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System of implicit nonconvex variationl inequality problems: A projection method approach

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Abstract

In this paper, we consider a new system of implicit nonconvex variational inequality problems in setting of prox-regular subsets of two different Hilbert spaces. Using projection method, we establish the equivalence between the system of implicit nonconvex variational inequality problems and a system of relations. Using this equivalence formulation, we suggest some iterative algorithms for finding the approximate solution of the system of implicit nonconvex variational inequality problems and its special case. Further, we establish some theorems for the existence and iterative approximation of the solutions of the system of implicit nonconvex variational inequality problems and its special case. The results presented in this paper are new and different form the previously known results for nonconvex variational inequality problems. These results also generalize, unify and improve the previously known results of this area.

Keywords: System of implicit nonconvex variational inequality problems, prox-regular set, projection method, iterative algorithm.
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1. Introduction

In 1985, Pang [25] showed that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities and discuss the convergence of method of decomposition for system of variational inequalities. Later, it was noticed that variational inequality over product sets and the system of variational inequalities both are equivalent, see

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for applications [3, 11, 21, 25]. Since then many authors, see for example [2, 10, 11, 13, 18, 21] studied the existence theory of various classes of system of variational inequalities by exploiting fixed-point theorems and minimax theorems. Recently, a number of iterative algorithms based on projection method and its variant forms have been developed for solving various systems of variational inequalities, see for instance [7, 12, 15, 16, 19, 27].

It is well known that the projection method and its variant forms based on projection operator over convex set are important tools for the study of existence and iterative approximation of solutions of various classes (systems) of variational inequality problems in the convexity settings, but these may not be applicable in general, when the sets are nonconvex. To overcome the difficulties that rise from the nonconvexity of underlying sets, the properties of projection operators over uniformly prox-regular sets are used.

In recent years, Bounkhel *et al.* [6], Moudafi [20], Wen [28], Kazmi *et al.* [17], Noor [[23, 24, 22] and the relevant references cited therein], Alimohammady *et al.* [1], Balooee *et al.* [4] suggested and analyzed iterative algorithms for solving some classes (systems) of nonconvex variational inequality problems in the setting of uniformly prox-regular sets.

On the other hand, to the best of our knowledge, the study of iterative algorithms for solving the systems of variational inequality problems considered in [7, 27] in nonconvex setting has not been done so far.

Motivated and inspired by research going on in this area, we introduce a system of implicit nonconvex variational inequality problems (in short, SINVIP) defined on the uniformly prox-regular sets in different two Hilbert spaces. SINVIP is different from those considered in [1, 4, 6, 16, 20, 22, 23, 24, 28] and includes the new and known systems of nonconvex (convex) variational inequality problems as special cases. Using the properties of projection operator over uniformly prox-regular sets, we suggest some iterative algorithms for finding the approximate solution of SINVIP and its an important special case. Further, we establish some theorems for the existence and iterative approximation of the solutions of the SINVIP and its special case. The results presented in this paper are different form the previously known results for nonconvex variational inequality problems. These results also generalize, unify and improve the previously known results of this area. The methods presented in this paper extend, unify and improve the methods presented in [1, 4, 6, 16, 14, 20, 22, 23, 24, 28].

2. Preliminaries

Let H be a real Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let K be a nonempty closed set in H, not necessarily convex.

First, we recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis, see [5, 8, 9, 26].

Definition 2.1. The *proximal normal cone* of K at $u \in H$ is given by

$$N^P(K;u) := \{\xi \in H : u \in P_K(u + \alpha\xi)\},$$

where $\alpha > 0$ is a constant and P_K is projection operator of H onto K, that is,

$$P_K(u) = \{u^* \in K : d_K(u) = \|u - u^*\|\},$$

where $d_K(u)$ is the usual distance function to the subset K, that is,

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

Lemma 2.2. *Let K be a nonempty closed subset of H. Then $\xi \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle \xi, v-u\rangle \ \leq \alpha\|v-u\|^2, \quad \forall v \in K.$$

Definition 2.3. The *Clarke normal cone*, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \bar{\text{co}}[N^P(K;u)],$$

where $\bar{\text{co}}A$ means the closure of the convex hull of A.

Poliquin *et al.* [26] and Clarke *et al.* [8] have introduced and studied a class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important role in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have

Definition 2.4. For a given $r \in (0, \infty]$, a subset K of H is said to be *normalized uniformly r-prox-regular* if and only if every nonzero proximal normal to K can be realized by any r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in \ N_P(K;u)$ with $\|\xi\| = 1$, one has

$$\langle \xi, \ v-u\rangle \ \leq \ \frac{1}{2r}\ \|v-u\|^2, \ \ \forall v \in K.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets, see [8, 9].

It is note that if $r = \infty$, then uniformly r-prox-regularity of K reduces to its convexity.

It is known that if K is a uniformly r-prox-regular set, the proximal normal cone $N_P(K;u)$ is closed as a set-valued mapping. Thus, we have $N_C(K;u) = N_P(K;u)$.

Now, let us state the following proposition which summarizes some important consequences of the uniformly prox-regularities:

Proposition 2.5. *Let $r > 0$ and let K_r be a nonempty closed and uniformly r-prox-regular subset of H. Set $U_r = \{x \in H : d(x, K_r) < r\}$.*

(i) For all $x \in U_r$, $\ P_{K_r}(x) \neq \emptyset$;

(ii) For all $r' \in (0,r)$, P_{K_r} is Lipschitz continuous with constant $\dfrac{r}{r-r'}$ on $U_{r'} = \{x \in H : d(x, K_r) < r'\}$.

3. System of implicit nonconvex variational inequality problems

Throughout the paper unless otherwise stated, we assume that for each $i \in \{1,2\}, K_{i,r_i}$ is a prox-regular subset of H_i. Let $N_i : H_1 \times H_2 \to H_i, \ g_i : H_i \to H_i$ be nonlinear mappings. For any constant $\rho_i > 0$, we consider the problem of finding $(x_1, x_2) \in H_1 \times H_2$ such that $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ and

$$\langle \rho_i N_i(x_1,x_2), y_i - g_i(x_i)\rangle_i + \frac{1}{2r_i}\|y_i - g_i(x_i)\|_i^2 \geq 0, \forall y_i \in K_{i,r_i}. \tag{3.1}$$

We call the problem (3.1), a system of implicit nonconvex variational inequality problems (SINVIP).

Some special cases of SINVIP (3.1):

(1) If $g_i = I_i$, the identity operator on H_i, then SINVIP (3.1) reduces to the system of nonconvex variational inequality problems (in short, SNVIP) of finding $(x_1, x_2) \in K_{1,r_1} \times K_{2,r_2}$ such that

$$\langle \rho_i N_i(x_1, x_2), y_i - x_i \rangle_i + \frac{1}{2r_i} \|y_i - x_i\|_i^2 \geq 0, \quad \forall y_i \in K_{i,r_i}, \tag{3.2}$$

which appears to be new.

(2) If $r_i = +\infty$, that is, $K_{i,r_i} = K_i$, the convex subset of H_i, and $g_i = I_i$, then SINVIP (3.1) reduces the system of variational inequality problems of finding $(x_1, x_2) \in K_{1,r_1} \times K_{2,r_2}$ such that

$$\langle N_i(x_1, x_2), y_i - x_i \rangle_i \geq 0, \quad \forall y_i \in K_i, \tag{3.3}$$

which has been considered and studied in [2, 27] using different approaches.

The following definitions are needed in the proof of main result.

Definition 3.1. A nonlinear mapping $g_1 : H_1 \to H_1$ is said to be k_1-*strongly monotone* if there exists a constant $k_1 > 0$ such that

$$\langle g_1(x_1) - g_1(y_1), x_1 - y_1 \rangle_1 \ \geq \ k_1 \|x_1 - y_1\|_1^2, \ \forall x_1, y_1 \in H_1.$$

Definition 3.2. Let $N_1 : H_1 \times H_2 \to H_1$ be a nonlinear mapping. Then N_1 is said to be

(i) δ_1-*strongly monotone* in the first argument if there exists a constant $\delta_1 > 0$ such that

$$\langle N_1(x_1, x_2) - N_1(y_1, x_2), x_1 - y_1 \rangle_1 \ \geq \ \delta_1 \|x_1 - y_1\|_1^2,$$

$$\forall x_1, y_1 \in H_1, \ x_2 \in H_2;$$

(iii) $(L_{(N_1,1)}, L_{(N_1,2)})$-*mixed Lipschitz continuous* if there exist constants $L_{(N_1,1)}, L_{(N_1,2)} > 0$ such that

$$\|N_1(x_1, x_2) - N_1(y_1, y_2)\|_1 \leq L_{(N_1,1)} \|x_1 - y_1\|_1 + L_{(N_1,2)} \|x_2 - y_2\|_2,$$

$$\forall x_1, y_1 \in H_1, \ x_2, y_2 \in H_2.$$

Similarly to Definition 3.2(i), we can define the strongly monotonicity of $N_2 : H_1 \times H_2 \to H_2$ in the second argument.

First we prove the following technical Lemmas.

Lemma 3.3. *SINVIP (3.1) is equivalent to the following system of implicit nonconvex variational inclusions of finding* $(x_1, x_2) \in H_1 \times H_2$ *such that* $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ *and*

$$\mathbf{0}_i \in \rho_i N_i(x_1, x_2) + N^p_{K_{i,r_i}}(g_i(x_i)), \tag{3.4}$$

for $\rho_i > 0, i = 1, 2,$ *where* $N^p_{K_{i,r_i}}(u)$ *denotes the proximal normal cone of* K_{i,r_i} *at* u *in the sense of nonconvex analysis and* $\mathbf{0}_i$ *is the zero vector in* H_i.

Proof. Let $(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ be a solution of SINVIP (3.1). If $N_i(x_1, x_2) = \mathbf{0}_i$, evidently the inclusion (3.4) follows for each $i = 1, 2$. If $N_i(x_1, x_2) \neq \mathbf{0}_i$, from (3.1) and Lemma 2.2, we get the inclusion (3.4) for each $i = 1, 2$. Converse part directly follows from Definition 2.4. □

Lemma 3.4. *$(x_1, x_2) \in H_1 \times H_2$ with $(g_1(x_1), g_2(x_2)) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SINVIP (3.1) if and only if it satisfies the system of relations*

$$g_i(x_i) = P_{K_{i,r_i}}\Big[g_i(x_i) - \rho_i N_i(x_1, x_2)\Big], \tag{3.5}$$

for $\rho_i > 0, i = 1, 2,$ where $P_{K_{i,r_i}}$ is the projection operator of H_i onto the prox-regular set K_{i,r_i}.

Proof. The proof is directly followed from Lemma 3.3 and from the fact that $P_{K_{i,r_i}} = (I_i + N^p_{K_{i,r_i}})^{-1}$. □

The alternative formulation (3.5) of SINVIP (3.1) enable us to suggest and analyze the following iterative algorithm for solving SINVIP (3.1).

Iterative algorithm 3.5. *For each $i = 1, 2$, given $z_i^0 \in H_i$, compute the iterative sequences $\{z_i^n\}$, $\{x_i^n\}$ defined by the iterative schemes:*

$$g_i(x_i^n) = P_{K_{i,r_i}}(z_i^n), \tag{3.6}$$

$$z_i^{n+1} = g_i(x_i^n) - \rho_i N_i(x_1^n, x_2^n), \tag{3.7}$$

for all $n = 0, 1, 2, \ldots..,$ and $\rho_i > 0$.

Remark 3.6. We observe that $\{g_i(x_i^n)\}_{n=0}^{\infty} \subset K_{i,r_i}$ for each $i = 1, 2$.

Taking $g_i = I_i$ in Iterative algorithm 3.5, we have the following iterative algorithm for solving SNVIP (3.2).

Iterative algorithm 3.7. *For each $i = 1, 2$, given $z_i^0 \in H_i$, compute the iterative sequences $\{z_i^n\}$, $\{x_i^n\}$ defined by the iterative schemes:*

$$x_i^n = P_{K_{i,r_i}}(z_i^n),$$

$$z_i^{n+1} = x_i^n - \rho_i N_i(x_1^n, x_2^n),$$

for all $n = 0, 1, 2, \ldots,$ and $\rho_i > 0$.

Now, we prove some theorems concerning the existence and iterative approximation of solutions of SINVIP (3.1) and SNVIP (3.2).

Theorem 3.8. *For each $i \in \{1, 2\}$, let K_{i,r_i} be closed and uniformly prox-regular subsets of real Hilbert space H_i; let $N_i : H_1 \times H_2 \to H_i$ be δ_i-strongly monotone in i^{th} argument and $(L_{(N_i,1)}, L_{(N_i,2)})$-mixed Lipschitz continuous and let $g_i : H_i \to H_i$ be k_i-strongly monotone and continuous. For each $i \in \{1, 2\}$ and $j \in \{1, 2\}\backslash\{i\}$, if the constant $\rho_i > 0$ satisfies the following condition:*

$$M_i - \Delta_i < \rho_i < \min\Big\{M_i + \Delta_i, \psi_i, \frac{r_i'}{1 + \|N_i(x_1^n, x_2^n)\|_i}, \frac{r_i'}{1 + \|N_i(x_1^{n+1}, x_2^{n+1})\|_i}\Big\}, \tag{3.8}$$

where

$$\Delta_i := \frac{\sqrt{(b_ik_i - a_ie_i)^2 - b_i^2(1 - e_i^2)(1 - a_i^2)}}{b_i^2(1 - e_i^2)}$$

$$M_i := \frac{b_ik_i - a_ie_i}{b_i^2(1 - e_i^2)}; \;\; a_i := \frac{1}{\mu_i} - \phi_i; \quad \phi_i := b_i\rho_jL_{(N_j,i)};$$

$$\psi_j < \frac{1}{\mu_ib_iL_{(N_j,i)}}; \;\; b_i := \frac{1}{\sqrt{2k_i + 3}}; \;\; \mu_i := \frac{r_i}{r_i - r_i'};$$

$$e_i := \sqrt{1-2\delta_i + L^2_{(N_i,i)}}; \quad b_ik_i > a_ie_i + b_i\sqrt{(1-e_i^2)(1-a_i^2)}$$

$$\frac{\rho_i^2 - 2\rho_ik_i}{2k_i+3} \in [-1,0); \; r_i' \in (0,r_i); \; r_i \in (0,\infty].$$

Then the sequences $\{x_i^n\}$ and $\{z_i^n\}$ generated by Iterative algorithm 3.5 converge strongly to $x_i, z_i \in H_i$, respectively, where $(x_1,x_2) \in H_1 \times H_2$ with $g_i(x_i) \in K_{i,r_i}$ is a solution of SINVIP (3.1).

Proof. From (3.6), we have

$$\begin{aligned}\|z_i^{n+2} - z_i^{n+1}\|_i &= \|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(N_i(x_1^{n+1},x_2^{n+1}) - N_i(x_1^n,x_2^n))\|_i \\ &\leq \|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(x_i^{n+1}-x_i^n)\|_i \\ &+ \rho_i\|N_i(x_1^{n+1},x_2^{n+1}) - N_i(x_1^n,x_2^n) - (x_i^{n+1}-x_i^n)\|_i. \end{aligned} \tag{3.9}$$

Since N_i is δ_i-strongly monotone in i^{th} argument and $(L_{(N_i,1)}, L_{(N_i,2)})$-mixed Lipschitz continuous, we have the following estimates:

$$\begin{aligned}&\|N_1(x_1^{n+1},x_2^{n+1}) - N_1(x_1^n,x_2^n) - (x_1^{n+1}-x_1^n)\|_1 \\ &\leq \|N_1(x_1^{n+1},x_2^{n+1}) - N_1(x_1^n,x_2^{n+1}) - (x_1^{n+1}-x_1^n)\|_1 \\ &+\|N_1(x_1^n,x_2^{n+1}) - N_1(x_1^n,x_2^n)\|_1 \\ &\leq \sqrt{1-2\delta_1 + L^2_{(N_1,1)}}\|x_1^{n+1}-x_1^n\|_1 + L_{(N_1,2)}\;\|x_2^{n+1}-x_2^n\|_2. \end{aligned} \tag{3.10}$$

$$\begin{aligned}&\|(N_2(x_1^{n+1},x_2^{n+1}) - N_2(x_1^n,x_2^n) - (x_2^{n+1}-x_2^n)\|_2 \\ &\leq \|(N_2(x_1^{n+1},x_2^{n+1}) - N_2(x_1^{n+1},x_2^n) - (x_2^{n+1}-x_2^n)\|_2 \\ &+\|(N_2(x_1^{n+1},x_2^n) - N_2(x_1^n,x_2^n)\|_2 \\ &\leq \sqrt{1-2\delta_2 + L^2_{(N_2,2)}}\|x_2^{n+1}-x_2^n\|_2 + L_{(N_2,1)}\;\|x_1^{n+1}-x_1^n\|_1. \end{aligned} \tag{3.11}$$

Next it follows from the conditions (3.8) on ρ_i that $z_i^n \in K_{i,r_i'}$ for all $n = 1,2,\ldots$. Hence using the k_i-strongly monotonicity of g_i, Proposition 2.5 and (3.6), we have the following estimates.

$$\begin{aligned}\|x_i^{n+1}-x_i^n\|_i^2 &\leq \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 - 2\langle g_i(x_i^{n+1}) - g_i(x_i^n) + x_i^{n+1} - x_i^n, x_i^{n+1}-x_i^n\rangle_i \\ &= \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 - 2\langle g_i(x_i^{n+1}) - g_i(x_i^n), x_i^{n+1}-x_i^n\rangle_i \\ &-2\langle x_i^{n+1}-x_i^n, x_i^{n+1}-x_i^n\rangle_i. \\ &\leq \mu_i^2\|z_i^{n+1}-z_i^n\|_i^2 - (2k_i+2)\|x_i^{n+1}-x_i^n\|_i^2. \end{aligned}$$

or

$$\|x_i^{n+1}-x_i^n\|_i \leq \frac{\mu_i}{\sqrt{2k_i+3}}\|z_i^{n+1}-z_i^n\|_i, \tag{3.12}$$

where $\mu_i = \frac{r_i}{r_i - r_i'}$.

$$\begin{aligned}&\|g_i(x_i^{n+1}) - g_i(x_i^n) - \rho_i(x_i^{n+1}-x_i^n)\|_i^2 \\ &\leq \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i^2 \\ &-2\rho_i\langle g_i(x_i^{n+1}) - g_i(x_i^n), x_i^{n+1}-x_i^n\rangle + \rho_i^2\|x_i^{n+1}-x_i^n\|_i^2 \\ &\leq \mu_i^2\|z_i^{n+1}-z_i^n\|_i^2 + (\rho_i^2 - 2\rho_ik_i)\|x_i^{n+1}-x_i^n\|_i^2. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\|g_i(x_i^{n+1})-g_i(x_i^n)-\rho_i((x_i^{n+1}-x_i^n)\|_i \le \mu_i\sqrt{1+\frac{\rho_i^2-2\rho_ik_i}{2k_i+3}}\,\|z_i^{n+1}-z_i^n\|_i. \tag{3.14}$$

Now, from (3.9)-(3.11) and (3.14), we have

$$\begin{aligned}\|z_i^{n+2}-z_i^{n+1}\|_i &\le \mu_i\sqrt{1+\frac{\rho_i^2-2\rho_ik_i}{2k_i+3}}\,\|z_i^{n+1}-z_i^n\|_i \\ &+\rho_i\Big[\frac{\mu_i}{\sqrt{2k_i+3}}\Big(\sqrt{1-2\delta_i+L^2_{(N_i,i)}}\Big)\,\|z_i^{n+1}-z_i^n\|_i+\frac{\mu_jL_{(N_i,j)}}{\sqrt{2k_j+3}}\|z_j^{n+1}-z_j^n\|_j\Big], \end{aligned} \tag{3.15}$$

for each $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$.

Define norm $\|\cdot\|_*$ on $H_1\times H_2$ by $\|(x_1,x_2)\|_* = \sum_{i=1}^{2}\|x_i\|_i$ for all $(x_1,x_2)\in H_1\times H_2$. We note that $(H_1\times H_2,\|\cdot\|_*)$ is a Banach space.

It follows from (3.15) that

$$\begin{aligned}\|(z_1^{n+2},z_2^{n+2})-(z_1^{n+1},z_2^{n+1})\|_* &= \sum_{i=1}^{2}\|z_i^{n+2}-z_i^{n+1}\|_i \\ &\le \theta\|(z_1^{n+1},z_2^{n+1})-(z_1^n,z_2^n)\|_*, \end{aligned} \tag{3.16}$$

where $\theta := \max\{\theta_1,\theta_2\}$; $\theta_i := \mu_i[p_i + b_i(\rho_ie_i+\rho_jd_j)]$;

$$p_i := \Big[1+\frac{\rho_i^2-2\rho_ik_i}{2k_i+3}\Big]^{\frac{1}{2}};\quad e_i := \sqrt{1-2\delta_i+L^2_{(N_i,i)}};$$

$$b_i := \frac{1}{\sqrt{2k_i+3}};\quad d_j := L_{(N_j,i)}.$$

From condition (3.8), we have $0<\theta<1$. Hence it follows from (3.16) that $\{(z_1^n,z_2^n)\}$ is a Cauchy sequence in $H_1\times H_2$. Assume that $(z_1^n,z_2^n)\to(z_1,z_2)$ in $H_1\times H_2$ as $n\to\infty$, that is, for each $i=1,2,\ z_i^n\to z_i$ in H_i as $n\to\infty$. Hence, we observe from (3.12) that $\{x_i^n\}$ is also a Cauchy sequence and hence assume that $x_i^n\to x_i$ in H_i as $n\to\infty$ for each $i=1,2$. Further the continuity of N_i, g_i and $P_{K_{i,r_i}}$ and Iterative algorithm 3.5 imply that

$$g_i(x_i) = P_{K_{i,r_i}}\Big[g_i(x_i)-\rho_iN_i(x_1,x_2)\Big].$$

Now it follows from Lemma 3.4 that $(x_1,x_2)\in H_1\times H_2$ with $g_i(x_i)\in K_{i,r_i}$ is a solution of SINVIP (3.1). This completes the proof. □

Theorem 3.9. *For each $i\in\{1,2\}$ and $j\in\{1,2\}\backslash\{i\}$, let K_{i,r_i}, N_i and g_i be same as in Theorem 3.8, and let the constant $\rho_i>0$ satisfy the condition:*

$$M_i-\Delta_i<\rho_i<\min\Big\{M_i+\Delta_i,\psi_i,\frac{r_i'}{1+\|N_i(x_1^n,x_2^n)\|_i},\frac{r_i'}{1+\|N_i(x_1^{n+1},x_2^{n+1})\|_i}\Big\}, \tag{3.17}$$

where

$$\Delta_i := \frac{\sqrt{\delta_i^2-L^2_{(N_i,i)}(1-a_i^2)}}{L^2_{(N_i,i)}};\quad a_i := \frac{1}{\mu_ib_i}-(2+\phi_i);\quad \phi_i := \rho_jL_{(N_i,i)};$$

$$M_i := \frac{\delta_i}{L^2_{(N_i,i)}}; \quad \psi_i < \frac{1}{\mu_j b_j d_i}; \quad \delta_i > L_{(N_i,i)}\sqrt{(1-a_i^2)}; \quad a_i^2 < 1;$$

$$b_i := \frac{1}{\sqrt{2k_i+3}}; \quad \mu_i := \frac{r_i}{r_i - r_i'}; \quad r_i' \in (0, r_i); \quad r_i \in (0, \infty].$$

Then the sequences $\{x_i^n\}$ and $\{z_i^n\}$ generated by Iterative algorithm 3.5 converge strongly to $x_i, z_i \in H_i$, respectively, where $(x_1, x_2) \in H_1 \times H_2$ with $g_i(x_i) \in K_{i,r_i}$ is a solution of SINVIP (3.1).

Proof. From (3.6), we have

$$\|z_i^{n+2} - z_i^{n+1}\|_i \leq \|g_i(x_i^{n+1}) - g_i(x_i^n) - (x_i^{n+1} - x_i^n)\|_i + \|(x_1^{n+1} - x_2^n) - \rho_i(N_i(x_1^{n+1}, x_2^{n+1}) - N_i(x_1^n, x_2^n))\|_i. \tag{3.18}$$

Using the same arguments used in proof of Theorem 3.8, we have the following estimates:

$$\|(x_1^{n+1} - x_1^n) - \rho_1(N_1(x_1^{n+1}, x_2^{n+1}) - N_1(x_1^n, x_2^n))\|_1 \leq \sqrt{1 - 2\rho_1\delta_1 + \rho_2 L^2_{(N_1,1)}}\ \|x_1^{n+1} - x_1^n\|_1 + \rho_1 L_{(N_1,2)}\|x_2^{n+1} - x_2^n\|_2. \tag{3.19}$$

$$\|(x_2^{n+1} - x_2^n) - \rho_2(N_2(x_1^{n+1}, x_2^{n+1}) - N_2(x_1^n, x_2^n))\|_2 \leq \sqrt{1 - 2\rho_2\delta_2 + \rho_2 L^2_{(N_2,2)}}\ \|x_2^{n+1} - x_2^n\|_2 + \rho_2 L_{(N_2,1)}\|x_1^{n+1} - x_1^n\|_1. \tag{3.20}$$

$$\|x_i^{n+1} - x_i^n\|_i \leq \mu_i b_i \|z_i^{n+1} - z_i^n\|_i, \tag{3.21}$$

where

$$\mu_i = \frac{r_i}{r_i - r_i'}; \quad b_i := \frac{1}{\sqrt{2k_i+3}}.$$

and

$$\|g_i(x_i^{n+1}) - g_i(x_i^n) - (x_i^{n+1} - x_i^n)\|_i \leq 2\mu_i b_i \|z_i^{n+1} - z_i^n\|_i. \tag{3.22}$$

Now, from (3.17)-(3.22), we have

$$\|z_i^{n+2} - z_i^{n+1}\|_i \leq 2\mu_i b_i\|z_i^{n+1} - z_i^n\|_i + \mu_i b_i e_i \|z_i^{n+1} - z_i^n\|_i + \rho_i \mu_j b_j L_{(N_i,j)}\|z_j^{n+1} - z_j^n\|_j, \tag{3.23}$$

where $e_i := \sqrt{1 - 2\rho_i\delta_i + \rho_i^2 L^2_{(N_i,i)}}$ for each $i \in \{1, 2\}$ and $j \in \{1, 2\}\backslash\{i\}$.

It follows from (3.23) that.

$$\|(z_1^{n+2}, z_2^{n+2}) - (z_1^{n+1}, z_2^{n+1})\|_* \leq \theta\|(z_1^{n+1}, z_2^{n+1}) - (z_1^n, z_2^n)\|_*, \tag{3.24}$$

where $\theta := \max\{\theta_1, \theta_2\}$ and $\theta_i := b_i\mu_i[2 + e_i + \rho_j d_j)]$; $d_j := L_{(N_j,i)}$.

From condition (3.17), we have $0 < \theta < 1$. Hence it follows from (3.24) that $\{(z_1^n, z_2^n)\}$ is a Cauchy sequence in $H_1 \times H_2$. The rest of the proof is same as the proof of Theorem 3.8. This is completes the proof. □

In the following theorem, we have relaxed the condition of strongly monotonicity from N_i.

Theorem 3.10. *For each $i \in \{1,2\}$, let K_{i,r_i} be closed and uniformly prox-regular subsets of real Hilbert space H_i; let N_i be $(L_{(N_i,1)}, L_{(N_i,2)})$-mixed Lipschitz continuous and let g_i be k_i-strongly monotone and continuous. For each $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$, if the constant $\rho_i > 0$ satisfies the condition:*

$$0 < \rho_i < \min\Big\{\Delta_i, \psi_i, \frac{r_i'}{1 + \|N_i(x_1^n, x_2^n)\|_i}, \frac{r_i'}{1 + \|N_i(x_1^{n+1}, x_2^{n+1})\|_i}\Big\}, \tag{3.25}$$

where

$$\Delta_i := \frac{1}{L_{(N_i,i)}}\left[\frac{1}{\mu_i b_i} - (1+\phi_i)\right] > 0; \quad \psi_i < \frac{1}{\mu_j b_j L_{(N_i,j)}}; \quad \phi_i := \rho_j L_{(N_j,i)}$$

$$b_i := \frac{1}{\sqrt{2k_i + 3}}; \quad \mu_i := \frac{r_i}{r_i - r_i'}; \quad r_i' \in (0, r_i); \quad r_i \in (0, \infty].$$

Then the sequences $\{x_i^n\}$ and $\{z_i^n\}$ generated by Iterative algorithm 3.5 converge strongly to $x_i, z_i \in H_i$, respectively, where $(x_1, x_2) \in H_1 \times H_2$ with $g_i(x_i) \in K_{i,r_i}$ is a solution of SINVIP (3.1).

Proof. Since N_i is $(L_{(N_i,1)}, L_{(N_i,2)})$-mixed Lipschitz continuous, we have

$$\begin{aligned}\|z_i^{n+2} - z_i^{n+1}\|_i &\le \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i + \rho_i\|N_i(x_1^{n+1}, x_2^{n+1}) - N_i(x_1^n, x_2^n)\|_i \\ &\le \|g_i(x_i^{n+1}) - g_i(x_i^n)\|_i + \rho_i\Big(L_{(N_i,1)}\|x_1^{n+1} - x_1^n\|_1 + L_{(N_i,2)}\|x_2^{n+1} - x_2^n\|_2\Big).\end{aligned}$$

Hence it follows from (3.21) that

$$\|z_i^{n+2} - z_i^{n+1}\|_i \le \mu_i b_i\|z_i^{n+1} - z_i^n\|_i + \rho_i L_{(N_i,1)}\mu_1 b_1\|z_1^{n+1} - z_1^n\|_1 + \rho_i L_{(N_i,2)}\mu_2 b_2\|z_2^{n+1} - z_2^n\|_2, \tag{3.26}$$

where

$$\mu_i = \frac{r_i}{r_i - r_i'}; \quad b_i := \frac{1}{\sqrt{2k_i + 3}},$$

which gives

$$\|(z_1^{n+2}, z_2^{n+2}) - (z_1^{n+1}, z_2^{n+1})\|_* \le \theta\|(z_1^{n+1}, z_2^{n+1}) - (z_1^n, z_2^n)\|_*, \tag{3.27}$$

where $\theta := \max\{\theta_1, \theta_2\}$ and $\theta_i := b_i\mu_i(1 + \rho_i L_{(N_i,i)} + \rho_j L_{(N_j,j)})$.

From condition (3.25), we have $0 < \theta < 1$. and hence (3.27) implies $\{(z_1^n, z_2^n)\}$ is a Cauchy sequence in $H_1 \times H_2$. The rest of the proof is same as the proof of Theorem 3.8. This is completes the proof. □

Finally we prove the following theorem for the existence and iterative approximation of solutions of SNVIP (3.2).

Theorem 3.11. *For each $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$, let K_{i,r_i} be closed and uniformly prox-regular subset of real Hilbert space H_i; let N_i be δ_i-strongly monotone in i^{th} argument and $(L_{(N_i,1)}, L_{(N_i,2)})$-mixed Lipschitz continuous. If $\rho_i > 0$ satisfies the following condition:*

$$\frac{\delta_i}{L^2_{(N_i,i)}} - \Delta_i < \rho_i < \min\left\{\frac{\delta_i}{L^2_{(N_i,i)}} + \Delta_i, \psi_i, \frac{r_i'}{1 + \|N_i(x_1^n, x_2^n)\|_i}, \frac{r_i'}{1 + \|N_i(x_1^{n+1}, x_2^{n+1})\|_i}\right\}, \tag{3.28}$$

where

$$\Delta_i := \frac{\sqrt{\delta_i^2 - L^2_{(N_i,i)}(1 - a_i^2)}}{L^2_{(N_i,i)}}; \quad \delta_i > L_{(N_i,i)}\sqrt{(1 - a_i^2)}; \quad a_i^2 := \frac{1}{\mu_i} - \phi_i; \quad a_i^2 < 1;$$

$$\phi_i := \rho_j L_{(N_j,i)}; \quad \psi_i < \frac{1}{\mu_j L_{(N_i,j)}}; \quad \mu_i := \frac{r_i}{r_i - r_i'}; \quad r_i' \in (0, r_i); \quad r_i \in (0, \infty].$$

Then the sequences $\{x_i^n\}$ and $\{z_i^n\}$ generated by Iterative algorithm 3.7 converge strongly to $x_i, z_i \in H_i$, respectively, where $(x_1, x_2) \in K_{1,r_1} \times K_{2,r_2}$ is a solution of SNVIP (3.2).

Proof. We easily observe that the condition (3.28) on ρ_i assures that $z_i^n \in K_{i,r'_i}$ for all $n = 1, 2, \dots$.

From Iterative algorithm 3.7 and Proposition 2.5, we have

$$\|x_i^{n+1} - x_i^n\|_i \leq \mu_i \|z_i^{n+1} - z_i^n\|_i, \tag{3.29}$$

where $\mu_i = \frac{r_i}{r_i - r'_i}$ and

$$\|z_i^{n+2} - z_i^{n+1}\|_i = \|x_i^{n+1} - x_i^n - \rho_i(N_i(x_1^{n+1}, x_2^{n+1}) - N_i(x_1^n, x_2^n))\|_i. \tag{3.30}$$

Since N_i is δ_i-strongly monotone in i^{th} argument and $(L_{(N_i,1)}, L_{(N_i,2)})$-mixed Lipschitz continuous, from (3.19),(3.20),(3.29) and (3.30), we have

$$\|z_i^{n+2} - z_i^{n+1}\|_i \leq \mu_i e_i \|z_i^{n+1} - z_i^n\|_i + \rho_i \mu_j L_{(N_i,j)} \|z_j^{n+1} - z_j^n\|_j, \tag{3.31}$$

where $e_i := \sqrt{1 - 2\rho_i \delta_i + \rho_i^2 L_{(N_i,i)}^2}$ for each $i \in \{1, 2\}$ and $j \in \{1, 2\} \backslash \{i\}$.

It follows from (3.31) that

$$\begin{aligned} \|(z_1^{n+2}, z_2^{n+2}) - (z_1^{n+1}, z_2^{n+1})\|_* &= \sum_{i=1}^{2} \|z_i^{n+2} - z_i^{n+1}\|_i \\ &\leq \theta \|(z_1^{n+1}, z_2^{n+1}) - (z_1^n, z_2^n)\|_*, \end{aligned} \tag{3.32}$$

where $\theta := \max\{\theta_1, \theta_2\}$; $\theta_i := \mu_i(e_i + \rho_j d_j)$; $d_j := L_{(N_j,i)}$.

From condition (3.28), we have $0 < \theta < 1$. Hence it follows from (3.32) that $\{(z_1^n, z_2^n)\}$ is a Cauchy sequence in $H_1 \times H_2$. The rest of the proof is same as the proof of Theorem 3.8. This is completes the proof. □

Remark 3.12. (i) The methods presented in this paper extend and unify the methods considered in [14, 15, 16, 17, 18, 20, 21, 23, 24] to the systems of nonconvex variational inequality problems defined on the product of two different Hilbert spaces.

(ii) The methods presented in this paper improve the methods considered in [21, 23, 24] in the sense that the continuity of g is required instead of the Lipschitz continuity.

(iii) One needs further research effort to extend the methods presented for solving the systems of nonconvex variational inequality problems involving set-valued mappings.

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