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A NEW REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS

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ABSTRACT. We study the incompressible Navier-Stokes equations in the entire three-dimensional space. We prove that if $\partial_3 u_3 \in L_t^{s_1} L_x^{r_1}$ and $u_1, u_2 \in L_t^{s_2} L_x^{r_2}$, then the solution is regular. Here $\frac{2}{s_1} + \frac{3}{r_1} \leq 1$, $3 \leq r_1 \leq \infty$, $\frac{2}{s_2} + \frac{3}{r_2} \leq 1$ and $3 \leq r_2 \leq \infty$.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we consider Cauchy problem for the incompressible Navier-Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p &= \nu \triangle u \\ \nabla \cdot u &= 0 \\ u(x,0) &= u_0(x) \end{cases} \tag{1.1}$$

in which $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the unknown velocity field, $u_0 = (u_1(x,0), u_2(x,0), u_3(x,0))$ is the initial velocity field with $\nabla \cdot u_0 = 0$, and $p(x,t)$ is a scalar pressure. While ν is the kinematic viscosity coefficient, we will assume that $\nu \equiv 1$ for simplicity in this paper. Here we use the classical notations

$$\nabla u = (\partial_1 u, \partial_2 u, \partial_3 u),\ \triangle u = \sum_{i=1}^{3} \partial_i^2 u,\ \nabla \cdot u = \sum_{i=1}^{3} \partial_i u_i,\ u \cdot \nabla u = \sum_{j=1}^{3} u_j \partial_j u.$$

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In 1934, Leray proved the existence of a global weak solution

$$u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \bigcap L^2(0,\infty; H^1(\mathbb{R}^3)),$$

on the condition that $u_0 \in L^2$ and $\nabla \cdot u_0 = 0$. It is called the Leray-Hopf weak solution which satisfies the energy inequality [1]. If $\|u(t)\|_{H^1}$ is continuous, then we say that $u(t)$ is regular. Up to the present, the regularity and uniqueness of the weak solutions is still an open problem. On the other hand, there are already many criterions which ensure that the weak solution is regular [2, 3].

In this paper, we give a regularity criterion concerning one derivative of one component of the velocity and other components of velocity.

2. Main results

Our main result can be stated as follows.

Theorem 2.1. *Let*

$$\partial_3 u_3 \in L_t^{s_1} L_x^{r_1}, \ \textit{where}\ \frac{2}{s_1} + \frac{3}{r_1} \leq 1,\ 3 \leq r_1 \leq \infty, \tag{2.1}$$

and

$$u_1, u_2 \in L_t^{s_2} L_x^{r_2}, \ \textit{where}\ \frac{2}{s_2} + \frac{3}{r_2} \leq 1,\ 3 \leq r_2 \leq \infty.$$

Then u is regular.

At first, let us recall the definition of the Leray-Hopf weak solution.

Definition 2.2. (Leray-Hopf weak solution) If a measurable vector u satisfies the following properties in $0 \leq T \leq \infty$:

(1) u is weakly continuous in $[0,T) \times L^2(\mathbb{R}^3)$,
(2) u verifies (1.1) in the sense of distribution

$$\int_0^T \int_{\mathbb{R}^3} (\frac{\partial \Phi}{\partial t} + u \cdot \nabla \Phi + \triangle \Phi) \cdot u dx dt + \int_{\mathbb{R}^3} u_0 \cdot \Phi(x,0) dx = 0,$$

$\forall \Phi \in C_0^\infty(\mathbb{R}^3 \times [0,T)),\ \nabla \cdot \Phi = 0,\ \Phi$ is a vector function,
(3) u satisfies the energy inequality

$$\|u(\cdot,t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(\cdot,s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2,\ 0 \leq t \leq T,$$

then u is called a *Leray-Hopf weak solution of the Navier-Stokes equations.*

We give three lemmas for the proof of the theorem.

Lemma 2.3. (The estimate for the pressure in $L^q(\mathbb{R}^3)$)

$$\|p\|_{L^q}(t) \leq C\|u\|_{L^{2q}}^2(t),\ 1 < q < \infty.$$

Proof. See [2]. □

Lemma 2.4. (Interpolation inequality) *Assume* $u \in L_t^\infty L_x^2(\mathbb{R}^3 \times I)$ *and* $\nabla u \in L_t^2 L_x^2(\mathbb{R}^3 \times I)$, *where* I *is an open interval. Then* $u \in L_t^s L_x^r(\mathbb{R}^3 \times I)$ *for all* r, s *such that*

$$\frac{2}{s} + \frac{3}{r} \leq \frac{3}{2}$$

and $2 \leq r \leq 6$. *Moreover,*

$$\|u\|_{L_t^s L_x^r} \leq C\|u\|_{L_t^\infty L_x^2}^{(6-r)/2r} \|\nabla u\|_{L_t^2 L_x^2}^{(3r-6)/2r}.$$

Proof. See [3]. $\square$

By Lemmas 2.3 and 2.4, we get the following.

Lemma 2.5. (The estimate for the pressure in $L_t^s L_x^r(\mathbb{R}^3 \times I)$)

$$\|p\|_{L_t^s L_x^r} \leq C\|u_0\|_{L^2}^2, \ \frac{2}{s} + \frac{3}{r} = 3, \ 1 < r \leq 3.$$

Proof. (Proof of Theorem 2.1) At first, let us multiply the equation for u_3 of (1.1) by $|u_3|u_3$ and integrate over $[0, T] \times \mathbb{R}^3$ $t \in (0, T)$

$$\begin{aligned} \int_0^t \int_{\mathbb{R}^3} \frac{\partial u_3}{\partial t} \cdot |u_3|u_3 - \int_0^t \int_{\mathbb{R}^3} \triangle u_3 \cdot |u_3|u_3 &= -\int_0^t \int_{\mathbb{R}^3} u \cdot \nabla u_3 \cdot |u_3|u_3 \\ &\quad -\int_0^t \int_{\mathbb{R}^3} \partial_3 p \cdot |u_3|u_3 \\ I_1 + I_2 &= I_3 + I_4. \end{aligned}$$

Calculating every term,

$$\begin{aligned} I_1 &= \frac{1}{3}\int_0^t \int_{\mathbb{R}^3} \frac{\partial |u_3|^3}{\partial t} = \frac{1}{3}\|u_3\|_{L^3}^3|_0^t = \frac{1}{3}\|u(\cdot,t)\|_{L^3}^3 - \frac{1}{3}\|u(\cdot,0)\|_{L^3}^3, \\ I_2 &= -\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} \partial_{ii} u_3 |u_3| u_3 = \frac{8}{9}\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} (\partial_i |u_3|^{3/2})^2 = \frac{8}{9}\|\nabla |u_3|^{3/2}\|_{L_t^2 L_x^2}^2, \\ I_3 &= -\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} u_i \partial_i u_3 |u_3| u_3 = \frac{1}{3}\sum_{i=1}^3 \int_0^t \int_{\mathbb{R}^3} \partial_i u_i |u_3|^3 = 0, \\ I_4 &= \int_0^t \int_{\mathbb{R}^3} p\partial_3(|u_3|u_3) \leq C\|p\|_{L_t^{s_2} L_x^{r_2}}^{3/2} \|\partial_3 u_3\|_{L_t^{s_1} L_x^{r_1}}^{3/2} + \frac{1}{6}\|u_3\|_{L_t^\infty L_x^3}^3, \end{aligned}$$

where

$$\begin{cases} \frac{1}{s_1} + \frac{1}{s_2} = 1 & \\ \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{3} & \\ \frac{2}{s_1} + \frac{3}{r_1} = 1 & 2 \leq s_1 \leq \infty, 3 \leq r_1 \leq \infty \\ \frac{2}{s_2} + \frac{3}{r_2} = 3 & 1 \leq s_2 \leq 2, \frac{3}{2} \leq r_2 \leq 3. \end{cases}$$

By Lemma 2.3, we get

$$\|p\|_{L_t^{s_2} L_x^{r_2}} \leq C\|u_0\|_{L^2}^2, \tag{2.2}$$

and so

$$I_4 \leq C\|u_0\|_{L^2}^3 \|\partial_3 u_3\|_{L_t^{s_1} L_x^{r_1}}^{3/2} + \frac{1}{6}\|u_3\|_{L_t^\infty L_x^3}^3. \tag{2.3}$$

Summarizing the above calculating of I_1, I_2, I_3 and I_4, we get

$$\frac{1}{6}\|u_3(\cdot,t)\|_{L^3}^3+\frac{8}{9}\|\nabla|u_3|^{\frac{3}{2}}\|_{L_t^2L_x^2}^2 \le C\|u_0\|_{L^2}^3\|\partial_3 u_3\|_{L_t^{s_1}L_x^{r_1}}^{\frac{3}{2}}+C\|u_0\|_{L^2}^{\frac{3}{2}}\|\nabla u_0\|_{L^2}^{\frac{3}{2}}.$$

By (2.1), we get

$$\|u_3(\cdot,t)\|_{L_t^\infty L_x^3}+\|\nabla|u_3|^{\frac{3}{2}}\|_{L_t^2L_x^2}\le C,$$

moreover,

$$\|u_3(\cdot,t)\|_{L_t^3L_x^9}\le C\|\nabla|u_3|^{\frac{3}{2}}\|_{L_t^2L_x^2}^{\frac{2}{3}},$$

and so

$$\|u_3(\cdot,t)\|_{L_t^\infty L_x^3}+\|u_3(\cdot,t)\|_{L_t^3L_x^9}\le C,$$

i.e., $u_3\in L_t^sL_x^r$, where $\frac{2}{s}+\frac{3}{r}\le 1,\ 3\le r\le 9$, so the solution is regular making use of the regularity criterion [4, 5]:

$$u\in L_t^sL_x^r,\ \text{where}\ \frac{2}{s}+\frac{3}{r}\le 1,\ 3\le r\le\infty,$$

which completes the proof. □

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