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EXISTENCE RESULTS FOR IMPULSIVE SYSTEMS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday
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Abstract. According to semigroup theories and Sadovskii fixed point theorem, this paper is mainly concerned with the existence of solutions for an impulsive neutral differential and integrodifferential systems with nonlocal conditions in Banach spaces. As an application of this main theorem, a practical consequence is derived for the sub-linear growth case. In the end, an example is also given to show the application of our result.

1. Introduction

Many evolution process are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus differential equations involving impulsive effects appear as a natural description of observed evolution phenomena of several real world problems. For more details

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on this theory and on its applications we refer to the monographs of Lakshmikantham et al. [14], and Samoilenko and Perestyuk [20] for the case of ordinary impulsive system and [5, 11, 12, 15, 17, 18, 21] for partial differential and partial functional differential equations with impulses.

The starting point of this paper is the work in papers [9, 10]. Especially, authors in [10] investigated the existence of solutions for the system

$$\begin{gathered}\frac{d}{dt}[x(t)+F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]+Ax(t)\\ =G(t,x(t),x(a_1(t)),\dots,x(a_n(t))),\quad 0\le t\le a,\\ x(0)+g(x)=x_0,\end{gathered}$$

by using fractional powers of operators and Sadovskii fixed point theorem. And in [9], authors studied the following neutral partial differential equations of the form

$$\begin{gathered}\frac{d}{dt}[x(t)-F(t,x(h_1(t)))]=-A[x(t)-F(t,x(h_1(t)))]+G(t,x(h_2(t))),\quad t\in J\\ x(0)+g(x)=x_0\in X,\end{gathered}$$

by using fractional powers of operators and Banach contraction fixed point theorem. Motivated by above mentioned works [9, 10], the main purpose of this paper is to prove the existence of mild solutions for the following impulsive neutral partial differential equations in a Banach space X:

$$\begin{gathered}\frac{d}{dt}[x(t)-F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]=A[x(t)-F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]\\ +G(t,x(t),x(a_1(t)),\dots,x(a_n(t))),\quad t\in J=[0,b],\quad t\ne t_k,\ k=1,2,\dots,m,\end{gathered}\tag{1.1}$$

$$\Delta x|_{t=t_k}=I_k(x(t_k^-)),\quad k=1,2,\dots,m,\tag{1.2}$$

$$x(0)+g(x)=x_0\in X,\tag{1.3}$$

where the linear operator A generates an analytic semigroup $\{T(t)\}_{t\ge 0}$, $\Delta x|_{t=t_k}=I_k(x(t_k^-))$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. F, G and g are given functions to be specified later.

Finally in section 3, we prove the existence results for the following impulsive neutral integrodifferential equations of the form

$$\begin{gathered}\frac{d}{dt}[x(t)-F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]=A[x(t)-F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]\\ +\int_0^t K(t,s)G(s,x(s),x(a_1(s)),\dots,x(a_n(s)))ds,\\ t\in J=[0,b],\ t\ne t_k,\ k=1,2,\dots,m,\end{gathered}\tag{1.4}$$

with the conditions (1.2)-(1.3), where $A,\ F,\ G,\ I_k$ are as defined in (1.1)-(1.3) and $K:D\to R,\ D=\{(t,s)\in J\times J: t\ge s\}$.

The nonlocal Cauchy problem was considered by Byszewski [3] and the importance of nonlocal conditions in different fields has been discussed in [3] and [8] and the references therein. For example, in [8] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(x) = \sum_{i=0}^{p} C_i x(t_i),$$

where $C_i,\ i = 0, 1, \dots, p$ are given constants and $0 < t_1 < t_2 < \cdots < t_p < b$. In this case the above equation allows the additional measurement at $t_i,\ i = 0, 1, \dots, p$. In the past several years theorems about existence, uniqueness of differential and impulsive functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by Byszewski and Lakshmikantham [4], by Akca et al. [1], by Anguraj et al. [2], by Fu et al. [9, 10] and by Chang et al. [6] and the references therein.

This paper has four sections. In the next section we recall some basic definitions and preliminary facts which will be used throughout this paper. In section 3 we prove the existence results for the system (1.1)-(1.3) without assume the boundedness condition on $I_k,\ k = 1, 2, \dots, m$, and also we prove the existence results for an impulsive neutral integrodifferential system (1.4) with the conditions (1.2)-(1.3). As an immediate result of the obtained theorem, a practical consequence is derived for the sub-linear growth of impulsive functions $I_k,\ k = 1, 2, \dots, m$ and the nonlocal initial function g. Finally, in section 4 an example is presented to illustrate the application of the obtained results. This paper extends and generalize the results of [9, 10].

2. Preliminaries

In this section, we introduce some results, notations and lemma which are needed to establish our main results.

Let X be a Banach space provided with norm $\|\cdot\|$. Let $A : D(A) \to X$ is the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operator $T(t)$ on X. It well known that there exist $M \geq 1$ and $w \in R$ such that $\|T(t)\| \leq Me^{wt}$ for every $t \geq 0$. If $\{T(t)\}_{t\geq 0}$ is uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression

$$\|x\|_\alpha = \|(-A)^\alpha x\|, \quad x \in D(-A)^\alpha,$$

defines a norm on $D(-A)^\alpha$. For more details of fractional power of operators and semigroup theory, we refer Pazy [16].

From the above theory, we define the following lemma.

Lemma 2.1 ([10, 16]). *The following properties hold:*

(i) *If $0 < \beta < \alpha \leq 1$, then $X_\alpha \hookrightarrow X_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.*

(ii) *For the every $0 < \alpha \leq 1$, there exists $C_\alpha > 0$ such that*

$$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t <\leq b.$$

We need the following fixed point theorem due to Sadovskii [19].

Theorem 2.1. *(Sadovskii) Let P be a condensing operator on a Banach space, that is, P is continuous and takes bounded sets into bounded sets, and let $\alpha(P(B)) \leq \alpha(B)$ for every bounded set B of X with $\alpha(B) > 0$ of $P(H) \subset H$ for a convex, closed, and bounded set H of X, then P has fixed point in H (where $\alpha(\cdot)$ denotes Kuratowski's measure of noncompactness).*

3. Existence results

In order to define the solution of the problem (1.1)-(1.3), we consider the following space

$$\Omega = \{x : J \to X,\ x_k \in C(J_k, X),\ k = 0, 1, \ldots, m,\ \text{and there exist } x(t_k^-) \text{ and } x(t_k^+),\\ k = 0, 1, 2, \ldots, m,\ \text{with } x(t_k^-) = x(t_k),\ x(0) + g(x) = x_0\},$$

which is a Banach space with the norm

$$\|x\|_\Omega = \max\{\|x_k\|_{J_k},\ k = 0, 1, \ldots, m\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}],\ k = 0, 1, \ldots, m.$

Now, we define the mild solution for the system (1.1)-(1.3).

Definition 3.1. *A function $x \in \Omega$ is said to be a mild solution of the system (1.1)-(1.3) if*

(i) $x(0) + g(x) = x_0$;

(ii) $\Delta x|_{t=t_k} = I_k(x(t_k^-)),\ k = 1, 2, \ldots, m$; *and*

$$x(t) = T(t)[x_0 - g(x) - F(0, x(0), x(b_1(0)), \ldots, x(b_m(0)))]\\ + F(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) + \int_0^t T(t-s)G(s, x(s), x(a_1(s)), \ldots, x(a_n(s)))ds\\ + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad t \in J.$$

is satisfied.

For the system (1.1)-(1.3) we assume that the following hypotheses are satisfied:

(H1) There exist constant $\beta \in (0, 1)$ such that $F : J \times X^{m+1} \to X$ is a continuous function, and $A^\beta F : J \times X^{m+1} \to X$ satisfies the Lipschitz condition, that is, there exists a constant $L > 0$ such that

$$\|A^\beta F(s_1, x_0, x_1, \ldots, x_m) - A^\beta F(s_2, \bar{x}_0, \bar{x}_1, \ldots, \bar{x}_m)\| \leq L(|s_1 - s_2| + \max_{i=0,1,\ldots,m} \|x_i - \bar{x}_i\|)$$

for any $0 \leq s_1, s_2 \leq b,\ x_i, \bar{x}_i \in X,\ i = 0, 1, \ldots, m$. Moreover, there exists a constant $L_1 > 0$ such that the inequality

$$\|A^\beta F(t, x_0, x_1, \ldots, , x_m)\| \leq L_1(\max\{\|x_i\| :\ i = 0, 1, \ldots, m\} + 1),$$

holds for any $(t, x_0, x_1, \ldots, x_m) \in J \times X^{m+1}$.

(H2) The function $G : J \times X^{n+1} \to X$ satisfies the following conditions:
   (i) For each $t \in J$, the function $G(t, \cdot) : X^{n+1} \to X$ is continuous and for each $(x_0, x_1, \dots, x_n) \times X^{n+1}$ the function $G(\cdot, x_0, x_1, \dots, x_n) : J \to X$ is strongly measurable;
   (ii) For each positive number $r \in N$, there is a positive function $g_r \in L^1(J)$ such that

$$\sup_{\|x_0\|,\dots,\|x_n\| \le r} \|G(t, x_0, x_1, \dots, x_n)\| \le g_r(t)$$

and

$$\lim_{r\to\infty} \inf \frac{\int_0^b g_r(s)ds}{r} = \mu < +\infty.$$

(H3) $a_i, b_j \in C(J, J),\ i = 1, 2, \dots, n,\ j = 1, 2, \dots, m.$
(H4) There exist positive constants L_2 and L_2' such that

$$\|g(x)\| \le L_2\|x\|_\Omega + L_2' \text{ for all } x \in \Omega,$$

and $g : \Omega \to X$ is completely continuous.
(H5) $I_k : X \to X$ is completely continuous and there exist continuous nondecreasing functions $L_k : R_+ \to R_+$ such that for each $x \in X$

$$\|I_k(x)\| \le L_k(\|x\|), \quad \lim_{r\to\infty} \inf \frac{L_k(r)}{r} = \lambda_k < +\infty.$$

(H6) For each $t \in J,\ K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = ess\sup\{|K(t, s)| : 0 \le s \le t\},$$

is bounded on J. The map $t \to K_t$ is continuous from J to $L^\infty(J, R)$, here $K_t(s) = K(t, s)$.

Our main result may be presented as the following theorem.

Theorem 3.1. *Assume the conditions (H1)-(H5) hold. Then the problem (1.1)-(1.3) admits at least one mild solution on J provided that*

$$L_0 = L[(M + 1)M_0] < 1 \tag{3.1}$$

and

$$M\Big[L_2 + M_0L_1 + \mu + \sum_{k=1}^{m} \lambda_k\Big] + M_0L_1 < 1, \tag{3.2}$$

where $M_0 = \|A^{-\beta}\|$.

Proof. For the sake of brevity, we rewrite that

$$(t, x(t), x(b_1(t)), \dots, x(b_m(t))) = (t, v(t))$$

and

$$(t, x(t), x(a_1(t)), \dots, x(a_n(t))) = (t, u(t))$$

Consider the operator $N : \Omega \to \Omega$ defined by

$$N(x) = \{\varphi \in \Omega : \varphi(t) = T(t)[x_0 - g(x) - F(0, v(0))] + F(t, v(t)) + \int_0^t T(t-s)G(s, u(s))ds$$
$$+ \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))\}, \quad t \in J.$$

Clearly the fixed points of N are mild solutions to (1.1)-(1.3). We shall show that N satisfies the hypotheses of Theorem 2.1. For better readability we break the proof into sequence of steps.

Step 1. There exists a positive integer $r \in N$ such that $N(B_r) \subset B_r$, where $B_r = \{x \in \Omega : \|x\| \leq r,\ 0 \leq t \leq b\}$.

For each positive number r, B_r is clearly a bounded closed convex set in Ω. We claim that there exists a positive r such that $N(B_r) \subset B_r$, where $N(B_r) = \cup_{x\in B_r} N(x)$. If it is not true, then for each positive integer r, there exist the functions $x_r(\cdot) \in B_r$ and $\varphi_r(\cdot) \in N(x_r)$, but $\varphi_r(\cdot) \notin B_r$, that is $\|\varphi_r(t)\| > r$ for some $t(r) \in J$, where $t(r)$ denotes t is independent of r. However, on the other hand, we have

$$\begin{aligned}
r &< \|\varphi_r(t)\| \\
&= \|T(t)[x_0 - g(x_r) - F(0, v_r(0))] + F(t, v_r(t)) + \int_0^t T(t-s)G(s, u_r(s))ds \\
&\quad + \sum_{0<t_k<t} T(t-t_k)I_k(x_r(t_k^-))\| \\
&\leq \|T(t)[x_0 - g(x_r) - A^{-\beta}A^{\beta}F(0, v_r(0))]\| + \|A^{-\beta}A^{\beta}F(t, v_r(t))\| \\
&\quad + \int_0^t \|T(t-s)G(s, u_r(s))\|ds + \sum_{0<t_k<t} \|T(t-t_k)I_k(x_r(t_k^-))\| \\
&\leq M[\|x_0\| + L_2 r + L_2' + M_0L_1(r+1)] + M_0L_1(r+1) + M\int_0^t g_r(s)ds + M\sum_{k=1}^m L_k(r) \\
&\leq M[\|x_0\| + L_2 r + L_2'] + (M+1)M_0L_1(r+1) + M\int_0^b g_r(s)ds + M\sum_{k=1}^m L_k(r).
\end{aligned}$$

Dividing on both sides by r and taking the lower limit limit as $t \to +\infty$, we get

$$M\Big[L_2 + M_0L_1 + \mu + \sum_{k=1}^m \lambda_k\Big] + M_0L_1 \geq 1.$$

This is a contradiction to (3.2). Hence for some positive integer $r, N(B_r) \subseteq B_r$.

Step 2. Next we show that the operator $N = N_1 + N_2$ (is condensing), N_1 is contraction and N_2 is compact.

The operators N_1, N_2 are defined on B_r respectively by

$$(N_1x)(t) = F(t, v(t)) - T(t)F(0, v(0)),$$

$$N_2x = \{\varphi \in \Omega : \varphi(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)G(s, u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))\}, \quad t \in J.$$

We will verify that N_1 is a contraction while N_2 is a completely continuous operator.

To prove that N_1 is a contraction, we take $x_1, x_2 \in B_r$ arbitrarily. Then for each $t \in J$ and by condition (H1) and (3.1), we have

$$\begin{aligned}\|(N_1x_1)(t) - (N_1x_2)(t)\| &\leq \|F(t, v_1(t)) - F(t, v_2(t))\| + \|T(t)[F(0, v_1(0)) - F(0, v_2(0))]\| \\ &= \|A^{-\beta}[A^{\beta}F(t, v_1(t)) - A^{\beta}F(t, v_2(t))]\| \\ &\quad + \|T(t)A^{-\beta}[A^{\beta}F(0, v_1(0)) - A^{\beta}F(0, v_2(0))]\| \\ &\leq M_0L \sup_{0\leq s\leq b} \|x_1(s) - x_2(s)\| + MM_0L \sup_{0\leq s\leq b} \|x_1(s) - x_2(s)\| \\ &\leq (M+1)M_0L \sup_{0\leq s\leq b} \|x_1(s) - x_2(s)\| \\ &\leq L_0 \sup_{0\leq s\leq b} \|x_1(s) - x_2(s)\|\end{aligned}$$

Thus

$$\|N_1x_1 - N_1x_2\| \leq L_0\|x_1 - x_2\|.$$

Therefore, by assumption $0 < L_0 < 1$, we see that N_1 is a contraction.

To prove that N_2 is compact, firstly we prove that N_2 is continuous on B_r. Let $\{x_n\}_{n=0}^{\infty} \subseteq B_r$ with $x_n \to x$ in B_r, then by (H2)(i) and (H5)

(i) $I_k,\ k = 1, 2, \ldots, m$ is continuous.
(ii) $G(s, u_n(s)) \to G(s, u(s)), \quad n \to \infty.$
Since

$$\|G(s, u_n(s)) - G(s, u(s))\| \leq 2g_r(s).$$

We have by the dominated convergence theorem,

$$\|N_2x_n - N_2x\| = \sup_{0\leq t\leq b} \|T(t)[g(x) - g(x_n)] + \int_0^t T(t-s)G(s, u_n(s)) - G(s, u(s))]ds + \sum_{0<t_k<t} T(t-t_k)[I_k(x_n(t_k^-)) - I_k(x(t_k^-))]\|$$

$$
\begin{aligned}
&\leq M\|g(x_n) - g(x)\| + M\int_0^b \|G(s, u_n(s)) - G(s, u(s))\|ds \\
&\quad + M\sum_{0<t_k<t} \|I_k(x_n(t_k^-)) - I_k(x(t_k^-))\| \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Thus, N_2 is continuous.

Next, we prove that $\{N_2x : x \in B_r\}$ is a family of equicontinuous functions. Let $x \in B_r$ and $\tau_1, \tau_2 \in J$. Then if $0 < \tau_1 \leq \tau_2 \leq b$ and $\varphi \in N_2(x)$, then for each $t \in J$, we have

$$\varphi(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)G(s, u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-))$$

Then

$$
\begin{aligned}
\|\varphi(\tau_2) - \varphi(\tau_1)\| \leq\ & \|T(\tau_2) - T(\tau_1)\|\|x_0 - g(s)\| \\
&+ \int_0^{\tau_1-\epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|\|G(s, u(s))\|ds \\
&+ \int_{\tau_1-\epsilon}^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|\|G(s, u(s))\|ds \\
&+ \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|\|G(s, u(s))\|ds \\
&+ \sum_{0<t_k<\tau_1} \|T(\tau_2 - t_k) - T(\tau_1 - t_k)\|\|I_k(x(t_k^-))\| \\
&+ \sum_{\tau_1\leq t_k<\tau_2} \|T(\tau_2 - t_k)\|\|I_k(x(t_k^-))\|.
\end{aligned}
$$

The right-hand side is independent of $x \in B_r$ and tends to zero as $\tau_2 - \tau_1 \to 0$, since the compactness of $\{T(t)\}_{t\geq 0}$ implies the continuity in the uniform operator topology. Similarly, using the ompactness of the set $g(B_r)$ we can prove that the functions $N_2x,\ x \in B_r$ are equicontinuous at $t = 0$. Hence N_2 maps B_r into a family of equicontinuous functions.

It remains to prove that $(N_2B_r)(t)$ is relatively compact for each $t \in J$, where $(N_2B_r)(t) = \{\varphi(t) : \varphi \in N_2(B_r)\},\ t \in J$.

Obviously, by condition (H4), $(N_2B_r)(t)$ is relatively compact in Ω for $t = 0$. Let $0 < t \leq b$ be fixed and $0 < \epsilon < t$. For $x \in B_r$ and $\varphi \in N_2(x)$, we have

$$\varphi(t) = T(t)[x_0 - g(x)] + \int_0^t T(t-s)G(s, u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(x(t_k^-)), \quad t \in J.$$

Define

$$\begin{aligned}\varphi_\epsilon(t) &= T(t)[x_0 - g(x)] + \int_0^{t-\epsilon} T(t-s)G(s,u(s))ds + \sum_{0<t_k<t-\epsilon} T(t-t_k)I_k(x(t_k^-)), \quad t \in J \\ &= T(t)[x_0 - g(x)] + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)G(s,u(s))ds \\ &\quad + T(\epsilon)\sum_{0<t_k<t-\epsilon} T(t-t_k-\epsilon)I_k(x(t_k^-)).\end{aligned}$$

Since $\{T(t)\}_{t\geq 0}$ is compact, the set $V_\epsilon(t) = \{\varphi_\epsilon(t) : \varphi \in N_2(B_r)\}$ is relatively compact in Ω for every ϵ, $0 < \epsilon < t$. Moreover, for every $\varphi \in N_2(B_r)$,

$$\begin{aligned}\|\varphi(t) - \varphi_\epsilon(t)\| &\leq \int_{t-\epsilon}^{t} \|T(t-s)G(s,u(s))\|ds + \sum_{t-\epsilon<t_k<} \|T(t-t_k)I_k(x(t_k^-))\| \\ &\leq M\int_{t-\epsilon}^{t} g_r(s)ds + M\sum_{t-\epsilon<t_k<t} L_k(r).\end{aligned}$$

Therefore, letting $\epsilon \to 0$, we see that, there are relatively compact sets arbitrarily close to the set $\{\varphi(t) : \varphi \in N_2(B_r)\}$ is relatively compact in Ω.

As a consequence of the above steps and the Arzela-Ascoli theorem, we can conlude that N_2 is a compact operator. These arguments enable us to conclude that $N = N_1 + N_2$ is a condensing map on B_r, and by the fixed point theorem of Sadovskii there exists a fixed point $x(\cdot)$ for N on B_r. Therefore, the nonlocal system (1.1)-(1.3) has a mild solution. The proof is now completed. $\square$

As an immediate result of Theorem 3.1, we can obtain the following corollary when g and I_k have sub-linear growth.

Corollary 3.1. *Suppose conditions of (H4)-(H5) in Theorem 3.1 are replaced by the following:*

(H4′) *There exist positive constants L_2, L_2' and $\theta \in [0,1)$ such that*

$$\|g(x)\| \leq L_2\|x\|_\Omega^\theta + L_2', \quad \textit{for } x \in \Omega$$

and $g : \Omega \to X$ is completely continuous;

(H5′) *$I_k \in C(X,X)$ and there exist nondecreasing functions $L_k : R_+ \to R_+$ and positive constants c_k, d_k and $\sigma \in [0,1)$ such that*

$$\|I_k(x)\| \leq c_k\|x\|^\sigma + d_k.$$

Then the problem (1.1)-(1.3) has at least one mild solution in J provided that

$$\Big[(M+1)M_0L_1 + M\mu\Big] < 1$$

and

$$\Big[(M+1)M_0L\Big] < 1.$$

Proof. In this case, we have $L_k(\|x\|) = c_k r^\sigma + d_k$ and $\|g(x)\| \leq L_2 r^\theta + L_2'$, $x \in B_r$ in the proof of Theorem 3.1. Thus (3.1) and (3.2) are reduced to $\Big[(M+1)M_0L_1 + M\mu\Big] < 1$ and $\Big[(M+1)M_0L\Big] < 1$, this completes the proof. □

Now, we define the mild solution for the system (1.4) with the conditions (1.2)-(1.3).

Definition 3.2. *A function $x \in \Omega$ is said to be a mild solution of the system (1.4) with the conditions (1.2)-(1.3) if*

(i) $x(0) + g(x) = x_0$;
(ii) $\Delta x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \ldots, m$; *and*

$$\begin{aligned} x(t) = &T(t)[x_0 - g(x) - F(0, x(0), x(b_1(0)), \ldots, x(b_m(0)))] \\ &+ F(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) \\ &+ \int_0^t T(t-s) \int_0^s K(s,\tau) G(\tau, x(\tau), x(a_1(\tau)), \ldots, x(a_n(\tau))) d\tau ds \\ &+ \sum_{0<t_k<t} T(t-t_k) I_k(x(t_k^-)), \quad t \in J. \end{aligned}$$

is satisfied.

Now we state the existence theorem for the system (1.4) with the conditions (1.2)-(1.3). The proof of the following theorem is similar to Theorem 3.1, so we omit it.

Theorem 3.2. *Assume the conditions (H1)-(H6) hold. Then the problem (1.4) with the conditions (1.2)-(1.3) admits at least one mild solution on J provided that*

$$L_0 = L[(M+1)M_0] < 1$$

and

$$M\Big[L_2 + M_0L_1 + b \sup_{t\in J} K(t)\mu + \sum_{k=1}^m \lambda_k\Big] + M_0L_1 < 1,$$

where $M_0 = \|A^{-\beta}\|$.

4. An Example

As an application of Theorem 3.1, we consider the following system

$$\frac{\partial}{\partial t}\Big[z(t,x)-\int_0^\pi b(y,x)z(t\sin t,y)dy\Big]=\frac{\partial^2}{\partial x^2}\Big[z(t,x)-\int_0^\pi b(y,x)z(t\sin t,y)dy\Big]$$
$$+h(t,z(t\sin t,x)),\quad 0\le t\le b,\ 0\le x\le\pi,\ t\ne t_k,\ k=1,2,\dots,m, \tag{4.1}$$

$$z(t,0)=z(t,\pi)=0, \tag{4.2}$$

$$z(t_k^+)-z(t_k^-)=I_k(z(t_k^-)),\ k=1,2,\dots,m, \tag{4.3}$$

$$z(0,x)+\sum_{i=0}^{p}\int_0^\pi k(x,y)z(t_i,y)dy=z_0(x),\quad 0\le x\le\pi, \tag{4.4}$$

where p is a positive integer, $0<t_0<\cdots<t_p<1$, and $0<t_1<t_2<\cdots<t_m<b$. The function $z_0(x)\in X=L^2([0,\pi])$ and A is defined by

$$Af=f'',$$

with the domain

$$D(A)=\{f(\cdot)\in X: f',\ f''\in X,\ f(0)=f(\pi)=0\}.$$

Then A generates a strongly continuous semigroup $T(\cdot)$ which is compact, analytic and self adjoint. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n\in N$, with the corresponding normalized eigenvectors $z_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$. Then the following properties hold:

(a) If $f\in D(A)$, then

$$Af=\sum_{n=1}^{\infty}n^2<f,z_n>z_n.$$

(b) The operator $A^{\frac{1}{2}}$ is given by

$$A^{\frac{1}{2}}f=\sum_{n=1}^{\infty}n<f,z_n>z_n,$$

on the space $D(A^{\frac{1}{2}})=\{f(\cdot)\in X,\ \sum_{n=1}^{\infty}n<f,z_n>z_n\in X\}$.

We assume that the following conditions hold:

(i) The function b is measurable and

$$\int_0^\pi\int_0^\pi b^2(y,x)dydx<\infty.$$

(ii) The function $\frac{\partial}{\partial x}b(y,x)$ is measurable, $b(y,0)=b(y,\pi)=0$, and let

$$N_1=\Big[\int_0^\pi\int_0^\pi\Big(\frac{\partial}{\partial x}b(y,x)\Big)^2dydx\Big]^{\frac{1}{2}}<\infty.$$

(iii) For the function $h:J\times R\to R$ the following three conditions are satisfied:
  (1) For each $t\in J$, $h(t,\cdot)$ is continuous.
  (2) For each $z\in X$, $h(\cdot,z)$ is measurable.

(3) There are positive functions $h_1, h_2 \in L^1(J)$ such that

$$|h(t,z)| \leq h_1(t)|z| + h_2(t), \quad \forall\ (t,z) \in J \times X.$$

(iv) The function $I_k(X,X),\ k = 1,2,\ldots,m$ and there exist nondecreasing functions $L_k \in (J, R_+),\ k = 1,2,\ldots,m$ such that for each $x \in X$

$$\|I_k(x)\| \leq L_k(\|x\|).$$

We define $F,\ G : X \times X \to X$ and $g : \Omega \to X$ by

$$\begin{aligned} F(t,z) &= Z_1(z), \\ G(t,z)(x) &= h(t,z(x)), \\ g(w(t)) &= \sum_{i=0}^{p} Kw(t_i), \quad w \in \Omega,\ (\Omega \text{ is defined in section 3}), \end{aligned}$$

respectively, where

$$Z_1(z)(x) = \int_0^\pi b(y,x)z(y)dy$$

and

$$K(z)(x) = \int_0^\pi k(x,y)z(y)dy.$$

Then G satisfies condition (H2) while g verifies (H4) (noting that $K : X \to X$ is completely continuous). From (i) it is clear that Z_1 is a bounded linear operators on X. Furthermore, $Z_1(z) \in D[A^{\frac{1}{2}}]$, and $\|A^{\frac{1}{2}} Z_1\| \leq N_1$. In fact, from the definition of Z_1 and (ii) it follows that

$$\begin{aligned} < Z_1(z), z_n > &= \int_0^\pi z_n(x)\Big[\int_0^\pi b(y,x)z(y)dy\Big]dx \\ &= \frac{1}{n}\sqrt{\frac{2}{\pi}} < Z(z), \cos(nx) >, \end{aligned}$$

where Z is defined by

$$Z(z)(x) = \int_0^\pi \frac{\partial}{\partial x} b(y,x)z(y)dy.$$

From (ii) we know that $Z : X \to X$ is a bounded linear operator with $\|Z\| \leq N_1$. Hence $\|A^{\frac{1}{2}} Z_1(z)\| = \|Z(z)\|$, which implies the assertion. Therefore, the conditions (H1)-(H5) are all satisfied. Hence from Theorem 3.1, system (4.1)-(4.4) admits a mild solution on J under the above assumptions additionally provided that (3.1) and (3.2) hold.

Remark 4.1. *Differential inclusions plays an important role in characterizing many social, physical, biological and engineering problems. In particular, the problems in physics, especially in solid mechanics, where non-monotone and multivalued constitutive laws lead to differential inclusions. The above results can* *be extended to study the existence results for an impulsive neutral differential inclusions with nonlocal conditions in Banach spaces by suitably introducing the multivalued map defined in* [6, 7, 13].

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