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CONTROLLABILITY OF IMPULSIVE QUASI-LINEAR FRACTIONAL MIXED VOLTERRA-FREDHOLM-TYPE INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

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Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday
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ABSTRACT. In this paper, we establish a sufficient condition for the controllability of impulsive quasi-linear fractional mixed Volterra-Fredholm-type integrodifferential equations in Banach spaces. The results are obtained by using Banach contraction fixed point theorem combined with the fractional calculus theory.

1. INTRODUCTION

The purpose of this paper is to establish the sufficient conditions for the controllability of impulsive quasi-linear fractional mixed Volterra-Fredholm-type integrodifferential equation of the form

$$
{}^cD^q x(t) = A(t,x)x(t) + Bu(t) + f\left(t, x(t), \int_0^t g(t,s,x(s))ds, \int_0^b k(t,s,x(s))ds\right),
$$

$$
t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\ldots,m, \tag{1.1}
$$

$$
\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\ldots,m, \tag{1.2}
$$

$$
x(0) = x_0, \tag{1.3}
$$

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where the state variable $x(\cdot)$ takes values in a Banach space $\mathbb{X}$ and control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Here $0 < q < 1$ and $A(t,x)$ is a bounded linear operator on a Banach space $\mathbb{X}$. Further $f : J \times \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X},\ g : \Omega \times \mathbb{X} \to \mathbb{X},\ k : \Omega \times \mathbb{X} \to \mathbb{X},\ I_k : \mathbb{X} \to \mathbb{X},\ \Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, for all $k = 1, 2, \dots, m;\ 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b;\ \Omega = \{(t,s),\ 0 \le s \le t \le b\}$.

Many processes studied in applied sciences are represented by differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics, theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology, medicine and so on. Adequate mathematical models of such processes are systems of differential equations with impulses. The theory of impulsive differential and integrodifferential equations is a new and important branch of differential equations, which has an extensive physical back ground; For instance, we refer [22, 31, 39, 40, 50, 45, 61].

Fractional differential equations have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc., (see[25, 41, 2, 37, 52]). There has been significant development in fractional differential equations in recent years; see the monographs of Kilbas et al. [43], Miller and Ross [53], Podlubny [58], Lakshmikantham et al. [47] and the papers [44, 38, 51, 3, 64, 4, 6, 7, 10, 23, 29, 46, 48, 49, 56, 65, 66, 19] and the references therein. Among previous research, little is concerned with differential equations with fractional order with impulses. Recently, Benchohra et al. [1, 20] establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential equations involving the Caputo fractional derivative of order $0 < \alpha \le 1$ and $1 < \alpha \le 2$. In [5], B. Ahmad et al. give some existence results for two-point boundary value problems involving nonlinear impulsive hybrid differential equations of fractional order $1 < \alpha \le 2$ where as Mophou [57] discussed the existence and uniqueness results for impulsive fractional differential equations. Very recently, K. Balachandran et al. [18, 19] studied the existence results for impulsive fractional differential and integrodifferential equations in Banach spaces by using standard fixed point theorems.

The existence of solutions of abstract quasi-linear evolution equations have been studied by several authors; see for instance [8, 11, 35, 62]. Bahuguna [9], Oka [59] and Oka and Tanaka [60] discussed the existence of solutions of quasilinear integrodifferential equations in Banach spaces. Kato [42] studied the nonhomogeneous evolution equations and Chandrasekaran [28] proved the existence of mild solutions of the nonlocal Cauchy problem for a nonlinear integrodifferential equation where as Dhakne and Pachpatte [34] established the existence of a unique strong solution of a quasi-linear abstract functional integrodifferential equation in Banach spaces. Recently Balachandran et al. [12, 13] studied the existence of solutions of nonlocal quasi-linear integrodifferential equations with or without impulsive conditions in Banach spaces. Further, Balachandran extends the works [12, 13] into controllability results [14] with impulsive conditions.

On the other hand, the most important qualitative behaviour of a dynamical system is controllability. It is well known that the issue of controllability plays an important role in control theory and engineering [15, 16, 24, 32] because they have close connections to pole assignment, structural decomposition, quadratic optimal control and observer design etc., In recent years, the problem of controllability for various kinds of fractional differential and integrodifferential equations have been discussed in [17, 21, 33, 63, 30]. The literature related to controllability of impulsive fractional integrodifferential equations and controllability of impulsive quasi-linear integrodifferential equations is limited, to our knowledge, to the recent works [14, 64]. The study of controllability of impulsive quasi-linear fractional mixed Volterra-Fredholm-type integrodifferential equations described in the general abstract form (1.1)-(1.3) is an untreated topic in the literature, and this fact, is the main motivation of our paper.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus which are used throughout this paper.

Let $\mathbb{X}$ be a Banach space and $\mathbb{R}_+ = [0,\infty)$. Suppose $f \in L_1(\mathbb{R}_+)$. Let $C(J,\mathbb{X})$ be the Banach space of continuous functions $x(t)$ with $x(t) \in \mathbb{X}$ for $t \in J = [0,b]$ and $\|x\|_{C(J,\mathbb{X})} = \max_{t\in J} \|x(t)\|$. Let $B(\mathbb{X})$ denote the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ with the norm $\|A\|_{B(\mathbb{X})} = \sup\{\|A(y)\| : |y\| = 1\}$. Also consider the Banach space

$$PC(J,\mathbb{X}) = \{x : J \to \mathbb{X} : x \in C((t_k,t_{k+1}],\mathbb{X}),\ k = 0,1,\ldots,m \quad \text{and there exist } x(t_k^-) \\ \text{and } x(t_k^+),\ k = 1,2,\ldots,m \quad \text{with } x(t_k^-) = x(t_k)\},$$

with the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|$. Denote $J' = [0,b] - \{t_1,t_2,\ldots,t_m\}$.

Definition 2.1. *The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n,\ n \in N$ is defined as*

$$^{(R-L)}D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 ([19])**.** *The Caputo fractional derivative of order $\alpha > 0,\ n-1 < \alpha < n$, is defined as*

$$^{c}D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$. If $0<\alpha<1$, then

$$^cD^{\alpha}_{0+}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f'(s)}{(t-s)^{\alpha}}ds,$$

where $f'(s)=Df(s)=\frac{df(s)}{ds}$ and f is an abstract function with values in $\mathbb{X}$.

For our convenience, let us take $^cD^{\alpha}_{0+}$ as $^cD^{\alpha}$. For more details on properties of I^{α}_{0+} and $^cD^{\alpha}_{0+}$, we refer [19].

Definition 2.4. *The impulsive integrodifferential system (1.1)-(1.3) is said to be controllable on the interval $J=[0,b]$ if for every $x_0,x_1\in\mathbb{X}$, there exists a control $u\in L^2(J,U)$ such that the solution $x(\cdot)$ of (1.1)-(1.3) satisfies $x(b)=x_1$.*

It is easy to prove that [19, 38] the equation (1.1)-(1.3) is equivalent to the following integral equation

$$\begin{aligned}x(t)=x_0&+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s,x)x(s)ds+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s,x)x(s)ds\\&+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\Big[Bu(s)+f\Big(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau,\int_0^b k(s,\tau,x(\tau))d\tau\Big)\Big]ds\\&+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\Big[Bu(s)+f\Big(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau,\int_0^b k(s,\tau,x(\tau))d\tau\Big)\Big]ds\\&+\sum_{0<t_k<t}I_k(x(t_k^-)).\end{aligned}$$

Let $B_r=\{x\in\mathbb{X}:\|x\|\le r\}$ for some $r>0$. We assume the following conditions to prove the controllability of the system (1.1)-(1.3).

(H1) $A:J\times\mathbb{X}\to B(\mathbb{X})$ is a continuous bounded linear operator and there exists a constant $L_1>0,\widetilde{L}_1>0$ such that

$$\|A(t,x)-A(t,y)\|_{B(\mathbb{X})}\le L_1\|x-y\|,\quad\text{for all } x,y\in B_r$$

and $\widetilde{L}_1=\max_{t\in J}\|A(t,0)\|$.

(H2) The nonlinear function $f:J\times\mathbb{X}\times\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ is continuous and there exist constants $L_2>0,\widetilde{L}_2>0$, such that

$$\|f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)\|\le L_2\Big[\|x_1-y_1\|+\|x_2-y_2\|+\|x_3-y_3\|\Big],\ \text{for } x_i,y_i\in B_r,\ i=1,2,3.$$

and $\widetilde{L}_2=\max_{t\in J}\|f(t,0,0,0)\|$.

(H3) The nonlinear function $g:\Omega\times\mathbb{X}\to\mathbb{X}$ is continuous and there exist constants $L_3>0,\widetilde{L}_3>0$, such that

$$\|g(t,s,x)-g(t,s,y)\|\le L_3\|x-y\|,\quad\text{for } x,y\in B_r$$

and $\widetilde{L}_3=\max_{t\in J}\|g(t,s,0)\|$.

(H4) The nonlinear function $k : \Omega \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist constants $L_4 > 0, \widetilde{L}_4 > 0$, such that

$$\|k(t,s,x) - k(t,s,y)\| \leq L_4\|x-y\|, \quad \text{for } x,y \in B_r$$

and $\widetilde{L}_4 = \max_{t\in J} \|k(t,s,0)\|$.

(H5) The functions $I_k : \mathbb{X} \to \mathbb{X}$ are continuous and there exist constants $L_5 > 0, \widetilde{L}_5 > 0$, such that

$$\|I_k(x) - I_k(y)\| \leq L_5\|x-y\|, \quad \text{for } x,y \in B_r \text{ and } k = 1,2,\ldots,m,$$

and $\widetilde{L}_5 = \|I_k(0)\|$.

(H6) The linear operator $W : L^2(J,U) \to \mathbb{X}$ defined by

$$Wu = \frac{1}{\Gamma(q)}\int_0^b (b-s)^{q-1}Bu(s)ds$$

has an inverse operator W^{-1}, which takes values in $L^2(J,U)/KerW$ and there exists a positive constant $K > 0$ such that $\|BW^{-1}\| \leq K$ for every $x \in B_r$.

(H7) There exists a constant $r > 0$ such that

$$\|x_0\| + (m+1)\gamma\Big[r(L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]) + (\widetilde{K} + \widetilde{L}_2) + L_2[\widetilde{L}_3 + \widetilde{L}_4]b\Big] + m(L_5 r + \widetilde{L}_5) \leq r,$$

where

$$\widetilde{K} = K\Big[\|x_1\| + \|x_0\| + (m+1)\gamma\Big[r(L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]) + L_2[\widetilde{L}_3 + \widetilde{L}_4]b + \widetilde{L}_2\Big] + m(L_5 r + \widetilde{L}_5)\Big]$$

with $\gamma = \frac{b^q}{\Gamma(q+1)}$.

3. Controllability result

In this section, we present and prove the controllability results for the system (1.1)-(1.3).

Theorem 3.1. *If the hypotheses (H1)-(H7) are satisfied, then the impulsive fractional integrodifferential system (1.1)-(1.3) is controllable on J provided*

$$\Lambda = \Big[(m+1)\gamma\Big[K\big[(m+1)\gamma\Big\{2L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]\Big\} + mL_5\big] + (2L_1 r + \widetilde{L}_1) + L_2[1 + L_3 b + L_4 b]\Big] + mL_5\Big] < 1.$$

Proof. Using the hypothesis (H6) for an arbitrary function $x(\cdot)$ define the control

$$
\begin{aligned}
u(t) = W^{-1}\Big[& x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x) x(s) ds - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} A(s,x) x(s) ds \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau)) d\tau, \int_0^b k(s,\tau,x(\tau)) d\tau\right) ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau)) d\tau, \int_0^b k(s,\tau,x(\tau)) d\tau\right) ds \\
& - \sum_{0<t_k<b} I_k(x(t_k^-))\Big](t).
\end{aligned}
$$

We have to show that when using this control, the operator $\Phi : PC(J, B_r) \to PC(J, B_r)$ defined by

$$
\begin{aligned}
(\Phi x)(t) = x_0 &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s,x)x(s)ds \\
&+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} BW^{-1}\Big[x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} A(s,x)x(s)ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&- \sum_{0<t_k<b} I_k(x(t_k^-))\Big](\eta)d\eta + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} BW^{-1}\Big[x_1 - x_0 \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} A(s,x)x(s)ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&- \sum_{0<t_k<b} I_k(x(t_k^-))\Big](\eta)d\eta \\
&+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&+ \sum_{0<t_k<t} I_k(x(t_k^-))
\end{aligned}
$$

has a fixed point. Since all the functions involved in the operator are continuous, therefore Φ is continuous. For our convenience, let us take

$$\begin{aligned}
G(\eta, x) = BW^{-1}\Big[&x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b - s)^{q-1} A(s,x)x(s)ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
&- \sum_{0<t_k<b} I_k(x(t_k^-))\Big]
\end{aligned}$$

and

$$\begin{aligned}
G(\eta, y) = BW^{-1}\Big[&x_1 - x_0 - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,y)y(s)ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b - s)^{q-1} A(s,y)y(s)ds \\
&- \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^b k(s,\tau,y(\tau))d\tau\right) ds \\
&- \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b - s)^{q-1} f\left(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^b k(s,\tau,y(\tau))d\tau\right) ds \\
&- \sum_{0<t_k<b} I_k(y(t_k^-))\Big].
\end{aligned}$$

From our assumptions, we get

$$
\begin{aligned}
\|G(\eta,x)\| \leq{}& K\Big[\|x_1\|+\|x_0\|+\frac{1}{\Gamma(q)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\big[\|A(s,x)-A(s,0)\|+\|A(s,0)\|\big]\\
&(\times)\|x(s)\|ds+\frac{1}{\Gamma(q)}\int_{t_k}^{b}(b-s)^{q-1}\big[\|A(s,x)-A(s,0)\|+\|A(s,0)\|\big]\|x(s)\|ds\\
&+\frac{1}{\Gamma(q)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\Big[\|f\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau,\int_0^b k(s,\tau,x(\tau))d\tau\right)\\
&-f(s,0,0,0)\|+\|f(s,0,0,0)\|\Big]ds\\
&+\frac{1}{\Gamma(q)}\int_{t_k}^{b}(b-s)^{q-1}\Big[\|f\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau,\int_0^b k(s,\tau,x(\tau))d\tau\right)\\
&-f(s,0,0,0)\|+\|f(s,0,0,0)\|\Big]ds+\sum_{0<t_k<b}\big[\|I_k(x(t_k^-))-I_k(0)\|+\|I_k(0)\|\big]\Big]\\
\leq{}& K\Big[\|x_1\|+\|x_0\|+(m+1)\gamma(L_1r+\widetilde{L}_1)r+(m+1)\gamma\big[L_2r+L_2(L_3r+\widetilde{L}_3)b\\
&+L_2(L_4r+\widetilde{L}_4)b+\widetilde{L}_2\big]+m[L_5r+\widetilde{L}_5]\Big]\\
\leq{}& K\Big[\|x_1\|+\|x_0\|+(m+1)\gamma\left\{r(L_1r+\widetilde{L}_1+L_2[1+L_3b+L_4b])+L_2(\widetilde{L}_3+\widetilde{L}_4)b+\widetilde{L}_2\right\}\\
&+m[L_5r+\widetilde{L}_5]\Big]\\
={}&\widetilde{K}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|G(\eta,x)-G(\eta,y)\|\\
&\leq K\Big[\frac{1}{\Gamma(q)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\big[\|A(s,x)x(s)-A(s,y)y(s)\|\big]ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_k}^{b}(b-s)^{q-1}\big[\|A(s,x)x(s)-A(s,y)y(s)\|\big]ds\\
&\quad+\frac{1}{\Gamma(q)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\Big[\|f\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau,\int_0^b k(s,\tau,x(\tau))d\tau\right)\\
&\quad-f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau,\int_0^b k(s,\tau,y(\tau))d\tau\right)\|\Big]ds\\
&\quad+\frac{1}{\Gamma(q)}\int_{t_k}^{b}(b-s)^{q-1}\Big[\|f\left(s,x(s),\int_0^s g(s,\tau,x(\tau))d\tau,\int_0^b k(s,\tau,x(\tau))d\tau\right)\\
&\quad-f\left(s,y(s),\int_0^s g(s,\tau,y(\tau))d\tau,\int_0^b k(s,\tau,y(\tau))d\tau\right)\|\Big]ds+\sum_{0<t_k<b}\|I_k(x(t_k^-))-I_k(y(t_k^-))\|\Big]\\
&\leq K\Big[(m+1)\gamma\left\{2L_1r+\widetilde{L}_1+L_2[1+L_3b+L_4b]\right\}+mL_5\Big]\|x-y\|.
\end{aligned}
$$

First, we show that Φ maps $PC(J, B_r)$ into $PC(J, B_r)$. Now

$$\begin{aligned}
\|(\Phi x)(t)\| \leq\ & \|x_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \big[\|A(s,x) - A(s,0)\| + \|A(s,0)\|\big] \|x(s)\| ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \big[\|A(s,x) - A(s,0)\| + \|A(s,0)\|\big] \|x(s)\| ds \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} \|G(\eta, x)\| d\eta + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} \|G(\eta, x)\| d\eta \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \Big[\Big\|f\left(s, x(s), \int_0^s g(s,\tau,x(\tau)) d\tau, \int_0^b k(s,\tau,x(\tau)) d\tau\right) \\
& - f(s,0,0,0)\Big\| + \|f(s,0,0,0)\|\Big] ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \Big[\Big\|f\left(s, x(s), \int_0^s g(s,\tau,x(\tau)) d\tau, \int_0^b k(s,\tau,x(\tau)) d\tau\right) \\
& - f(s,0,0,0)\Big\| + \|f(s,0,0,0)\|\Big] ds + \sum_{0<t_k<t} \big[\|I_k(x(t_k^-)) - I_k(0)\| + \|I_k(0)\|\big] \\
\leq\ & \|x_0\| + (m+1)\gamma \Big[r(L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]) + (\widetilde{K} + \widetilde{L}_2) + L_2[\widetilde{L}_3 + \widetilde{L}_4] b\Big] \\
& + m(L_5 r + \widetilde{L}_5) \\
\leq\ & r.
\end{aligned}$$

From assumption (H7), one gets $\|(\Phi x)(t)\| \leq r$. Therefore Φ maps $PC(J, B_r)$ into $PC(J, B_r)$. Moreover, if $x, y \in PC(J, B_r)$, then

$$
\begin{aligned}
&\|(\Phi x)(t) - (\Phi y)(t)\| \\
&\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \big[\|A(s,x)x(s) - A(s,y)y(s)\|\big] ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \big[\|A(s,x)x(s) \\
&\quad - A(s,y)y(s)\|\big] ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} \big[\|G(\eta, x) - G(\eta, y)\|\big] d\eta \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} \big[\|G(\eta, x) - G(\eta, y)\|\big] d\eta \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \Big[\Big\| f\Big(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\Big) \\
&\quad - f\Big(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^b k(s,\tau,y(\tau))d\tau\Big)\Big\|\Big] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \Big[\Big\| f\Big(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\Big) \\
&\quad - f\Big(s, y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^b k(s,\tau,y(\tau))d\tau\Big)\Big\|\Big] ds + \sum_{0<t_k<t} \|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\
&\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \Big[(\|A(s,x) - A(s,0)\| + \|A(s,0)\|)\|x(s) - y(s)\| + \|A(s,x) \\
&\quad - A(s,y)\|\|y(s)\|\Big] ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \Big[(\|A(s,x) - A(s,0)\| + \|A(s,0)\|)\|x(s) - y(s)\| \\
&\quad + \|A(s,x) - A(s,y)\|\|y(s)\|\Big] ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} \|G(\eta, x) - G(\eta, y)\| d\eta \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} \|G(\eta, x) - G(\eta, y)\| d\eta \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} L_2 \Big[\|x(s) - y(s)\| + \int_0^s \|g(s,\tau,x(\tau)) - g(s,\tau,y(\tau))\| d\tau \\
&\quad + \int_0^b \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\| d\tau\Big] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} L_2 \Big[\|x(s) - y(s)\| + \int_0^s \|g(s,\tau,x(\tau)) - g(s,\tau,y(\tau))\| d\tau \\
&\quad + \int_0^b \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\| d\tau\Big] ds + \sum_{0<t_k<t} \|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \\
&\leq \Big[(m+1)\gamma\Big[K\big[(m+1)\gamma\Big\{2L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]\Big\} + mL_5\big] + (2L_1 r + \widetilde{L}_1) \\
&\quad + L_2[1 + L_3 b + L_4 b]\Big] + mL_5\Big] \|x - y\| \\
&= \Lambda \|x - y\|
\end{aligned}
$$

Since $0 \leq \Lambda < 1$, then Φ is a contraction and so by Banach fixed point theorem there exists a unique fixed point $x \in PC(J, B_r)$ such that $(\Phi x)(t) = x(t)$. This fixed point is then a solution of the system (1.1)-(1.3) and clearly, $x(b) = (\Phi x)(b) = x_1$, which implies that the system is controllable on J. □

4. Nonlocal controllability result

In this section, we discuss the controllability system (1.1)-(1.2) with a nonlocal condition of the form

$$x(0) + h(x) = x_0. \tag{4.1}$$

where $h : PC(J, \mathbb{X}) \to \mathbb{X}$ is a given function.

The nonlocal condition can be applied in physics with better effect than the classical initial condition $x(0) = x_0$. For example, $h(x)$ may be given by

$$h(x) = \sum_{i=1}^{m} c_i x(t_i),$$

where $c_i (i = 1, 2, \ldots, m)$ are given constants and $0 < t_1 < t_2 < \cdots < t_m < b$. Nonlocal conditions were initiated by Byszewski [26] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski and Lakshmikantham [27], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. For more details on fractional order with nonlocal condition, we refer [17, 18, 19, 36, 54, 55].

For the study of the controllability system (1.1)-(1.2) with (4.1), we need the following hypotheses:

(H8) $h : PC(J, \mathbb{X}) \to \mathbb{X}$ is continuous and there exist constants $L_6 > 0, \widetilde{L}_6 > 0$, such that

$$\|h(x) - h(y)\| \leq L_6 \|x - y\|_{PC}, \quad \text{for } x, y \in PC(J, \mathbb{X}),$$

and $\widetilde{L}_6 = \|h(0)\|$.

(H8) There exists a constant $r > 0$ such that

$$\|x_0\| + (mL_5 + L_6)r + (m\widetilde{L}_5 + \widetilde{L}_6) + (m+1)\gamma\Big[r(L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]) + (\widetilde{K} + \widetilde{L}_2) + L_2[\widetilde{L}_3 + \widetilde{L}_4]b\Big] \leq r,$$

where

$$\widetilde{K} = K\Big[\|x_1\| + \|x_0\| + (mL_5 + L_6)r + (m\widetilde{L}_5 + \widetilde{L}_6) + (m+1)\gamma\Big[r(L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]) + L_2[\widetilde{L}_3 + \widetilde{L}_4]b + \widetilde{L}_2\Big]\Big]$$

with $\gamma = \frac{b^q}{\Gamma(q+1)}$.

Definition 4.1. *The impulsive integrodifferential system (1.1)-(1.2) with the condition (4.1) is said to be controllable on the interval J if for every $x_0, x_1 \in \mathbb{X}$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of (1.1)-(1.2) with (4.1) satisfies $x(0) + h(x) = x_0$ and $x(b) = x_1$.*

Theorem 4.1. *If the hypotheses (H1)-(H6) and (H8)-(H9) are satisfied, then the impulsive fractional integrodifferential system (1.1)-(1.2) with the conditions (4.1) is controllable on J provided*

$$\Lambda' = \Big[(mL_5 + L_6) + (m+1)\gamma\Big[K\big[(m+1)\gamma\Big\{2L_1 r + \widetilde{L}_1 + L_2[1 + L_3 b + L_4 b]\Big\} + (mL_5 + L_6)\big] + (2L_1 r + \widetilde{L}_1) + L_2[1 + L_3 b + L_4 b]\Big]\Big] < 1.$$

Proof. Using the hypothesis (H6) for an arbitrary function $x(\cdot)$ define the control

$$\begin{aligned} u(t) = W^{-1}\Big[&x_1 - (x_0 - h(x)) - \frac{1}{\Gamma(q)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k - s)^{q-1}A(s,x)x(s)ds \\ &- \frac{1}{\Gamma(q)}\int_{t_k}^{b}(b-s)^{q-1}A(s,x)x(s)ds \\ &- \frac{1}{\Gamma(q)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k - s)^{q-1}f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right)ds \\ &- \frac{1}{\Gamma(q)}\int_{t_k}^{b}(b-s)^{q-1}f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right)ds \\ &- \sum_{0<t_k<b} I_k(x(t_k^-))\Big](t). \end{aligned}$$

We have to show that when using this control, the operator $\Psi : PC(J, B_r) \to PC(J, B_r)$ defined by

$$
\begin{aligned}
(\Psi x)(t) = {} & x_0 - h(x) + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s,x)x(s)ds \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - \eta)^{q-1} BW^{-1} \Big[x_1 - (x_0 - h(x)) \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} A(s,x)x(s)ds \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
& - \sum_{0<t_k<b} I_k(x(t_k^-)) \Big] (\eta) d\eta + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-\eta)^{q-1} BW^{-1} \Big[x_1 - (x_0 - h(x)) \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s,x)x(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} A(s,x)x(s)ds \\
& - \frac{1}{\Gamma(q)} \sum_{0<t_k<b} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
& - \frac{1}{\Gamma(q)} \int_{t_k}^{b} (b-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
& - \sum_{0<t_k<b} I_k(x(t_k^-)) \Big] (\eta) d\eta \\
& + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau, \int_0^b k(s,\tau,x(\tau))d\tau\right) ds \\
& + \sum_{0<t_k<t} I_k(x(t_k^-))
\end{aligned}
$$

has a fixed point. This fixed point is then a solution of the control problem (1.1)-(1.2) with (4.1). Clearly, $(\Psi x)(b) = x_1$, which means that the control u steers the system (1.1)-(1.2) with (4.1) from the initial state x_0 to x_1 in time b provided we can obtain a fixed point of the operator Ψ. The rest of the proof is similar to Theorem 3.1, hence omitted. $\square$

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