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Caristi's fixed point theorem on C^*-algebra valued metric spaces

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Abstract

We present the extension of Caristi's fixed point theorem for mappings defined on C^*-algebra valued metric spaces. We prove the existence of fixed point using the concept of minimal element in C^*-algebra valued metric space by introducing the notion of partial order on X.

Keywords: Caristi's theorem, C^*-algebra, metric space.
2010 MSC: 47H10, 54H25.

1. Introduction

This work is motivated by the recent work on extension of Banach contraction principle on C^*-algebra valued metric spaces, which has been done by Ma *et al.* [6]. Caristi's fixed point theorem [5] is a beautiful extension of Banach Contraction principle [1]. This theorem states that if (X, d) is a complete metric space then a mapping $T\colon X \to X$ has a fixed point if there exist a lower semi continuous map $\phi\colon X \to [0, \infty)$ such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \text{ for each } x \in X.$$

The proof of Caristi's theorem varies and uses different directions and techniques see [5], [4]. It is also worth mentioning that because of Caristi theorem's close connection with Ekeland's variational principle

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[3] many authors refer it to as Caristi-Ekeland fixed point result. Several authors extended this result on different type of distance spaces for example in [4]. Khamsi gave a characterization to the existence of minimal element in a partially ordered set in terms of fixed point of multivalued map. Then he showed how Caristi's theorem may be characterized.

In this paper we introduce the notion of lower semi continuity in the context of C^*-algebra valued metric spaces [6]. Also, we define a partial order on C^*-algebra valued metric space. Taking advantage offered by this framework, we extend the Caristi's fixed point theorem in context of C^*-algebra valued metric space.

2. Preliminaries

We now recollect some basic definitions, notations, and results that will be used subsequently. For the details, we refer to [2], [7].

An algebra $\mathbb{A}$ together with a conjugate linear involution map $*\colon \mathbb{A} \to \mathbb{A}$ defined by $a \mapsto a^*$ such that for all $a, b \in \mathbb{A}$ we have $(ab)^* = b^*a^*$ and $(a^*)^* = a$, is called a $*$-algebra. Moreover, if $\mathbb{A}$ contains an identity element $1_{\mathbb{A}}$ then the pair $(\mathbb{A}, *)$ is called a unital $*$-algebra. A unital $*$-algebra $(\mathbb{A}, *)$ together with a complete sub multiplicative norm satisfying $\|a^*\| = \|a\|$ for all $a \in \mathbb{A}$ is called a Banach $*$-algebra. A C^*-algebra is a Banach $*$-algebra $(\mathbb{A}, *)$ such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathbb{A}$. An element $a \in \mathbb{A}$ is called a positive element if $a = a^*$ and $\sigma(a) \subset \mathbb{R}_+$, where $\sigma(a) = \{\lambda \in \mathbb{R} : \lambda 1_{\mathbb{A}} - a \text{ is non invertible}\}$. If $a \in \mathbb{A}$ is positive, we write it as $a \succeq 0_{\mathbb{A}}$. Using positive elements, one can define a partial ordering on $\mathbb{A}$ as follows: $b \succeq a$ if and only if $b - a \succeq 0_{\mathbb{A}}$. Each positive element a of a C^*-algebra $\mathbb{A}$ has a unique positive square root. Subsequently, $\mathbb{A}$ will denote a unital C^*-algebra with the identity element $1_{\mathbb{A}}$. Further, $\mathbb{A}_+$ is the set $\{a \in \mathbb{A} : a \succeq 0_{\mathbb{A}}\}$ of positive elements of $\mathbb{A}$ and $(a^*a)^{1/2} = |a|$.

Using the concept of positive elements in $\mathbb{A}$, a C^*-algebra valued metric space is defined in the following way [6].

Definition 2.1. Let X be a non-empty set. A C^*-algebra valued metric on X is a mapping $d\colon X \times X \to \mathbb{A}$ satisfying the following conditions:

(I) $0_{\mathbb{A}} \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_{\mathbb{A}} \Leftrightarrow x = y$;

(II) $d(x, y) = d(y, x)\ \forall x, y \in X$;

(III) $d(x, y) \preceq d(x, z) + d(z, y)\ \forall x, y, z \in X$.

The triplet $(X, \mathbb{A}, d)$ is called a C^*-algebra valued metric space.

Remark 2.2. The set of positive elements in a C^*-algebra forms a positive cone in the C^*-algebra but the underlying vector space need not to be, in general, a real vector space. For example, let $\mathbb{A}$ be the set of all 2×2 matrices having entries from complex numbers, then $\mathbb{A}$ is a vector space over the field of complex numbers. Therefore, the notion of a C^*-valued metric space seems to be general than the notion of a cone metric space.

Definition 2.3. A sequence $\{x_n\}$ in $(X, \mathbb{A}, d)$ is said to converges to $x \in X$ with respect to $\mathbb{A}$, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|d(x_n, x)\| < \epsilon$, for all $n > N$. We write it as $\lim_{n\to\infty} x_n = x$.

Definition 2.4. A sequence $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$ if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|d(x_n, x_m)\| < \epsilon$ for all $n, m > N$.

The triplet $(X, \mathbb{A}, d)$ is said to be a complete C^*-algebra valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

Definition 2.5. Let$(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space.

A mapping $T\colon X \to X$ is said to be a C^*-valued contraction [6] mapping on X if there exist an $a \in \mathbb{A}$ with $\|a\| < 1$ such that

$$d(Tx, Ty) \preceq a^* d(x, y) a, \quad \text{for all } x, y \in X. \tag{2.1}$$

3. Main Results

We begin this section by introducing the notion of lower semi continuity in the context of C^*-algebra valued metric spaces.

Definition 3.1. Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space. A mapping $\phi\colon X \to \mathbb{A}$ is said to be lower semi continuous at x_0 with respect to $\mathbb{A}$ if

$$\|\phi(x_0)\| \leq \lim_{x \to x_0} \inf \|\phi(x)\|.$$

Example 3.2. Let $X = [-1, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be the C^*-algebra with $\|(a_1, a_2)\| = \sqrt{|a_1|^2 + |a_2|^2}$. Define an order $\preceq$ on $\mathbb{A}$ as follows:

$$(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \leq x_2 \text{ and } y_1 \leq y_2,$$

where "$\leq$" is the usual order on the elements of $\mathbb{R}$. It is easy to see that $\preceq$ is a partial order on $\mathbb{A}_+$. Consider $d\colon X \times X \to \mathbb{A}$ defined by $d(x, y) = (|x - y|, 0)$,then clearly $(X, \mathbb{A}, d)$ is a C^*-algebra valued metric space. Define a map

$$\phi\colon X \to \mathbb{A}, \quad \phi(x) = \begin{cases} (\frac{x}{2}, 0) & \text{if } x \geq 0 \\ (1, 0) & \text{otherwise.} \end{cases}$$

Then it is easy to see that ϕ is lower semi continuous at $x_0 = 0$.

It is straightforward to prove the following lemma.

Lemma 3.3. *Let $(X, \mathbb{A}, d)$ be a C^*-algebra valued metric space and let $\phi\colon X \to \mathbb{A}_+$ be a map. Define the order $\preceq_\phi$ on X by*

$$x \preceq_\phi y \Longleftrightarrow d(x, y) \preceq \phi(y) - \phi(x) \text{ for any } x, y \in X. \tag{3.1}$$

Then $\preceq_\phi$ is a partial order on X.

Theorem 3.4. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi-continuous map. Then $(X, \preceq_\phi)$ has a minimal element, where $\preceq_\phi$ is defined by* (3.1).

Proof. Let $x_1 \succeq_\phi x_2 \succeq_\phi x_3 \succeq_\phi \cdots$ be a non-increasing sequence in X, then from (3.1) we have

$$o_{\mathbb{A}} \preceq d(x_2, x_1) \preceq \phi(x_1) - \phi(x_2), \ \ o_{\mathbb{A}} \preceq d(x_3, x_2) \preceq \phi(x_2) - \phi(x_3), \ \ldots$$

$$\Rightarrow \ \phi(x_1) \succeq \phi(x_2) \succeq \phi(x_3) \succeq \ldots.$$

Hence $\{\phi(x_\alpha) : \alpha \in I\}$ is a decreasing chain in $\mathbb{A}_+$, where I is an indexing set.

Let $\{\alpha_n\}$ be an increasing sequence of elements from the indexing set I such that

$$\lim_{n \to \infty} \phi(x_{\alpha_n}) = \inf\{\phi(x_\alpha) \ : \ \alpha \in I\}. \tag{3.2}$$

Take $m > n$ then $x_{\alpha_n} \succeq_\phi x_{\alpha_m}$. It follows from (3.1) that

$$d(x_{\alpha_m}, x_{\alpha_n}) \preceq \phi(x_{\alpha_n}) - \phi(x_{\alpha_m})$$

$$\Rightarrow \|d(x_{\alpha_m}, x_{\alpha_n})\| \leq \|\phi(x_{\alpha_n}) - \phi(x_{\alpha_m})\|,$$

taking $\lim n \to \infty$ then together with (3.2), it further implies that

$$\begin{aligned} \lim_{n \to \infty} \|d(x_{\alpha_m}, x_{\alpha_n})\| &\leq \lim_{n \to \infty} \|\phi(x_{\alpha_n}) - \phi(x_{\alpha_m})\| \\ &= \|\inf\{\phi(x_\alpha) \ : \ \alpha \in I\} - \inf\{\phi(x_\alpha) \ : \ \alpha \in I\}\| \\ &= 0. \end{aligned}$$

Therefore $\{x_{\alpha_n}\}$ is a Cauchy sequence in X by (2.4). As X is complete, there exists $x \in X$ such that $x_{\alpha_n} \to x$. Since $\{x_{\alpha_n}\}$ is decreasing chain in X, it follows that $x \preceq_\phi x_{\alpha_n}$ for all $n \geq 1$. This implies that x is a lower bound for $\{x_{\alpha_n}\}_{n\geq 1}$.

We claim that x is a lower bound for $\{x_\alpha\}_{\alpha \in I}$.

Let $\beta \in I$ be such that $x_\beta \preceq_\phi x_{\alpha_n}$ for all $n \geq 1$. Then

$$0_{\mathbb{A}} \preceq d(x_{\alpha_n}, x_\beta) \preceq \phi(x_{\alpha_n}) - \phi(x_\beta).$$

Taking limit $n \to \infty$ implies

$$\phi(x_\beta) \preceq \inf \{\phi(x_\alpha) : \alpha \in I\}. \tag{3.3}$$

Since $\beta \in I$, we have

$$\inf \{\phi(x_\alpha) : \alpha \in I\} \preceq \phi(x_\beta). \tag{3.4}$$

Combining (3.3) and (3.4) we get

$$\inf \{\phi(x_\alpha) : \alpha \in I\} = \phi(x_\beta). \tag{3.5}$$

As $x_\beta \preceq_\phi x_{\alpha_n}$ it follows from (3.1) that $d(x_\beta, x_{\alpha_n}) \preceq \phi(x_{\alpha_n}) - \phi(x_\beta)$. Using (3.5) and the fact that $\{x_{\alpha_n}\}$ is decreasing chain in X we get

$$0_{\mathbb{A}} \preceq d(\lim_{n\to\infty} x_{\alpha_n}, x_\beta) \preceq \lim_{n\to\infty} \phi(x_{\alpha_n}) - \phi(x_\beta) = \phi(x_\beta) - \phi(x_\beta) = 0_{\mathbb{A}}.$$

Thus

$$d(\lim_{n\to\infty} x_{\alpha_n}, x_\beta) = 0_{\mathbb{A}}.$$

Hence $\lim_{n\to\infty} x_{\alpha_n} = x_\beta$. It follows from the uniqueness of limit that $x_\beta = x$. That is, x is a lower bound of $\{x_\alpha : \alpha \in I\}$. Hence by using Zorn's Lemma we conclude that X has a minimal element. □

As a consequence of the above theorem we have the following fixed-point result.

Theorem 3.5. *Let $(X, \mathbb{A}, d)$ be a complete C^*-algebra valued metric space and $\phi\colon X \to \mathbb{A}_+$ be a lower semi continuous map. Let $T\colon X \to X$ be such that for all $x \in X$*

$$d(x, Tx) \preceq \phi(x) - \phi(Tx). \tag{3.6}$$

Then T has at least one fixed point.

Proof. Let $a \in X$ be a minimal element of X. Since $Ta \in X$, it follows that $a \preceq_\phi x$ for all $x \in X$ in particular

$$a \preceq_\phi Ta. \tag{3.7}$$

By combining (3.7) and the condition (3.6) we have $Ta = a$, that is T has a fixed point. □

Example 3.6. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ be a C^*-algebra with the partial order as given in Example 3.2. Define $d\colon X \times X \to \mathbb{A}$ by $d(x, y) = (|x - y|, 0)$. Let $\phi\colon X \to \mathbb{A}_+$, $\phi(x) = (x, 0)$ be continuous map, and $T\colon X \to X$ given by the formula $T(x) = x^2$. Then it is easy to see that all the conditions of Theorem 3.5 are satisfied and T has a fixed point. Note that contractive theorem stated in [6] is not applicable here, since contractive condition (2.1) does not hold.

4. Conclusions

In this paper, we presented an extension of Caristi's fixed point theorem for mappings defined on C^*-algebra valued metric spaces. We proved the existence of fixed point using the concept of minimal element in C^*-algebra valued metric space by introducing the notion of partial order on X.

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