J. Nonlinear Sci. Appl. 4 (2011), no. 3, 193–199

The Journal of Nonlinear Sciences and Applications
http://www.tjnsa.com

INTERNAL NONLOCAL AND INTEGRAL CONDITION PROBLEMS OF THE DIFFERENTIAL EQUATION $x' = f(t, x, x')$

A. M. A. EL-SAYED[1], E. M. HAMDALLAH[2] AND KH. W. ELKADEKY[3]

Abstract. In this work, we are concerned with the existence of at least one absolutely continuous solution of the Cauchy problem for the differential equation $x' = f(t, x, x')$, $t \in (0, 1)$ with the internal nonlocal condition $\sum_{k=1}^{m} a_k x(\tau_k) = x_o$, $\tau_k \in (c, d) \subseteq (0, 1)$. The problem of the integral condition $\int_c^d x(s)\, dg(s) = x_o$ will be considered.

1. Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1]- [6] and [9] - [15] and references therein.
Here we are consisted with the nonlocal problem

$$\frac{dx(t)}{dt} = f(t, x(t), \frac{dx(t)}{dt}), \ a.e, \ t \in (0, 1), \tag{1.1}$$

$$\sum_{k=1}^{m} a_k \, x(\tau_k) = x_o, \ \sum_{k=1}^{m} a_k \neq 0 \text{ and } \tau_k \in (c, d) \subseteq (0, 1). \tag{1.2}$$

The existence of at least one solution $x \in AC[0, 1]$ will be studied when the function f is measurable in $t \in [0, 1]$,for any $(u_1, u_2) \in R^2$ and continuous in $(u_1, u_2) \in R^2$, for $t \in [0, 1]$. As a consequence of our result, the problem of the

Date: Received: 1 Jan, 2011; Revised: 20 April, 2011.
*Corresponding author

2000 *Mathematics Subject Classification.* Primary 26A33; Secondary 34A12, 34A40.
Key words and phrases. Nonlocal conditions, integral condition, existence of solution, fixed-point theorem.

differential equation (1.1) with integral condition

$$\int_c^d x(s)\ dg(s) = x_o, \tag{1.3}$$

where g is a nondecreasing function, will be studied.
It must be noticed that the nonlocal condition (1.2) and the integral condition (1.3) are more general than the following ones

$$x(\tau) \ = \ x_o,\ \tau \in (c,d), \tag{1.4}$$

$$\sum_{k=1}^{m} a_k x(\tau_k) \ = \ 0,\ \tau_k \in (a,c), \tag{1.5}$$

and

$$\int_a^c x(s)\ dg(s) \ = \ 0. \tag{1.6}$$

The following theorems will be needed.
Theorem (Kolmogorov Compactness Criterion) see[8]
Let $\Omega \subseteq L^P(0,1),\ 1 \leq P < \infty$. If

(i) Ω is bounded $L^p(0,1)$,
(ii) $x_h \to x$ as $h \to 0$ uniformly with respect to $x \in \Omega$, then Ω is relatively compact in $L^P(0,1)$, where

$$x_h(t) = \frac{1}{h}\int_t^{t+h} x(s)\ ds.$$

Theorem (Schauder) see[12]
Let U be a convex subset of a Banach space X, and $T : U \to U$ is compact, continuous map. Then T has at least one fixed point in U .

2. Existence of solution

The following Lemma gives the integral equation representation for the nonlocal problem (1.1)-(1.2).

Lemma 2.1 The solution of the nonlocal problem (1.1)-(1.2) can be expressed by the integral equation

$$x(t) = ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + \int_0^t y(s)\ ds \tag{2.1}$$

where y is the solution of the functional integral equation

$$y(t) = f(t, ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + \int_0^t y(s)\ ds,\ y(t)),\ t \in (0,1). \tag{2.2}$$

Proof. Let $\frac{dx(t)}{dt} = y(t)$ in equation (1), then

$$y(t) \ = \ f(t, x(0) + \int_0^t y(s)\ ds, y(t)) \tag{2.3}$$

where

$$x(t) = x(0) + \int_0^t y(s)\ ds. \tag{2.4}$$

Let $t = \tau_k$ in (2.4), we obtain

$$x(\tau_k) = x(0) + \int_0^{\tau_k} y(s)\ ds$$

and

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k x(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds. \tag{2.5}$$

Substitute from (2) into (2.5), we get

$$x_0 = \sum_{k=1}^{m} a_k x(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds$$

and

$$x(0) = a(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds) \tag{2.6}$$

where $a = (\sum_{k=1}^{m} a_k)^{-1}$.
Substitute from (2.6) into (2.4) and (2.3), we obtain (2.1) and (2.2).

Consider the functional integral equation (2.2) with the following assumptions

(i) $f : [0,1] \times R^2 \to R^+$ is measurable in $t \in [0,1]$ for any $(u_1, u_2) \in R^2$ and continuous in $(u_1, u_2) \in R^2$ for almost all $t \in [0,1]$.

(ii) There exists a function $a \in L_1[0,1]$ and two constants $b_i > 0$, $i = 1,2$ such that

$$|f(t, u_1, u_2)| \leq |a(t)| + \sum_{i=1}^{2} b_i |u_i|,\ \forall (t, u_1, u_2) \in [0,1] \times R^2.$$

(iii)

$$(2b_1 + b_2) < 1.$$

Now we have the following theorem.

Theorem 2.1 Assume that the assumptions (i) - (iii) are satisfied. Then the functional integral equation (2.2) has at least one solution $y \in L_1(0,1)$.
Proof. Let $y \in B_r \subset L^1$, $B_r = \{y : ||y||_{L_1} \leq r,\ r > 0\}$, $r = \frac{||a|| + ab_1 x_0}{1-(2b_1+b_2)}$.
Clearly B_r is nonempty, convex and closed .
Define the operator H by

$$Hy(t) = f(t, ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\ ds + \int_0^t y(s)\ ds,\ y(t)),\ t \in (0,1). \tag{2.7}$$

From assumptions (i) and (iii), we obtain

$$\begin{aligned}
||Hy||_{L_1} &= \int_0^1 |(Hy)(t)|\ dt \\
&= \int_0^1 |f(t, ax_0 - a\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)\ ds + \int_0^t y(s)\ ds,\ y(t))|\ dt \\
&\leq \int_0^1 (|a(t)| + b_1|ax_o - a\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)\ ds + \int_0^t y(s)\ ds| + b_2|y(t)|)\ dt \\
&\leq \int_0^1 (|a(t)|dt + \int_0^1 b_1 ax_o dt + ab_1 \sum_{k=1}^m a_k \int_0^1 \int_0^{\tau_k} |y(s)|\ dsdt \\
&+ b_1 \int_0^1 \int_0^t |y(s)|\ dsdt + b_2 \int_o^1 |y(t)|)\ dt \\
&\leq ||a|| + ab_1 x_o + b_1\ ||y|| + b_1||y|| + b_2||y|| \\
&\leq ||a|| + b_1 ax_0 + (2b_1 + b_2)||y||\ \leq r.
\end{aligned}$$

Then $||Hy||_{L_1} \leq r$, which implies that the operator H maps B_r into itself. Assumption (i) implies that H is continuous.

Now, let Ω be a bounded subset of B_r , therefore $H(\Omega)$ is bounded in $L_1(0,1)$, $i.e$ condition (i) of Kolmogorav compactness criterion is satisfied, it remains to show $(Hy)_h \to (Hy)$, in $L_1(0,1]$.
Let $y \in \Omega \subset L_1(0,1)$, then we have the following

$$\begin{aligned}
||(Hy)_h - (Hy)||_{L_1} &= \int_0^1 |(Hy)_h(t) - (Hy)(t)|\ dt \\
&= \int_0^1 |\frac{1}{h}\int_t^{t+h} (Hy)(s)\ ds - (Hy)(t)|\ dt \\
&\leq \int_0^1 (\frac{1}{h}\int_t^{t+h} |(Hy)(s) - (Hy)(t)|\ ds)\ dt \\
&\leq \int_0^1 \frac{1}{h}\int_t^{t+h} |f(s, ax_0 - a\sum_{k=1}^m a_k \int_0^{s_k} y(\tau)d\tau + \int_0^s y(\tau)d\tau,\ y(s)) \\
&- f(t, ax_0 - a\sum_{k=1}^m a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds,\ y(t))|\ ds\ dt.
\end{aligned}$$

Since $y \in \Omega \subset L_1$, and (assumption (ii) implies that) $f \in L_1[0,1]$, it follows that

$$\frac{1}{h}\int_t^{t+h} |f(s, ax_0 - a\sum_{k=1}^m a_k \int_0^{s_k} y(\tau)d\tau\ +\ \int_0^s y(\tau)d\tau,\ y(s))$$

$$-f(t, ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds,\ y(t))|\ ds\ \to 0\ as\ h \to 0,\ t \in (0,1).$$

Hence $(Hy)_h \to (Hy)$, uniformly as $h \to 0$.
Then by Kolmogorav compactness criterion, $H(\Omega)$ is relatively compact.
That is H has a fixed point in B_r, then there exist at least one solution $y \in L_1(0,1)$ of the functional equation (2.3).

Now, consider the nonlocal problem (1.1)-(1.2).

Theorem 2.2 Let the assumptions of Theorem 2.1 are satisfied. Then the nonlocal problem (1.1)-(1.2) has at least one solution $x \in AC[0,1]$.
Proof. Form Theorem 2.1 and equations (2.1) and (2.6) we deduce that there exist at least one solution $x \in AC[0,1]$ of equation (2.1) where

$$x(0) \ = \ \lim_{t\to 0}\ x(t) \ = \ ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k}\ y(s)\ ds \tag{2.8}$$

and

$$x(1) \ = \ \lim_{t\to 1}\ x(t) \ = \ ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k}\ y(s)\ ds + \int_0^1\ y(s)\ ds \tag{2.9}$$

To complete the proof, we prove that equation (2.1) satisfies nonlocal problem (1.1)-(1.2).
Differentiating (2.1), we get

$$\frac{dx}{dt} = y(t) = f(t, x(t), \frac{dx}{dt})$$

Let $t = \tau_k$ in (2.1) ,we get

$$x(\tau_k) = ax_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k}\ y(s)\ ds + \int_0^{\tau_k}\ y(s)\ ds$$

$$= ax_0 + (1 - a\sum_{k=1}^{m} a_k)\ \int_0^{\tau_k}\ y(s)\ ds.$$

Then

$$\sum_{k=1}^{m}\ a_k x(t_k) \ = \ \sum_{k=1}^{m}\ a_k a x_0 + \sum_{k=1}^{m}\ a_k(1 - a\sum_{k=1}^{m} a_k)\ \int_0^{\tau_k}\ y(s)\ ds \ = \ x_o.$$

This complete the proof of the equivalent between the nonlocal problem (1.1)-(1.2) and the integral equation (2.1) .
This implies that there exist at least one solution $x \in AC[0,1]$ of the nonlocal problem (1.1)-(1.2).

3. Nonlocal integral condition

Let $x \in AC[0,1]$ be the solution of the nonlocal problem (1.1)-(1.2). Let $a_k = g(t_k) - g(t_{k-1})$, g is a nondecreasing function, $\tau_k \in (t_{k-1}, t_k)$, $c = t_0 < t_1 < t_2, ... < t_n = d$, then the nonlocal condition (1.2) will be

$$\sum_{k=1}^{m} (g(t_k) - g(t_{k-1}))\ x(\tau_k) \ = \ x_o.$$

From the continuity of the solution x of the nonlocal problem (1.1)-(1.2) we can obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} (g(t_k) - g(t_{k-1}))\ x(\tau_k) \ = \ \int_c^d x(s)\ dg(s).$$

and the nonlocal condition (2) transformed to the integral one

$$\int_c^d x(s)\ dg(s) \ = \ x_o.$$

Now, we have the following Theorem

Theorem 3.1 Let the assumptions of Theorem 2.1 are satisfied. Then there exists at least one solution $x \in AC[0,1]$ of the nonlocal problem with integral condition,

$$\frac{dx(t)}{dt} \ = \ f(t, x(t), \frac{dx(t)}{dt}),\ a.e,\ t \in \ (0,1],$$

$$\int_c^d x(s)\ dg(s) \ = \ x_o.$$

References

[1] M. Benchohra, E.P. Gatsori and S.K. Ntouyas, Existence results for seme-linear integrodifferential inclusions with nonlocal conditions. Rocky Mountain J. Mat. Vol. 34, No. 3, Fall 2004

[2] M. Benchohra, S. Hamani, S. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Analysis Vol.71 (2009) 23912396

[3] Boucherif, A. First-order differential inclusions with nonlocal initial conditions, Applied Mathematics Letters Vol.15 (2002) 409-414.

[4] A. Boucherif, Nonlocal Cauchy problems for first-order multivalued differential equations, Electronic Journal of Differential Equations Vol. 2002 (2002), No. 47, pp. 1-9.

[5] Boucherif, A and Precup, R. On The nonlocal Initial Value Problem for First Order Differential Equations, Fixed Point Theory,Volume 4,No 2,(2003)205-212.

[6] Boucherif, A. Semilinear evolution inclusions with nonlocal conditions, Applied Mathematics Letters Vol.22 (2009) 1145-1149.

[7] Curtain, R. F. and Pritchard, A. J. Functional Analysis in modern, Applied Mathematics Academic Press (1977).

[8] Dugundji, J. and Grans, A. Fixed Point Theory, Monografie Mathematyczne, PWN, Warsaw (1963).

[9] El-Sayed, A. M. A. and Abd El-Salam, Sh. A. On the stability of a fractional order differential equation with nonlocal initial condtion, EJQTDE, 29(2008)1-8.

[10] El-Sayed, A.M.A and Elkadeky, Kh. W. Caratheodory theorem for a nonlocal problem of the differential equation $x^{'} = f(t, x^{'})$, Alexandria j. of Math. Vol. 1 No. 2 (2010) In press.
[11] Gatsori. E, Ntouyas. S. K, and Sficas. Y.G. On a nonlocal cauchy problem for differential inclusions, Abstract and Applied Analysis (2004) 425-434.
[12] Goebel. K and Kirk W. A. Topics in Metric Fixed point theory, Cambridge University press, Cambridge (1990).
[13] Guerekata, G. M. A Cauchy problem for some fractional abstract differential equation with non local conditions, Nonlinear Analysis 70 (2009) 1873-1876.
[14] Liu. H. and Jiang. D. Two-point boundary value problem for first order implicit differential equations, Hiroshima Math.J.30(2000) 21-27.
[15] Ma. R. Existence and Uniqueness of Solutions to First - Order Three - Point Boundary Value Problems,Applied Mathematics Letters, 15(2002) 211-216.

[1]Department of Mathematics, Alexandria University, Alexandria, Egypt
E-mail address: amasayed@hotmail.com

[2]Department of Mathematics, Alexandria University, Alexandria, Egypt
E-mail address: emanhamdalla@hotmail.com

[3] Department of Mathematics, Faculty of Science, Garyounis University, Benghazi, Libya
E-mail address: k-welkadeky@yahoo.com