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ON THE (p, q)-GROWTH OF ENTIRE FUNCTION SOLUTIONS OF HELMHOLTZ EQUATION

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Dedicated to Themistocles M. Rassias on the occasion of his sixtieth birthday
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ABSTRACT. The (p, q)-growth of entire function solutions of Helmholtz equations in R^2 has been studied. We obtain some lower bounds on order and type through function theoretic formulae related to those of associate. Our results extends and improve the results studied by McCoy [10].

1. Introduction and preliminaries

The Helmholtz equation be given in the form

$$[\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + F(r^2)]\phi(r, \theta) = 0, \tag{1.1}$$

where (r, θ) are polar co-ordinates in R^2 and $F(r^2) \neq 0$ is a real valued entire function with analytic continuation as an entire function of $z \in C$.

Regular solution of (1.1) at the origin has a local representation via the Bergman operator [1,7] of the first kind

$$\phi(r, \theta) = B_2(f(z)) = \int_{-1}^{1} E(r^2, t) f(z\frac{(1-t^2)}{2})(1-t^2)^{-1/2}dt, \tag{1.2}$$

where

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$$E(r^2,t)=1+\sum_{n=1}^{\infty}t^{2n}Q^{(2n)}(r^2) \tag{1.3}$$

is a real valued analytic function for $t\in[-1,+1]$ that is entire for $r\in[0,\infty)$ and known as Bergman $'EFunction'$. the coefficients $Q^{(2n)}$ are themselves entire functions defined from the recurrences

$$\partial_r(Q^{(2)}(\tau))+2F(\tau)=0, Q^{(2)}(0)=0,$$

$$(2n+1)\partial_r(Q^{(2n+2)})+2\partial_r(\tau Q^{(2n)})+F(\tau)Q^{(2n)}-n\partial_r(Q^{2n})=0,$$

$$Q^{(2n+2)}(\tau)|_{\tau=0}=0$$

for $n=1,2,3,\cdots$. Thus the solution of (1.1) has an expansion in a neighborhood of the origin as

$$\phi(r,\theta)=\sum_{n=0}^{\infty}a_n\phi_n(r,\theta),$$

where

$$\phi_n(r,\theta)=(re^{i\theta}/2)^nG_n(r); G_n(r)=\int_{-1}^{1}E(r^2,t)(1-t^2)^{n-1/2}dt$$

for $n=0,1,2,\cdots$, and the B_2 associate of ϕ is given by

$$f(z)=\sum_{n=0}^{\infty}a_nz^n. \tag{1.4}$$

Since the analytic continuation of the perturbation term, $F(z^2)$ is taken as an entire function, it can be easily seen from Gilbert and Coltan [4] that $\phi(r,\theta)$ is an entire function if, and only if, the associate $f(z)$ is taken as an entire function. The formulae (1.1)–(1.4) hold throughout the C-plane and an identifying characteristic of entire function solutions $\phi(r,\theta)$, as in function theory, is that

$$\limsup_{n\to\infty}|a_n|^{1/n}=0.$$

For classifying entire functions by their growth, the concept of order was introduced. If the order is a (finite) positive number, then the concept of type permits a sub classification. For the classes of order $\delta=0$ and $\delta=\infty$ no sub classification is possible. For example, all entire functions that grow at least as fast as $\exp(\exp(z))$ has to be kept in one class. For this reason, numerous attempts have been made to refine the concept of order and type. Amongst them, proposals by Lindelof and by Valiron have gained some attention.

Function theoretic methods extended these properties to harmonic functions in several variables (see Gelbert [2, 3] and McCoy [8]). McCoy [9] studied the growth on the disk of regularity for solutions of the Helmholtz equation in R^2. The method modified Bergman's integral operator of the first kind [1] to develop a suitable basis for function theoretic extension. Kreyezig and Kracht [7] discussed entire function solutions in terms of associated analytic functions of two complex variables. They obtained an upper bound on the order that was computed from coefficients of the associate. The type of a solution was not studied in that paper. McCoy [10] studied the fast growth of entire function solution $\phi(r,\theta)$ in terms of order δ and type τ using the concept of index k, i.e., $\delta(k-1)=\infty$ and $\delta(k)<\infty$. Moreover, they obtained bounds on the order and type of $\phi(r,\theta)$ that reflect their antecedents in the theory of analytic functions of a single complex variable. It has been noticed that his results do not give any precise information about the growth of those functions for which $\delta(k-1)=\infty$ and $\delta(k)=0$. To refine this scale, in the present paper we pick up a concept of (p,q)-order and (p,q)-type introduced by Juneja et al. [5, 6]. Roughly speaking, this concept is a modification of the classical definition of order and type, obtained by replacing logarithms by iterated logarithms, where the degrees of iteration are determined by p and q. In univariate case, important results are formulae for order and type of an entire function in terms of Taylor's coefficients about the origin. Our approach unifies the above approaches and at the same time is applicable to every entire function, whether of slow or fast growth. Moreover, we make an attempt to characterize (p,q)-growth of $\phi(r,\theta)$ and obtain some bounds on order and type through function theoretic formulae related to those of the associate $f(z)$.

2. Measures of (p,q)-growth

We define the (p,q)-order of an entire function $h(z)$ by

$$\delta(p,q)=\limsup_{r\to\infty}\frac{\log^{[p]}M(r,h)}{\log^{[q]}r}, \tag{2.1}$$

where p and q are integers such that $p\geq q\geq 0$, $M(r,h)=\max\{|h(z)|:|z|=r\}$. If $b\leq\delta(p,q)\leq\infty$, where $b=1$ if $p=q$ and $b=0$ if $p>q$, then the (p,q)-type is

$$T(p,q,h)=\limsup_{r\to\infty}\frac{\log^{[p-1]}M(r,h)}{(\log^{[q-1]}r)^{\delta(p,q,h)}}, 0\leq T(p,q,h)\leq\infty \tag{2.2}$$

and $\log^{[m]}x=\exp^{[-m]}x=\log(\log^{[m-1]}x)=\exp(\exp^{[-m-1]}x), m=0,\pm1,\pm2,\cdots$, provided that $0<\log^{[m-1]}x<\infty$ with $\log^{[0]}x=\exp^{[0]}x=x$. The entire function $h(z)$ is said to be of (p,q)-order δ if it is of index-pair (p,q) and δ is given by (2.1). (For the definition of index-pair etc., see [5]).

3. (p,q)-Growth of solutions

Theorem 3.1. *Let $\phi(r,\theta)$ be an entire function, which is a solution of the Helmholtz equation with expansion*

$$\phi(r,\theta)=\Sigma_{n=0}^{\infty}a_n\phi_n(r,\theta).$$

Let ϕ and B_2, associated with f, be entire functions of (p,q)-order $\delta(p,q,\phi)$ and $\delta(p,q,f)$ for a pair of integers (p,q), $p \geq 2$, $q \geq 1$. Then the (p,q)-order of ϕ is bounded below by

(1) $\frac{1}{[1/\delta(2,1,f)+1/\sigma(2,1,E)]} \leq \delta(2,1,\phi)$
(2) $\{\sigma(2,2,E)+1 \leq \delta(2,2,\phi);\ if\ \delta(2,2,\phi) < \delta(2,2,f)\ \}$
$\{\sigma^*(2,2,E)+1 \geq \delta(2,2,f);\ if\ \delta(2,2,\phi) > \delta(2,2,f)\}$
(3) $\sigma(p,q,E,f)+1 \leq \delta(p,q,\phi);\ for\ 3 \leq p = q \geq \infty,$

where

$$\frac{1}{\delta(p,q,f)} = \liminf_{n\to\infty} \frac{\log^{[q]}|a_n|^{-1/n}}{\log^{[p-1]} n},$$

$$\frac{1}{\sigma(2,1,E)} = \liminf_{n\to\infty} \frac{\log|G_n(\zeta_{n,2,1})|^{-1/n}}{\log n}$$

$$\frac{1}{\sigma(2,2,E)} = \liminf_{n\to\infty} \frac{\log\log|G_n(\zeta_{n,2,2})|^{-1/n}}{\log n}$$

$$\frac{1}{\sigma^*(2,2,E)} = \liminf_{n\to\infty} \frac{\log\log|G_n(\zeta_{n,2,2})|^{1/n}}{\log n}$$

and

$$\frac{1}{\sigma(p,q,E,f)} = \liminf_{n\to\infty} \frac{\log^{[q-2]}\{2e|G_n(\zeta_{n,p,q})|^{-1/n}\exp^{[q-3]}(log^{[p-2]}n)^{\frac{1}{\delta(p,q)}}\}}{\log^{[p-1]} n}, \quad 3 \leq p = q < \infty,$$

with

$$\zeta_{n,2,1} = (n/\delta(2,1,\phi))^{\frac{1}{\delta(2,1,\phi)}}, \zeta_{n,2,2} = \exp(\frac{n}{\delta(2,2,\phi)})^{\frac{1}{(\delta(2,2,\phi))-1}}$$

and

$$\zeta_{n,p,q} = \exp^{[q-1]}(log^{[p-2]}n)^{\frac{1}{\delta(p,q,\phi)}}, n = 2,3,\cdots.$$

Proof. By Cauchy estimates we have $|a_n G_n(r)/2^n| = |\phi_n(r,\theta)/r^n| = |\frac{1}{2\pi}\int_0^{2\pi}\phi(r,\theta)e^{-in\phi}d\phi| \leq \frac{M(r,\phi)}{r^n}, r > 0,\ n = 0,1,2,\cdots$. Now using the definition of (p,q)-order of $\phi(r,\theta)$, we have for $\varepsilon > 0$, there exists an $A(\varepsilon) > 0$ such that

$$|a_n G_n(r)| \leq 2^n A(\varepsilon)\exp^{[p-1]}\{(log^{[q-1]}r)^{\delta(p,q,\phi)+\varepsilon}\}r^{-n} \tag{3.1}$$

for $(p,q) = (2,1)$ it gives

$$|a_n G_n(r)| \leq 2^n A(\varepsilon)\exp(r^{\delta(2,1,\phi)+\varepsilon})r^{-n}. \tag{3.2}$$

The right hand side is minimized at the sequence of points

$$\zeta'_{n,2,1} = \zeta'_{n,2,1}(\delta(2,1,\phi)) = (n/\delta(2,1,\phi)+\varepsilon)^{\frac{1}{\delta(2,1,\phi)+\varepsilon}}, n = 2,3,....,$$

with $\zeta_{n,2,1} = \lim_{\varepsilon\to 0} \zeta^{'}_{n,2,1}$ (see Juneja et al [5]). This leads with (3.2) to

$$\begin{aligned}&\log|a_n| + \log|G_n(\zeta^{'}_{n,2,1})|\\ &\le \frac{n}{\delta(2,1,\theta)+\varepsilon} - \frac{n}{\delta(2,1,\phi)+\varepsilon}\log\frac{n}{(\delta(2,1,\phi)+\varepsilon)} + n\log 2 + logA(\varepsilon)\end{aligned}$$

or

$$\log|a_n|^{-1/n} + \log|G_n(\zeta^{'}_{n,2,1})|^{-1/n} \ge \frac{\log(n/\delta(2,1,\phi))}{\delta(2,1,\phi)+\varepsilon}[1-O(1)]. \tag{3.3}$$

In order to connect the $(2,1)$-orders of ϕ and f, we use the order coefficient formulae for the associate [5].

$$\log|a_n|^{-1/n} \le \log n^{\frac{1}{(\delta(2,1,f)-\varepsilon)}}$$

is valid on some sequence of indices. Using (3.3) we obtain

$$\frac{1}{\delta(2,1,f)-\varepsilon} + \frac{\log|G_n(\zeta^{'}_{n,2,1})|^{-1/n}}{\log n} \ge \frac{1}{\delta(2,1,\phi)+\varepsilon} + O(1)$$

for infinitely many indices. Proceeding to limits it gives

$$\frac{1}{\delta(2,1,f)} + \liminf_{n\to\infty}\frac{\log|G_n(\zeta_{n,2,1})|^{-1/n}}{\log n} \ge \frac{1}{\delta(2,1,\phi)}$$

or

$$\frac{1}{\delta(2,1,f)} + \frac{1}{\sigma(2,1,E)} \ge \frac{1}{\delta(2,1,\phi)}.$$

This completes the proof of (1).

(2) Substituting $(p,q)=(2,2)$ in (3.1), we get

$$|a_nG_n(r)| \le 2^n A(\varepsilon)\exp\{(\log r)^{\delta(2,2,\phi)+\varepsilon}\}r^{-n}. \tag{3.4}$$

The larger factor is minimized at the sequence of points

$$\zeta^{'}_{n,2,2} = \zeta^{'}_{n,2,2}(\delta(2,2,\phi)) = \exp(\frac{n}{\delta(2,2,\phi)+\varepsilon})^{\frac{1}{(\delta(2,2,\phi)-1+\varepsilon)}}, n = 2,3,...,$$

with $\zeta_{n,2,2} = \lim_{\varepsilon\to 0}\zeta^{'}_{n,2,2}$ (see Juneja et al. [5]). In view of (3.4) it gives

$$\begin{aligned}&\log|a_n|^{-1/n} + \log|G_n(\zeta^{'}_{n,2,2})|^{-1/n}\\ &\ge (\frac{n}{\delta(2,2,\phi)+\varepsilon})^{\frac{1}{(\delta(2,2,\phi))-1+\varepsilon}}(\frac{\delta(2,2,\phi)-1+\varepsilon}{\delta(2,2,\phi)+\varepsilon}) - \log 2 - \frac{1}{n}\log A(\varepsilon).\end{aligned} \tag{3.5}$$

The $(2,2)$-order coefficient formulae for the associate [5] is

$$\log|a_n|^{-1/n} \le n^{\frac{1}{(\delta(2,2,f)-1-\varepsilon)}}$$

valid on some sequence of indices. Using this in (3.5), we get

$$\begin{aligned}\log|G_n(\zeta^{'}_{n,2,2})|^{-1/n} \quad &\ge (\frac{n}{\delta(2,2,\phi)+\varepsilon})^{\frac{1}{(\delta(2,2,\phi))-1+\varepsilon}}(\frac{\delta(2,2,\phi)-1+\varepsilon}{\delta(2,2,\phi)+\varepsilon})\\ &-n^{\frac{1}{\delta(2,2,f)-1-\varepsilon}} - \log 2 - \frac{1}{n}\log A(\varepsilon)\end{aligned}$$

or

$$\geq n^{\frac{1}{(\delta(2,2,\phi)-1+\varepsilon)}}[\frac{\delta(2,2,\phi)-1+\varepsilon}{(\delta(2,2,\phi)+\varepsilon)^{\frac{\delta(2,2,\phi)+\varepsilon}{\delta(2,2,\phi)-1+\varepsilon}}}-n^{\frac{\delta(2,2,\phi)-\delta(2,2,f)+2\varepsilon}{(\delta(2,2,\phi)-1+\varepsilon)(\delta(2,2,f)+1+\varepsilon)}}+O(1)]$$

If $\delta(2,2,\phi)<\delta(2,2,f)$, then

$$\frac{\log\log|G_n(\zeta_{n,2,2}^{'})|^{-1/n}}{\log n}\geq\frac{1}{\delta(2,2,\phi)-1+\varepsilon}+o(1).$$

Proceeding to limits, we get

$$\liminf_{n\to\infty}\frac{\log\log|G_n(\zeta_{n,2,2})|^{-1/n}}{\log n}\geq\frac{1}{\delta(2,2,\phi)-1}$$

or

$$\delta(2,2,\phi)\geq\sigma(2,2,E)+1.$$

Also if $\delta(2,2,\phi)>\delta(2,2,f)$, then after a simple calculation we get

$$\liminf_{n\to\infty}\frac{\log\log|G_n(\zeta_{n,2,2})|^{1/n}}{\log n}\leq\frac{1}{\delta(2,2,f)-1}$$

or

$$\delta(2,2,f)\leq 1+\sigma^*(2,2,E),\frac{1}{\sigma^*(2,2,E)}=\liminf_{n\to\infty}\frac{\log\log|G_n(\zeta_{n,2,2})|^{1/n}}{\log n}.$$

(3) For $3\leq p=q<\infty$, the right hand side of (3.1) is minimized at the sequence of points

$$\zeta_{n,p,q}^{'}=\zeta_{n,p,q}^{'}(\delta(p,q,\phi))=\exp^{[q-1]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,\phi)+\varepsilon}}$$

with $\zeta_{n,p,q}=\lim_{\varepsilon\to 0}\zeta_{n,p,q}^{'}$ [5]. Now from (3.1) we have

$$\log|a_n|+\log|G_n(\zeta^{'},p,q)|\leq n-n\exp^{[q-2]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,\phi)+\varepsilon}}+n\log 2+logA(\varepsilon)$$

or

$$\begin{aligned}&\log|a_n|^{-1/n}+\log|G_n(\zeta^{'},p,q)|^{-1/n}\\&\geq\exp^{[q-2]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,\phi)+\varepsilon}}-1-\log 2-\frac{1}{n}logA(\varepsilon).\end{aligned}$$

The (p,q)-order coefficient formulae for the associate [5] is

$$\log|a_n|^{-1/n}\leq\exp^{[q-2]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,f)-\varepsilon}},3\leq p=q<\infty,$$

valid on some sequence of indices. Now we have

$$\begin{aligned}&\log|G_n(\zeta^{'},p,q)|^{-1/n}\\&\geq\exp^{[q-2]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,\phi)+\varepsilon}}-\exp^{[q-2]}(\log^{[p-2]}n)-1-\log 2-\frac{1}{n}\log A(\varepsilon)\\&=\log\frac{\exp^{[q-3]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,\phi)+\varepsilon}}}{2e(A(\varepsilon))^{1/n}\exp^{[q-3]}(\log^{[p-2]}n)^{\frac{1}{\delta(p,q,f)-\varepsilon}}}\end{aligned}$$

or

$$\liminf_{n\to\infty} \frac{\log^{[q-2]}\{2e|G_n(\zeta_{n,p,q})|^{-1/n}\exp^{[q-3]}(\log^{[p-2]} n)^{\frac{1}{\delta(p,q,f)}}\}}{\log^{[p-1]} n} \geq \frac{1}{\delta(p,q,\phi)}.$$

This completes the proof of (iii). □

Theorem 3.2. *Let $\phi(r,\theta)$ be an entire function, which is a solution of the Helmholtz equation with expansion*

$$\phi(r,\theta) = \sum_{n=0}^{\infty} a_n\phi_n(r,\theta).$$

Let $\phi(r,\theta)$ and B_2-associate $f(z)$ have the same index-pair (p,q). Then the (p,q)-types satisfy

(1) $[T(2,1,\phi)]^{\frac{1}{\delta(2,1,\phi)}}$
$\geq \frac{[\delta(2,1,f)]^{\frac{1}{\delta(2,1,f)}}}{[\delta(2,1,\phi)]^{\frac{1}{\delta(2,1,\phi)}}}[T(2,1,f)]^{\frac{1}{\delta(2,1,f)}} \liminf_{n\to\infty}[(\frac{n}{e})^{\frac{1}{\delta(2,1,\phi)}-\frac{1}{\delta(2,1,f)}}|G_n(\xi_{n,2,1})|^{1/n}]/2$

(2) $(\frac{M}{T(2,2,f)})^{\frac{1}{(\delta(2,2,f)-1)}} \geq (\frac{M'}{T(2,2,\phi)})^{\frac{1}{(\delta(2,2,\phi)-1)}} \liminf_{n\to\infty}[n^{(\frac{1}{\delta(2,1,\phi)-1}-\frac{1}{\delta(2,1,f)-1})}]$
$+ \liminf_{n\to\infty}[n^{-(\frac{\delta(2,2,f)}{\delta(2,2,f)-1})}\log|G_n(\xi_{n,2,1})|]$, *where* $M = \frac{(\delta(2,2,f)-1)^{(\delta(2,2,f)-1)}}{(\delta(2,2,f))^{\delta(2,2,f)}}$ *and* $M' = \frac{(\delta(2,2,\phi)-1)^{(\delta(2,2,\phi)-1)}}{(\delta(2,2,\phi))^{\delta(2,2,\phi)}}$.

(3) *For* $3 \leq p = q < \infty$,

$$T(p,q,\phi) \geq \liminf_{n\to\infty} \frac{\log^{[p-2]} n}{\log^{[q-1]}\{|G_n(\xi_{n,p,q})|^{-1/n}\exp^{[q-1]}[(\frac{\log^{[p-2]} n}{T(p,q,f)})]^{\frac{1}{\delta(p,q,f)}}\}},$$

implicity here

$$\xi_{n,2,1} = (\frac{n}{\delta(2,1,\phi)(T(2,1,\phi))})^{\frac{1}{\delta(2,1,\phi)}}, \xi_{n,2,2} = \exp(\frac{n}{\delta(2,2,\phi)T(2,2,\phi)})^{\frac{1}{\delta(2,1,\phi)-1}}$$

and

$$\xi_{n,p,q} = \exp^{[q-][\frac{\log^{[p-2]} n}{T(p,q,\phi)}]^{\frac{1}{\delta(p,q,\phi)}}}, n = 2,3....$$

Proof. **(1)** The approach is similar to Theorem 3.1. Let $\phi(r,\theta)$ have (p,q)-order $\delta(p,q,\phi)$ and (p,q)-type $T(p,q,\phi)$. Let $\varepsilon > 0$ be given. Then there is a $B(\varepsilon) > 0$ such that

$$M(r,\phi) \leq B(\varepsilon)\exp^{[p-1]}\{(T(p,q,\phi)+\varepsilon)(\log^{[q-1]} r)^{\delta(p,q,\phi)}\} \tag{3.6}$$

for all $r > 0$. This bound is placed in the Cauchy estimate as in Theorem 3.1, we get

$$|a_nG_n(r)| \leq 2^n B(\varepsilon)\exp^{[p-1]}\{(T(p,q,\phi)+\varepsilon)(\log^{[q-1]} r)^{\delta(p,q,\phi)}\}r^{-n}. \tag{3.7}$$

For $(p,q) = (2,1)$ the expression an right hand side is minimized at the points

$$\xi'_{n,2,1} = \xi'_{n,2,1}(T(2,1,\phi)) = [\frac{n}{\delta(2,1,\phi)(T(2,1,\phi)+\varepsilon)}]^{\frac{1}{\delta(2,1,\phi)}}.$$

Now from (3.7), we obtain

$$\begin{aligned}&\log|a_nG_n(\xi'_{n,2,1})|\\&\leq \frac{n}{\delta(2,1,\phi)}-\frac{n}{\delta(2,1,\phi)}\log(\frac{n}{\delta(2,1,\phi)(T(2,2,\phi)+\varepsilon)})+n\log 2+\log B(\varepsilon)\end{aligned}$$

or

$$\begin{aligned}&\log|a_n|^{-1/n}+\log|G_n(\xi'_{n,2,1})|^{-1/n}\\&\geq \frac{1}{\delta(2,1,\phi)}\log(\frac{n}{\delta(2,1,\phi)(T(2,1,\phi)+\varepsilon)})-\frac{1}{\delta(2,1,\phi)}-\log 2+O(1).\end{aligned}$$

Using the $(2,1)$-type coefficient formulae [6] to obtain the bound

$$\log|a_n|^{-1/n}\leq \frac{1}{\delta(2,1,f)}\log(\frac{n}{\ell\delta(2,1,f)(T(2,2,f)-\varepsilon)})$$

valid for the subsequence of indices, we get

$$\begin{aligned}&\frac{1}{\delta(2,1,f)}\log(\frac{n}{e\delta(2,1,f)(T(2,1,f)-\varepsilon)})+\log|G_n(\xi'_{n,2,1})|^{-1/n}\\&\geq \frac{1}{\delta(2,1,\phi)}\log(\frac{n}{\delta(2,1,\phi)(T(2,1,\phi)+\varepsilon)})-\frac{1}{\delta(2,1,\phi)}-\log 2+O(1).\end{aligned}$$

After a simple calculation we get

$$\begin{aligned}&(T(2,1,\phi)^{\frac{1}{\delta(2,1,\phi)}}\\&\geq \frac{(\delta(2,1,f))^{\frac{1}{\delta(2,1,f)}}}{(\delta(2,1,\phi))^{\frac{1}{\delta(2,1,\phi)}}}(T(2,1,f))^{\frac{1}{\delta(2,1,f)}}\liminf_{n\to\infty}[(\frac{n}{e})^{(\frac{1}{\delta(2,1,\phi)}-\frac{1}{\delta(2,1,f)})}|G_n(\xi_{n,2,1})|^{1/n}]/2.\end{aligned}$$

(2) For $(p,q)=(2,2)$ the right hand factor of (3.7) is minimized a the points

$$\xi'_{n,2,1}=\xi'_{n,2,1}(T(2,2,\phi))=\exp(\frac{n}{\delta(2,2,\phi)(T(2,2,\phi)+\varepsilon)})^{\frac{1}{\delta(2,2,\phi)-1+\varepsilon)}}.$$

Substituting this in (3.7) for $(p,q)=(2,2)$, we obtain

$$|a_nG_n(\xi'_{n,2,1})|\leq 2^nB(\varepsilon)\exp\{(T(2,2,\phi)+\varepsilon)(\frac{n}{\delta(2,2,\phi)(T(2,2,\phi)+\varepsilon)})^{(\frac{\delta(2,2,\phi)}{\delta(2,2,\phi)-1+\varepsilon})}\}$$

$$[\exp(\frac{n}{\delta(2,2,\phi)(T(2,2,\phi)+\varepsilon)})^{\frac{1}{\delta(2,2,\phi)-1+\varepsilon)}}]^{-n}$$

or it gives

$$\begin{aligned}&\log|a_n|^{-1/n}+\log|G_n(\xi'_{n,2,1})|^{-1/n}\\&\geq(\frac{n}{T(2,2,\phi)+\varepsilon})^{(\frac{1}{\delta(2,2,\phi)-1+\varepsilon})}[(\frac{1}{\delta(2,2,\phi)})^{(\frac{1}{\delta(2,2,\phi)-1+\varepsilon})}\\&(\frac{\delta(2,2,\phi)-1}{\delta(2,2,\phi)})(\frac{1}{n(T(2,2,\phi)+\varepsilon)})^{\frac{\varepsilon}{\delta(2,2,\phi)-1+\varepsilon}}-\log 2+O(1)].\end{aligned}$$

The $(2,2)$-type coefficient formulae

$$\log|a_n|^{-1/n}\leq(\frac{n}{\frac{T(2,2,f)-\varepsilon}{M}})^{\frac{1}{(\delta(2,2,f)-1)}}$$

valid for many infinitely many indices (Juneja et al. [6]). Putting this information in above we get the relation

$$\begin{aligned}
&\log|G_n(\xi'_{n,2,1})|^{-1/n}\\
&\geq n^{\frac{1}{\delta(2,2,f)-1}}[(\frac{M'}{T(2,2,\phi)+\varepsilon})n^{(\frac{1}{\delta(2,2,\phi)-1}-\frac{1}{\delta(2,2,f)-1})}\frac{1}{n(T(2,2,\phi)+\varepsilon)}\\
&-(\frac{M}{T(2,2,f)-\varepsilon})^{\frac{1}{\delta(2,2,f)-1}}]-\log 2+O(1)
\end{aligned}$$

it gives

$$\begin{aligned}
(\frac{M}{T(2,2,f)})^{\frac{1}{(\delta(2,2,f)-1)}} \geq (\frac{M'}{T(2,2,\phi)})^{\frac{1}{\delta(2,2,\phi)-1}} \liminf\{n^{(\frac{1}{\delta(2,2,\phi)-1}-\frac{1}{\delta(2,2,f)-1})}\}\\
+\liminf_{n\to\infty}\{\frac{1}{n^{(\delta(2,2,f)/\delta(2,2,f)-1)}}\log|G_n(\xi_{n,2,2})|\}.
\end{aligned}$$

(3) For $3 \leq p = q < \infty$ following the same technique as earlier and using

$$\xi'_{n,p,q} = \xi'_{n,p,q}(T(p,q,\phi)) = \exp^{[q-1]}[\frac{\log^{[p-2]} n}{T(p,q,\phi)+\varepsilon}]^{\frac{1}{(\delta(p,q,\phi)+\varepsilon)}}$$

in (3.7) with (p,q)-type coefficient formulae [6]

$$\log|a_n|^{-1/n} \leq \exp^{[q-2]}\{(\frac{1}{(T(p,q,f)-\varepsilon)}\log^{[p-2]} n)^{\frac{1}{\delta(p,q,f)}}\}$$

valid for subsequence of infinite indices, we can easily prove the result (3).

Therefore, the proof of Theorem 3.2 is completed. □

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