_n), g(x_m), g(x_m)) = 0.$$

Thus $\{gx_n\}$ is G-Cauchy in $g(X)$. Similarly, we may show that $\{gy_n\}$ is G-Cauchy in $g(X)$.

Since $g(X)$ is G-complete, we get $\{gx_n\}$ and $\{gy_n\}$ are G-convergent to some $x \in X$ and $y \in X$ respectively. Since g is G-continuous, we have $\{g(gx_n)\}$ is G-convergent to gx and $\{g(gy_n)\}$ is G-convergent to gy, i.e,

$$\lim_{n\to+\infty} g(g(x_n)) = g(x) \text{ and } \lim_{n\to+\infty} g(g(y_n)) = g(y). \tag{3.13}$$

Also, from commutativity of F and g, we have

$$F(g(x_n), g(y_n)) = g(F(x_n, y_n)) = g(g(x_{n+1})) \tag{3.14}$$

and

$$F(g(y_n), g(x_n)) = g(F(y_n, x_n)) = g(g(y_{n+1})). \tag{3.15}$$

Next, we claim that (x, y) is a coupled coincidence point of F and g.

Now, from the condition (3.1), we have:

$$G(ggx_{n+1}, F(x, y), F(x, y)) = G(F(gx_n, gy_n), F(x, y), F(x, y)) \leq k[G(ggx_n, gx, gx) + G(ggy_n, gy, gy)].$$

Letting $n \to +\infty$, and using the fact that G is continuous on its variables, we get that

$$G(gx, F(x,y), F(x,y)) \leq k[G(gx, gx, gx) + G(gy, gy, gy)] = 0.$$

Hence $gx = F(x,y)$. Similarly, we may show that $gy = F(y,x)$.

Finally, we claim that x is a common fixed point of F and g.

Since (x,y) is a coupled coincidence point of the mappings F and g, we have $gx = F(x,y)$ and $gy = F(y,x)$. Assume $gx \neq gy$. Then by (3.1), we get

$$G(gx, gy, gy) = G(F(x,y), F(y,x), F(y,x)) \leq k[G(gx, gy, gy) + G(gy, gx, gx)].$$

Also by (3.1), we have

$$G(gy, gx, gx) = G(F(y,x), F(x,y), F(x,y)) \leq k[G(gy, gx, gx) + G(gx, gy, gy)].$$

Therefore

$$G(gx, gy, gy) + G(gy, gx, gx) \quad \leq \quad 2k[G(gx, gy, gy) + G(gy, gx, gx)].$$

Since $2k < 1$, we get

$$G(gx, gy, gy) + G(gy, gx, gx) \quad < \quad G(gx, gy, gy) + G(gy, gx, gx),$$

which is a contradiction. So $gx = gy$, and hence

$$F(x,y) = gx = gy = F(y,x).$$

Since $\{gx_{n+1}\}$ is subsequence of $\{gx_n\}$ we have $\{gx_{n+1}\}$ is G-convergent to x. Thus

$$\begin{aligned} G(gx_{n+1}, gx, gx) &= G(gx_{n+1}, F(x,y), F(x,y)) \\ &= G(F(x_n, y_n), F(x,y), F(x,y)) \\ &\leq k[G(gx_n, gx, gx) + G(gy_n, gy, gy)]. \end{aligned}$$

Letting $n \to +\infty$, and use the fact that G is continuous on its variables, we get

$$G(x, gx, gx) \leq k[G(x, gx, gx) + G(y, gy, gy)].$$

Similarly, we may show that

$$G(y, gy, gy) \leq k[G(x, gx, gx) + G(y, gy, gy)).$$

Thus

$$G(x, gx, gx) + G(y, gy, gy) \leq 2k[G(x, gx, gx) + G(y, gy, gy)].$$

Since $2k < 1$, the last inequality happenes only if $G(x, gx, gx) = 0$ and $G(y, gy, gy) = 0$. Hence $x = gx$ and $y = gy$. Thus we get $gx = F(x,x) = x$. Thus F and g have a common fixed point. This completes the proof of the theorem. □

Now, our second result is the following.

Theorem 3.2. *If in the above theorem, in the place of condition* (ii)*, we assume the following conditions in the complete G-metric space X, namely,*

$$\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \to x \text{ in } X, \text{ then } x_n \preceq x \text{ for all } n \tag{3.16}$$

and

$$\text{if } \{y_n\} \subset X \text{ is a nondecreasing sequence with } y_n \to y \text{ in } X, \text{ then } y_n \succeq y \text{ for all } n. \tag{3.17}$$

Then, we have the conclusions of Theorem 3.1, provided g is nondecreasing.

Proof. Proceeding exactly as in Theorem 3.1, we have $\{gx_n\}$ and $\{gy_n\}$ are G-Cauchy in X. Since (X,G) is a complete metric space, there exists $(x,y) \in X \times X$ such that

$$\lim_{n\to+\infty} F(x_n,y_n) = \lim_{n\to+\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to+\infty} F(y_n,x_n) = \lim_{n\to+\infty} g(y_n) = y. \tag{3.18}$$

Therefore, from (iii) we arrive at (3.13), (3.14) and (3.15). Since $\{g(x_n)\}$ is a non-decreasing sequence and $g(x_n) \to x$, and as $\{g(y_n)\}$ is a non-increasing sequence and $g(y_n) \to y$, by assumption (3.16) and (3.17) we have, $g(gx_n) \preceq g(x)$ and $g(gy_n) \succeq g(y)$ for all $n \geq 0$. If $g(gx_n) = g(x)$ and $g(gy_n) = g(y)$ for some n, then, by construction, $g(gx_{n+1}) = g(x)$ and $g(gy_{n+1}) = g(y)$ and (x,y) is a coupled fixed point. So we assume either $g(gx_n) \neq g(x)$ or $g(gy_n) \neq g(y)$. Applying the contractive condition (3.1), we have

$$\begin{aligned}
& G(F(x,y), gx, gx) \\
\leq\ & G(F(x,y), F(g(x_n), g(y_n)), F(g(x_n), g(y_n))) + G(F(g(x_n), g(y_n)), gx, gx) \\
=\ & G(F(g(x_n), g(y_n)), F(g(x_n), g(y_n)), F(x,y)) + G(gF(x_n,y_n), gx, gx) \\
\leq\ & \frac{k}{2}[G(g(gx_n), g(gx_n), gx) + G(g(gy_n), g(gy_n), gy)] + G(g(gx_{n+1}), gx, gx).
\end{aligned}$$

Taking $n \to +\infty$ in the above inequality we obtain $G(F(x,y), gx, gx) = 0$, that is, $F(x,y) = g(x)$. Similarly, we have that $F(y,x) = g(y)$. Remaining part of the proof follows from Theorem 3.1. Hence, we have $g(x) = F(x,x) = x$. This completes the proof of the theorem. □

Corollary 3.3. *Let $(X,G,\preceq)$ be a partially ordered G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0,y_0)$ and $g(y_0) \succeq F(y_0,x_0)$. Suppose that there exists a $k \in [0, \frac{1}{2})$ such that for $x,y,u,v \in X$, the following holds:*

$$G(F(x,y), F(u,v), F(u,v)) \leq k[G(gx,gu,gu) + G(gy,gv,gv)] \tag{3.19}$$

for all $gx \succeq gu$ and $gy \preceq gv$. We assume the following hypotheses:

(a) $F(X \times X) \subseteq g(X)$,

(b) g is G-continuous and commutes with F.

Then there exists a $x \in X$ such that $gx = F(x,x) = x$, provided either of the following conditions is satisfied:

(c) $g(X)$ is G-complete,

(d) g is nondecreasing with

$$\text{if } \{x_n\} \subset X \text{ is a nondecreasing sequence with } x_n \to x \text{ in } X, \text{ then } x_n \preceq x \text{ for all } n$$

and

$$\text{if } \{y_n\} \subset X \text{ is a nondecreasing sequence with } y_n \to y \text{ in } X, \text{ then } y_n \succeq y \text{ for all } n.$$

Proof. Follows from Theorem 3.1 by taking $z = u$ and $v = w$. □

Corollary 3.4. *Let $(X,G,\preceq)$ be a partially ordered G-metric space. Let $F : X \times X \to X$ be a mapping having mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Assume that there exists a $k \in [0, \frac{1}{2})$ such that for $x,y,u,v \in X$, the following holds:*

$$G(F(x,y), F(u,v), F(u,v)) \leq k[G(x,u,u) + G(y,v,v)] \tag{3.20}$$

for all $x \succeq u$ *and* $y \preceq v$*. Then there exists an* $x \in X$ *such that* $F(x, x) = x$*, provided*

$$\textit{if } \{x_n\} \subset X \textit{ is a nondecreasing sequence with } x_n \to x \textit{ in } X, \textit{ then } x_n \preceq x \textit{ for all } n$$

and

$$\textit{if } \{y_n\} \subset X \textit{ is a nondecreasing sequence with } y_n \to y \textit{ in } X, \textit{ then } y_n \succeq y \textit{ for all } n.$$

Proof. Define $g : X \to X$ by $gx = x$. Then F and g satisfy all the hypotheses of Corollary 3.3. Hence the result follows. □

Now, our third result is the following.

Theorem 3.5. *Let* $(X, G, \preceq)$ *be a partially ordered* G*-metric space. Let* $F : X \times X \to X$ *and* $g : X \to X$ *be mappings such that* F *has the mixed* g*-monotone property on* X *and* $F(x, y) \preceq F(y, x)$ *whenever* $x \preceq y$*. Suppose*

(I) $F(X \times X) \subseteq g(X)$*,*

(II) g *is* G*-continuous and commutes with* F*.*

Assume that there exists a $k \in [0, \frac{1}{2})$ *such that for* $x, y, z, u, v, w \in X$*, the inequality (3.1) holds, whenever* $gx \succeq gu \succeq gz$ *and* $gy \preceq gv \preceq gw$ *where either* $gu \neq gz$ *or* $gv \neq gw$*. If there exist two elements* $x_0, y_0 \in X$ *such that*

$$g(x_0) \preceq g(y_0), \;\; g(x_0) \preceq F(x_0, y_0) \;\; \textit{and} \;\; g(y_0) \succeq F(y_0, x_0)$$

then, we have the conclusions of Theorem 3.1, provided either of the following conditions holds:

(III) $g(X)$ *is* G*-complete,*

(IV) g *is nondecreasing in the complete* G*-metric space* (X, G) *with*

$$\textit{if } \{x_n\} \subset X \textit{ is a nondecreasing sequence with } x_n \to x \textit{ in } X, \textit{ then } x_n \preceq x \textit{ for all } n$$

and

$$\textit{if } \{y_n\} \subset X \textit{ is a nondecreasing sequence with } y_n \to y \textit{ in } X, \textit{ then } y_n \succeq y \textit{ for all } n.$$

Proof. By the condition of the theorem there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. We define $x_1, y_1 \in X$ as $g(x_0) \preceq F(x_0, y_0) = g(x_1)$ and $g(y_0) \succeq F(y_0, x_0) = g(y_1)$.

Since $g(x_0) \preceq g(y_0)$, we have, by a condition of the theorem, $F(x_0, y_0) \preceq F(y_0, x_0)$.

Hence $g(x_0) \preceq g(x_1) = F(x_0, y_0) \preceq F(y_0, x_0) = g(y_1) \preceq g(y_0)$.

Continuing the above procedure we have two sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ recursively as follows:

$$\text{for all } n \geq 1, g(x_n) = F(x_{n-1}, y_{n-1}) \text{ and } g(y_n) = F(y_{n-1}, x_{n-1}), \tag{3.21}$$

such that

$$\begin{aligned} g(x_0) \preceq F(x_0, y_0) = g(x_1) \preceq \cdots \preceq F(x_{n-1}, y_{n-1}) = g(x_n) \preceq \cdots \\ \preceq g(y_n) = F(x_{n-1}, y_{n-1}) \preceq \cdots \preceq g(y_1) = F(y_0, x_0) \preceq g(y_0). \end{aligned} \tag{3.22}$$

In particular, we have for all $n \geq 0$,

$$g(x_n) \preceq F(x_n, y_n) = g(x_{n+1}) \preceq g(y_{n+1}) = F(y_n, x_n) \preceq g(y_n).$$

If $x_n = y_n = c$ (say) for some n, then $g(c) \preceq F(c,c) \preceq F(c,c) \preceq g(c)$. This shows that $g(c) = F(c,c)$. Thus (c,c) is a coupled fixed point. Hence we assume that

$$g(x_n) \prec g(y_n), \text{ for all } n \geq 0. \tag{3.23}$$

Further, for the same reason as stated in Theorem 3.1, we assume that $(x_n, y_n) \neq (x_{n+1}, y_{n+1})$. Then, in view of (3.23), for all $n \geq 0$, the inequality (3.1) will hold with

$$x = x_{n+2}, u = x_{n+1}, w = x_n, y = y_n, v = y_{n+1} \textit{ and } z = y_{n+2}.$$

The rest of the proof is completed by repeating the same steps as in Theorem 3.1 and Theorem 3.2 . □

Now, we present examples to illustrate our obtained results given by Theorems 3.1 and Theorem 3.2 and to show that they are proper extension of some known results.

Example 3.6. Let $X = \mathbb{R}$ be ordered by the following relation

$$x \preceq y \iff x = y \quad \text{or} \quad (x, y \in [0,1] \quad \text{and} \quad x \leq y).$$

Let a G-metric on X be defined by

$$G(x,y,x) = |x-y| + |y-z| + |z-x|.$$

Then, $(X, G, \preceq)$ is a complete regular ordered G-metric space.

Let $g : X \to X$ and $F : X \times X \to X$ be defined by

$$gx = \begin{cases} \frac{1}{20}x, & x < 0 \\ \frac{x}{2}, & x \in [0,1] \\ \frac{1}{20}x + \frac{9}{20}, & x > 1; \end{cases} \qquad \text{and} \qquad F(x,y) = \frac{x+y}{20}.$$

Take $k = \frac{1}{10}$. We will check that condition (3.1) of Theorem 3.2 is fulfilled for all $x, y, u, v, z, w \in X$ satisfying $(gz \preceq gu \preceq gx$ and $gw \succeq gv \succeq gy)$ or $(gx \preceq gu \preceq gz$ and $gy \succeq gv \succeq gw)$. The only nontrivial case is when $x, y, u, v, z, w \in [0,1]$ and $(z \leq u \leq x$ and $w \geq v \geq y)$ or $(x \leq u \leq z$ and $y \geq v \geq w)$. Then,

$$\begin{aligned} G(F(x,y),&F(u,v), F(z,w)) \\ &= |F(x,y) - F(u,v)| + |F(u,v) - F(z,w)| + |F(z,w) - F(x,y)| \\ &= \left|\tfrac{x+y}{20} - \tfrac{u+v}{20}\right| + \left|\tfrac{u+v}{20} - \tfrac{z+w}{20}\right| + \left|\tfrac{z+w}{20} - \tfrac{x+y}{20}\right| \\ &= \left|\tfrac{x-u}{20} + \tfrac{y-v}{20}\right| + \left|\tfrac{u-z}{20} + \tfrac{v-w}{20}\right| + \left|\tfrac{z-x}{20} + \tfrac{w-y}{20}\right| \\ &\leq \frac{1}{20}\{[|x-u| + |u-z| + |z-x|] + [|y-v| + |v-w| + |w-y|]\} \\ &= \frac{1}{10}[G(gx,gu,gz) + G(gy,gv,gw)] \\ &= k[G(gx,gu,gz) + G(gy,gv,gw)], \end{aligned} \tag{3.24}$$

and the condition holds. We conclude that all the conditions of Theorems 3.1 and 3.2 are satisfied. Obviously, the mappings g and F have a unique common coupled fixed point $(0,0)$.

Note however that these theorems cannot be used in non-ordered case to reach this conclusion. Indeed, take $x = 2$ and $y = u = v = z = w = 0$. Then condition (3.1) does not hold since

$$G(F(2,0), F(0,0), F(0,0)) = G(\tfrac{1}{10}, 0, 0) = \frac{1}{5},$$

while

$$k[G(g2, g0, g0) + G(g0, g0, g0)] = \frac{1}{10}[G(\tfrac{11}{20}, 0, 0) + \frac{1}{10} \cdot 0] = \frac{1}{10} \cdot 2 \cdot \frac{11}{20} = \frac{11}{100} < \frac{1}{5},$$

and obviously contractive condition (3.1) is not fulfilled.

Example 3.7. Let $X = [0, +\infty)$ be equipped with the G-metric $G(x, y, z) = |x - y| + |y - z| + |z - x|$ and the order $\preceq$ defined by

$$x \preceq y \iff x = y \vee (x, y \in [0, 1] \wedge x \leq y).$$

Then $(X, G, \preceq)$ is a complete partially ordered G-metric space. Consider the (continuous) mapping $F : X^2 \to X$ given by

$$F(x, y) = \begin{cases} \frac{1}{6}x, & x \in [0, 1], y \in X, \\ x - \frac{5}{6}, & x > 1, y \in X, \end{cases}$$

and take $g : X \to X$ given by $gx = x$. Obviously, F has the g-mixed monotone property. Let $x, y, u, v, z, w \in X$ be such that $x \succeq u \succeq z$ and $y \preceq v \preceq w$. Then the following cases are possible.

1) All of these variables belong to $[0, 1]$ and, hence $x \geq u \geq z$ and $y \leq v \leq w$. If we denote by L and R, respectively, the left-hand and right-hand side (with, say, $k = \frac{1}{4}$) of inequality (3.1), then

$$\begin{aligned} L &= G(\tfrac{1}{6}x, \tfrac{1}{6}u, \tfrac{1}{6}z) \\ &= \frac{1}{6}(|x - u| + |u - z| + |z - x|) \\ &\leq \frac{1}{4}(|x - u| + |u - z| + |z - x| + |y - v| + |v - w| + |w - y|) = R. \end{aligned}$$

2) $x, u, z \in [0, 1]$ (and $x \geq u \geq z$) and $y, v, w > 1$ (and $y = v = w$). Then we have

$$\begin{aligned} L = G(\tfrac{1}{6}x, \tfrac{1}{6}u, \tfrac{1}{6}z) &= \frac{1}{6}(|x - u| + |u - s| + |s - x|) \\ &\leq \frac{1}{4}(|x - u| + |u - z| + |z - x|) = R. \end{aligned}$$

The case when $x, u, z > 1$ and $y, v, w \in [0, 1]$ is treated similarly.

3) $x, u, z, y, v, w > 1$. Then $x = u = z$, $y = v = w$ and $L = R = 0$.

Thus, all the conditions of Theorem 3.1 are fulfilled and F and g have a common coupled fixed point (which is $(0, 0)$).

However, consider the same G-metric space (X, G) without order. Take $(x, y) = (2, 2)$, $(u, v) = (2, 3)$ and $(z, w) = (3, 3)$. Then we have

$$L = G(F(2, 2), F(2, 3), F(3, 3)) = G(\tfrac{7}{6}, \tfrac{7}{6}, \tfrac{13}{6}) = 2,$$

and

$$R = k[G(2, 2, 3) + G(2, 3, 3)] = k[2 + 2] < 2,$$

i.e., $L > R$ whatever $k \in [0, \frac{1}{2})$ is chosen, and the contractive condition cannot be satisfied.

Remark 3.8. Our results generalize the results of Shatanawi [43] and Aydi et al. [8, Corollary 3.1 and 3.2] in G-metric spaces. Our results generalize the results of Choudhury and Maity [13] for a pair of commutative maps.

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