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ISOMORPHISMS AND GENERALIZED DERIVATIONS IN PROPER CQ^*-ALGEBRAS

CHOONKIL PARK[1,*] AND DEOK-HOON BOO[2]

Dedicated to Professor Themistocles M. Rassias on the occasion of his sixtieth birthday

Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of homomorphisms in proper CQ^*-algebras and of generalized derivations on proper CQ^*-algebras for the following Cauchy-Jensen additive mappings:

$$\begin{aligned} f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y+z}{2}\right) &= f(x)+f(z), \\ f\left(\frac{x+y+z}{2}\right)-f\left(\frac{x-y+z}{2}\right) &= f(y), \\ 2f\left(\frac{x+y+z}{2}\right) &= f(x)+f(y)+f(z), \end{aligned}$$

which were introduced and investigated in [3, 30].

This is applied to investigate isomorphisms in proper CQ^*-algebras.

1. Introduction and preliminaries

In a series of papers [1, 2], [4]–[9] and [46]–[48], many authors have considered a special class of quasi $*$-algebras, called *proper CQ^*-algebras*, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\|\cdot\|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a *proper CQ^*-algebra* if

(i) A_0 is dense in A with respect to its norm $\|\cdot\|$;

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*Corresponding author

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(ii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(xy)^* = y^* x^*$ for all $x, y \in A$ whenever the multiplication is defined;

(iii) $\|y\|_0 = \sup_{x \in A, \|x\| \le 1} \|xy\|$ for all $y \in A_0$.

Definition 1.1. Let (A, A_0) and (B, B_0) be proper CQ^*-algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *proper CQ^*-algebra homomorphism* if $H(x) \in B_0$ and $H(xz) = H(x)H(z)$ for all $x \in A_0$ and all $z \in A$. If, in addition, the mapping $H : A \to B$ and the mapping $H|_{A_0} : A_0 \to B_0$ are bijective, then the mapping $H : A \to B$ is called a *proper CQ^*-algebra isomorphism*.

Definition 1.2. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *generalized derivation* if

$$\delta(xyz) = \delta(xy)z + x\delta(y)z + x\delta(yz)$$

for all $x, y, z \in A_0$ (see [13]).

Ulam [49] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \to G'$ an *approximate homomorphism*.

Hyers [20] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

Th.M. Rassias [38] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.3. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. *If* $p < 0$ *then inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$. *Also, if for each* $x \in E$ *the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is* $\mathbb{R}$*-linear.*

Th.M. Rassias [39] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [14] following the same approach as in Th.M. Rassias [38], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [14], as well as by Th.M. Rassias and P. Šemrl [44] that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. The counterexamples of Gajda [14], as well as of Th.M. Rassias and P. Šemrl [44] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [15], who among others studied the Hyers-Ulam-Rassias stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [38] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of P. Czerwik [10, 11], D.H. Hyers, G. Isac and Th.M. Rassias [21]).

Beginning around the year 1980 the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. D.H. Hyers and Th.M. Rassias [22], Th.M. Rassias [42] and the references therein).

J.M. Rassias [32] following the spirit of the innovative approach of Th.M. Rassias [38] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [33] for a number of other new results).

Găvruta [15] provided a further generalization of Th.M. Rassias' Theorem. In 1996, G. Isac and Th.M. Rassias [21] applied the Hyers-Ulam-Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [22], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers-Ulam stability of mappings. During the several papers have been published on various generalizations and applications of Hyers-Ulam stability and Hyers-Ulam-Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Navier-Stokes equations. Several mathematician have contributed works on these subjects (see [12], [16]–[19], [23]–[37], [40]–[43], [45]).

Throughout this paper, assume that (A, A_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{A_0}$, norm $\|\cdot\|_A$ and unit e, and that (B, B_0) is a proper CQ^*-algebra with C^*-norm $\|\cdot\|_{B_0}$, norm $\|\cdot\|_B$ and unit e'.

The purpose of this paper is to investigate the Hyers-Ulam-Rassias stability of homomorphisms in proper CQ^*-algebras and of generalized derivations on proper CQ^*-algebras.

This paper is organized as follows: In Sections 2 and 4, we prove the Hyers-Ulam-Rassias stability of homomorphisms in proper CQ^*-algebras and of generalized derivations on proper CQ^*-algebras for the Cauchy-Jensen additive mappings.

In Section 3, we investigate isomorphisms in proper CQ^*-algebras, associated to the Cauchy-Jensen additive mappings.

2. Stability of homomorphisms in proper CQ^*-algebras

For a given mapping $f : A \to B$, we define

$$C_\mu f(x, y, z) := f\left(\frac{\mu x + \mu y + \mu z}{2}\right) + \mu f\left(\frac{x - y + z}{2}\right) - \mu f(x) - \mu f(z)$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and all $x, y, z \in A$.

We prove the Hyers-Ulam-Rassias stability of homomorphisms in proper CQ^*-algebras for the functional equation $C_\mu f(x, y, z) = 0$.

Theorem 2.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that $f(x_0) \in B_0$ and*

$$\|C_\mu f(x, y, z)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{2.1}$$

$$\|C_1 f(x_0, y_0, z_0)\|_{B_0} \leq \theta(\|x_0\|_{A_0}^r + \|y_0\|_{A_0}^r + \|z_0\|_{A_0}^r), \tag{2.2}$$

$$\|f(x_0 z) - f(x_0)f(z)\|_B \leq \theta(\|x_0\|_A^{2r} + \|z\|_A^{2r}) \tag{2.3}$$

for all $\mu \in \mathbb{T}^1$, all $x_0, y_0, z_0 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper CQ^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{(2^r + 2)\theta}{2^r - 2}\|x\|_A^r \tag{2.4}$$

for all $x \in A$.

Proof. Letting $\mu = -1$ and $x = y = z = 0$ in (2.1), we get $f(0) = 0$. Letting $\mu = 1$ and $y = 2x$ and $z = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\|_B \leq (2^r + 2)\theta\|x\|_A^r \tag{2.5}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \leq \frac{(2^r + 2)\theta}{2^r}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B &\leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \\ &\leq \frac{(2^r + 2)\theta}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\|x\|_A^r \end{aligned} \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.6) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.1) that

$$\begin{aligned}
&\left\| H\left(\frac{x+y+z}{2}\right) + H\left(\frac{x-y+z}{2}\right) - H(x) - H(z)\right\|_B \\
&\quad = \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|_B \\
&\quad \le \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$H\left(\frac{x+y+z}{2}\right) + H\left(\frac{x-y+z}{2}\right) = H(x) + H(z) \tag{2.7}$$

for all $x, y, z \in A$.

Letting $y = 0$ in (2.7), we get

$$2H\left(\frac{x+z}{2}\right) = H(x) + H(z) \tag{2.8}$$

for all $x, z \in A$.

Since $H(0) = \lim_{n\to\infty} 2^n f\left(\frac{0}{2^n}\right) = \lim_{n\to\infty} 2^n f(0) = 0$, by letting $y = 2x$ and $z = x$ in (2.7), we get

$$H(2x) = 2H(x)$$

for all $x \in A$.

Replacing x by $2x$ and z by $2z$ in (2.8), we get

$$H(x + z) = H(x) + H(z)$$

for all $x, z \in A$. Hence $H : A \to B$ is Cauchy additive.

Letting $y = 0$ and $z = x$ in (2.1), we get

$$\|f(\mu x) - \mu f(x)\|_B \le \theta \|x\|_A^r$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So

$$H(\mu x) = \lim_{n\to\infty} 2^n f\left(\frac{\mu x}{2^n}\right) = \lim_{n\to\infty} \mu \cdot 2^n f\left(\frac{x}{2^n}\right) = \mu H(x) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

By the same reasoning as in the proof of Theorem 2.1 of [29], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that $H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \in B_0$ for all $x \in A_0$. So it follows from (2.3) that

$$\begin{aligned} \|H(xz) - H(x)H(z)\|_B &= \lim_{n\to\infty} 4^n \left\| f\left(\frac{xz}{4^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{z}{2^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} \frac{4^n\theta}{4^{nr}} (\|x\|_A^{2r} + \|z\|_A^{2r}) = 0 \end{aligned}$$

for all $x \in A_0$ and all $z \in A$. So

$$H(xz) = H(x)H(z)$$

for all $x \in A_0$ and all $z \in A$.

Now, let $T : A \to B$ be another Cauchy-Jensen additive mapping satisfying (2.4). Then we have

$$\begin{aligned} \|H(x) - T(x)\|_B &= 2^n \left\| H\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_B \\ &\le 2^n \left(\left\| H\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_B \right) \\ &\le \frac{2(2^r+2)}{2^r-2} \cdot \frac{2^n \cdot \theta}{2^{nr}} \|x\|_A^r, \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $H(x) = T(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique proper CQ^*-algebra homomorphism satisfying (2.4). □

Theorem 2.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.1), (2.2) *and* (2.3) *such that $f(x) \in B_0$ for all $x \in A_0$. Then there exists a unique proper CQ^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{(2+2^r)\theta}{2-2^r} \|x\|_A^r \tag{2.10}$$

for all $x \in A$.

Proof. It follows from (2.5) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \le \frac{(2+2^r)\theta}{2} \|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_B &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x)\right\|_B \\ &\le \frac{(2+2^r)\theta}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|_A^r \end{aligned} \tag{2.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.11) that the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ is a Cauchy sequence for all $x \in A$. Since B

is complete, the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.11), we get (2.10).

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that $f(x_0) \in B_0$ and*

$$\|C_\mu f(x, y, z)\|_B \le \theta \cdot \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}}, \tag{2.12}$$

$$\|C_1 f(x_0, y_0, z_0)\|_{B_0} \le \theta \cdot \|x_0\|_{A_0}^{\frac{r}{3}} \cdot \|y_0\|_{A_0}^{\frac{r}{3}} \cdot \|z_0\|_{A_0}^{\frac{r}{3}}, \tag{2.13}$$

$$\|f(x_0 z) - f(x_0)f(z)\|_B \le \theta \cdot \|x_0\|_A^r \cdot \|z\|_A^r \tag{2.14}$$

for all $\mu \in \mathbb{T}^1$, all $x_0, y_0, z_0 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper CQ^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{2^{\frac{r}{3}}\theta}{2^r - 2} \|x\|_A^r \tag{2.15}$$

for all $x \in A$.

Proof. Letting $\mu = -1$ and $x = y = z = 0$ in (2.12), we get $f(0) = 0$. So, letting $\mu = 1$ and $y = 2x$ and $z = x$ in (2.12), we get

$$\|f(2x) - 2f(x)\|_B \le 2^{\frac{r}{3}}\theta\|x\|_A^r \tag{2.16}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \le \frac{\theta}{4^{\frac{r}{3}}}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \\ &\le \frac{\theta}{4^{\frac{r}{3}}} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}} \|x\|_A^r \end{aligned} \tag{2.17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.17) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.17), we get (2.15).

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

Theorem 2.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.12), (2.13) *and* (2.14) *such that $f(x) \in B_0$ for all $x \in A_0$. Then there exists a unique proper CQ^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{2^{\frac{r}{3}}\theta}{2 - 2^r}\|x\|_A^r \tag{2.18}$$

for all $x \in A$.

Proof. It follows from (2.16) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_B \le \frac{2^{\frac{r}{3}}\theta}{2}\|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\|_B &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\|_B \\ &\le \frac{2^{\frac{r}{3}}\theta}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\|x\|_A^r \end{aligned} \tag{2.19}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.19) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.19), we get (2.18).

The rest of the proof is similar to the proof of Theorem 2.1. □

3. Isomorphisms in proper CQ^*-algebras

For a given mapping $f : A \to B$, we define

$$D_\mu f(x, y, z) := f\left(\frac{\mu x + \mu y + \mu z}{2}\right) - \mu f\left(\frac{x - y + z}{2}\right) - \mu f(y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$.

We investigate isomorphisms in proper CQ^*-algebras, associated to the functional equation $D_\mu f(x, y, z) = 0$.

Theorem 3.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping such that $f(x_0) \in B_0$ and*

$$\|D_\mu f(x, y, z)\|_B \le \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{3.1}$$

$$\|D_1 f(x_0, y_0, z_0)\|_{B_0} \le \theta(\|x_0\|_{A_0}^r + \|y_0\|_{A_0}^r + \|z_0\|_{A_0}^r), \tag{3.2}$$

$$f(x_0 z) = f(x_0)f(z) \tag{3.3}$$

for all $\mu \in \mathbb{T}^1$, all $x_0, y_0, z_0 \in A_0$ and all $x, y, z \in A$. If $f|_{A_0} : A_0 \to B_0$ is bijective and $\lim_{n\to\infty} 2^n f(\frac{e}{2^n}) = e'$, then the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism.

Proof. Letting $\mu = 1$, $y = x$ and $z = 2x$ in (3.1), we get

$$\|f(2x) - 2f(x)\|_B \leq (2^r + 2)\theta\|x\|_A^r \tag{3.4}$$

for all $x \in A$. So

$$\left\|f(x) - 2f(\frac{x}{2})\right\|_B \leq \frac{(2^r + 2)\theta}{2^r}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B &\leq \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \\ &\leq \frac{(2^r + 2)\theta}{2^r} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}}\|x\|_A^r \end{aligned} \tag{3.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.5) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get

$$\|f(x) - H(x)\|_B \leq \frac{(2^r + 2)\theta}{2^r - 2}\|x\|_A^r$$

for all $x \in A$.

It follows from (3.1) that

$$\begin{aligned}&\left\|H\left(\frac{x+y+z}{2}\right) - H\left(\frac{x-y+z}{2}\right) - H(y)\right\|_B \\ &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x-y+z}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right\|_B \\ &\leq \lim_{n\to\infty} \frac{2^n\theta}{2^{nr}}(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0\end{aligned}$$

for all $x, y, z \in A$. So

$$H\left(\frac{x+y+z}{2}\right) - H\left(\frac{x-y+z}{2}\right) = H(y) \tag{3.6}$$

for all $x, y, z \in A$.

Letting $z = x + y$ in (3.6), we get

$$H(x + y) = H(x) + H(y)$$

for all $x, y \in A$. Hence the mapping $H : A \to B$ is Cauchy additive.

Letting $x = 0$ and $z = y$ in (3.1), we get

$$\|f(\mu y) - \mu f(y)\|_B \le 2\theta \|y\|_A^r$$

for all $\mu \in \mathbb{T}^1$ and all $y \in A$. So

$$H(\mu x) = \lim_{n\to\infty} 2^n f\left(\frac{\mu x}{2^n}\right) = \lim_{n\to\infty} \mu \cdot 2^n f\left(\frac{x}{2^n}\right) = \mu H(x) \tag{3.7}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

By the same reasoning as in the proof of Theorem 2.1 of [29], the mapping $H : A \to B$ is $\mathbb{C}$-linear.

Since $f(xz) = f(x)f(z)$ for all $x \in A_0$ and all $z \in A$,

$$H(xz) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n} \cdot \frac{z}{2^n}\right) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \cdot 2^n f\left(\frac{z}{2^n}\right) = H(x)H(z)$$

for all $x \in A_0$ and all $z \in A$. So the mapping $H : A \to B$ is a proper CQ^*-algebra homomorphism.

It follows from (3.2) that $H(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) \in B_0$ for all $x \in A_0$. So it follows from (3.3) that

$$\begin{aligned} H(x) &= H(ex) = \lim_{n\to\infty} 2^n f\left(\frac{ex}{2^n}\right) = \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}x\right) = \lim_{n\to\infty} 2^n f\left(\frac{e}{2^n}\right) f(x) \\ &= e' f(x) = f(x) \end{aligned}$$

for all $x \in A$. Hence the bijective mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism. □

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (3.1), (3.2) *and* (3.3) *such that $f(x) \in B_0$ for all $x \in A_0$. If $f|_{A_0} : A_0 \to B_0$ is bijective and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism.*

Proof. It follows from (3.4) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \le \frac{(2+2^r)\theta}{2} \|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_B &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|_B \\ &\le \frac{(2+2^r)\theta}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|_A^r \end{aligned} \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.8) that the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get

$$\|f(x) - H(x)\|_B \le \frac{(2+2^r)\theta}{2-2^r} \|x\|_A^r$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 3.1. □

Theorem 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (3.3) *such that $f(x_0) \in B_0$ and*

$$\|D_\mu f(x,y,z)\|_B \le \theta \cdot \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}}, \tag{3.9}$$

$$\|D_1 f(x_0,y_0,z_0)\|_{B_0} \le \theta \cdot \|x_0\|_{A_0}^{\frac{r}{3}} \cdot \|y_0\|_{A_0}^{\frac{r}{3}} \cdot \|z_0\|_{A_0}^{\frac{r}{3}}, \tag{3.10}$$

for all $\mu \in \mathbb{T}^1$, all $x_0, y_0, z_0 \in A_0$ and all $x, y, z \in A$. If $f|_{A_0} : A_0 \to B_0$ is bijective and $\lim_{n\to\infty} 2^n f(\frac{e}{2^n}) = e'$, then the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism.

Proof. Letting $\mu = 1$, $y = x$ and $z = 2x$ in (3.9), we get

$$\|f(2x) - 2f(x)\|_B \le 2^{\frac{r}{3}}\theta \|x\|_A^r \tag{3.11}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \le \frac{\theta}{4^{\frac{r}{3}}} \|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_B \\ &\le \frac{\theta}{4^{\frac{r}{3}}} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}} \|x\|_A^r \end{aligned} \tag{3.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.12) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get

$$\|f(x) - H(x)\|_B \le \frac{2^{\frac{r}{3}}\theta}{2^r - 2} \|x\|_A^r$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorem 2.3 and 3.1. □

Theorem 3.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (3.3), (3.9) *and* (3.10) *such that $f(x) \in B_0$ for all $x \in A_0$. If $f|_{A_0} : A_0 \to B_0$ is bijective and $\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e'$, then the mapping $f : A \to B$ is a proper CQ^*-algebra isomorphism.*

Proof. It follows from (3.11) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\|_B \le \frac{2^{\frac{r}{3}}\theta}{2} \|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\|_B &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|_B \\ &\le \frac{2^{\frac{r}{3}}\theta}{2} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|_A^r \end{aligned} \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (3.13) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get

$$\|f(x) - H(x)\|_B \le \frac{2^{\frac{r}{3}}\theta}{2 - 2^r} \|x\|_A^r$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorem 2.4 and 3.1. □

4. Stability of generalized derivations on proper CQ^*-algebras

For a given mapping $f : A \to B$, we define

$$E_\mu f(x, y, z) := 2f\left(\frac{\mu x + \mu y + \mu z}{2}\right) - \mu f(x) - \mu f(y) - \mu f(z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$.

We prove the Hyers-Ulam-Rassias stability of generalized derivations on proper CQ^*-algebras for the functional equation $E_\mu f(x, y, z) = 0$.

Theorem 4.1. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\|E_\mu f(x, y, z)\|_A \le \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{4.1}$$

$$\begin{aligned} \|f(x_0 y_0 z_0) - f(x_0 y_0) z_0 &- x_0 f(y_0) z_0 - x_0 f(y_0 z_0)\|_A \\ &\le \theta(\|x_0\|_A^{3r} + \|y_0\|_A^{3r} + \|z_0\|_A^{3r}) \end{aligned} \tag{4.2}$$

for all $\mu \in \mathbb{T}^1$, all $x_0, y_0, z_0 \in A_0$ and all $x, y, z \in A$. Then there exists a unique generalized derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{(2^r + 2)\theta}{2^r - 2} \|x\|_A^r \tag{4.3}$$

for all $x \in A$.

Proof. Letting $\mu = 1$, $y = 2x$ and $z = x$ in (4.1), we get

$$\|f(2x) - 2f(x)\|_A \le (2^r + 2)\theta\|x\|_A^r \tag{4.4}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_A \le \frac{(2^r+2)\theta}{2^r}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_A &\le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_A \\ &\le \frac{(2^r+2)\theta}{2^r}\sum_{j=l}^{m-1}\frac{2^j}{2^{rj}}\|x\|_A^r \end{aligned}\tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (4.5) that the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\{2^n f\left(\frac{x}{2^n}\right)\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

It follows from (4.1) that

$$\begin{aligned}&\left\|2\delta\left(\frac{x+y+z}{2}\right) - \delta(x) - \delta(y) - \delta(z)\right\|_A \\ &= \lim_{n\to\infty} 2^n\left\|2f\left(\frac{x+y+z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|_A \\ &\le \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r) = 0\end{aligned}$$

for all $x, y, z \in A$. So

$$2\delta\left(\frac{x+y+z}{2}\right) = \delta(x) + \delta(y) + \delta(z) \tag{4.6}$$

for all $x, y, z \in A$. Letting $x = y = z = 0$ in (4.6), we get $\delta(0) = 0$.

Letting $z = x + y$ in (4.6), we get

$$\delta(x+y) = \delta(x) + \delta(y)$$

for all $x, y \in A$. Hence the mapping $\delta : A \to A$ is Cauchy additive.

Letting $y = x$ and $z = 0$ in (4.1), we get

$$\|f(\mu x) - \mu f(x)\|_A \le \theta\|x\|_A^r$$

for all $\mu \in \mathbb{T}^1$ and all $y \in A$. So

$$\delta(\mu x) = \lim_{n\to\infty} 2^n f\left(\frac{\mu x}{2^n}\right) = \lim_{n\to\infty} \mu \cdot 2^n f\left(\frac{x}{2^n}\right) = \mu\delta(x) \tag{4.7}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

By the same reasoning as in the proof of Theorem 2.1 of [29], the mapping $\delta : A \to A$ is $\mathbb{C}$-linear.

It follows from (4.2) that

$$\begin{aligned}
&\|\delta(xyz) - \delta(xy)z - x\delta(y)z - x\delta(yz)\|_A \\
&\quad = \lim_{n\to\infty} 8^n \left\| f\left(\frac{xyz}{8^n}\right) - f\left(\frac{xy}{4^n}\right)\frac{z}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}\right)\frac{z}{2^n} - \frac{x}{2^n} f\left(\frac{yz}{4^n}\right)\right\|_A \\
&\quad \le \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}) = 0
\end{aligned}$$

for all $x, y, z \in A_0$. So

$$\delta(xyz) = \delta(xy)z + x\delta(y)z + x\delta(yz)$$

for all $x, y, z \in A_0$.

Now, let $T : A \to A$ be another Cauchy-Jensen additive mapping satisfying (4.3). Then we have

$$\begin{aligned}
\|\delta(x) - T(x)\|_A &= 2^n \left\|\delta\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right)\right\|_A \\
&\le 2^n \left(\left\|\delta\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_A + \left\|T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_A\right) \\
&\le \frac{2(2^r+2)\theta}{(2^r-2)2^{nr}}\|x\|_A^r,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $\delta(x) = T(x)$ for all $x \in A$. This proves the uniqueness of δ. Thus the mapping $\delta : A \to A$ is a unique generalized derivation satisfying (4.3). $\square$

Theorem 4.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (4.1) *and* (4.2)*. Then there exists a unique generalized derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{(2+2^r)\theta}{2-2^r}\|x\|_A^r \tag{4.8}$$

for all $x \in A$.

Proof. It follows from (4.4) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_A \le \frac{(2+2^r)\theta}{2}\|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned}
\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\|_A &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\|_A \\
&\le \frac{(2+2^r)\theta}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\|x\|_A^r
\end{aligned} \tag{4.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (4.9) that the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is a Cauchy sequence for all $x \in A$. Since A

is complete, the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.9), we get (4.8).

The rest of the proof is similar to the proof of Theorem 4.1. $\square$

Theorem 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\|E_\mu f(x, y, z)\|_A \le \theta \cdot \|x\|_A^{\frac{r}{3}} \cdot \|y\|_A^{\frac{r}{3}} \cdot \|z\|_A^{\frac{r}{3}}, \tag{4.10}$$

$$\begin{aligned} \|f(x_0y_0z_0) - f(x_0y_0)z_0 \quad &- \quad x_0 f(y_0)z_0 - x_0 f(y_0z_0)\|_A \\ &\le \quad \theta \cdot \|x_0\|_A^r \cdot \|y_0\|_A^r \cdot \|z_0\|_A^r \end{aligned} \tag{4.11}$$

for all $\mu \in \mathbb{T}^1$, all $x_0, y_0, z_0 \in A_0$ and all $x, y, z \in A$. Then there exists a unique generalized derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{2^{\frac{r}{3}}\theta}{2^r - 2}\|x\|_A^r \tag{4.12}$$

for all $x \in A$.

Proof. Letting $\mu = 1$, $y = 2x$ and $z = x$ in (4.10), we get

$$\|f(2x) - 2f(x)\|_A \le 2^{\frac{r}{3}}\theta\|x\|_A^r \tag{4.13}$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_A \le \frac{\theta}{4^{\frac{r}{3}}}\|x\|_A^r$$

for all $x \in A$. Hence

$$\begin{aligned} \left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_A &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|_A \\ &\le \frac{\theta}{4^{\frac{r}{3}}} \sum_{j=l}^{m-1} \frac{2^j}{2^{rj}}\|x\|_A^r \end{aligned} \tag{4.14}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (4.14) that the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.14), we get (4.12).

The rest of the proof is similar to the proof of Theorem 4.1. $\square$

Theorem 4.4. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (4.10) *and* (4.11). *Then there exists a unique generalized derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{2^{\frac{r}{3}}\theta}{2 - 2^r}\|x\|_A^r \tag{4.15}$$

for all $x \in A$.

Proof. It follows from (4.13) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_A \leq \frac{2^{\frac{r}{3}}\theta}{2}\|x\|_A^r$$

for all $x \in A$. So

$$\begin{aligned} \left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\|_A &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1}x)\right\|_A \\ &\leq \frac{2^{\frac{r}{3}}\theta}{2}\sum_{j=l}^{m-1}\frac{2^{rj}}{2^j}\|x\|_A^r \end{aligned} \tag{4.16}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (4.16) that the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is a Cauchy sequence for all $x \in A$. Since A is complete, the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ converges. So one can define the mapping $\delta : A \to A$ by

$$\delta(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.16), we get (4.15).

The rest of the proof is similar to the proofs of Theorems 4.1 and 4.3. □

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[1]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, South Korea
E-mail address: baak@hanyang.ac.kr

[2]Department of Mathematics, Chungnam National University, Daejeon 305-764, South Korea
E-mail address: dhboo@cnu.ac.kr