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J. M. RASSIAS PRODUCT-SUM STABILITY OF AN EULER-LAGRANGE FUNCTIONAL EQUATION

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Abstract. In 1940 (and 1964) S. M. Ulam proposed the well-known Ulam stability problem. In 1941 D. H. Hyers solved the Hyers-Ulam problem for linear mappings. In 1992 and 2008, J. M. Rassias introduced the Euler-Lagrange quadratic mappings and the JMRassias "product-sum" stability, respectively. In this paper we introduce an Euler-Lagrange type quadratic functional equation and investigate the JMRassias "product-sum" stability of this equation. The stability results have applications in Mathematical Statistics, Stochastic Analysis and Psychology.

1. Introduction and Preliminaries

In 1940 (and 1964) Stanislaw M. Ulam [9] proposed the following stability problem, well-known as *Ulam stability problem*:

"When is true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true?"

In particular he stated the stability question:

"Let G_1 be a group and G_2 a metric group with the metric $\rho(.,.)$. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then a unique homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941 D. H. Hyers [3] solved this problem for linear mappings as follows:

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Theorem 1.1. *(D. H. Hyers, 1941: [3]). If a mapping $f : E \to E'$ satisfies the approximately additive inequality*

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon, \tag{1.1}$$

for some fixed $\varepsilon > 0$ and all $x, y \in E$, where E and E' are Banach spaces, then there exists a unique additive mapping $A : E \to E'$, satisfying the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), \tag{1.2}$$

and inequality

$$||f(x) - A(x)|| \le \varepsilon \tag{1.3}$$

for some fixed $\varepsilon > 0$ and all $x \in E$.

No continuity conditions are required for this result.

In 1992, Euler-Lagrange functional equations were introduced ([5],[6]).

Theorem 1.2. *(J. M. Rassias, 1992: [5]). Let X be a normed linear space, Y a Banach space, and $f : X \to Y$. If there exist $\alpha, b :\ 0 \le a + b < 2$, and $c_2 \ge 0$ such that*

$$||f(x+y) + f(x-y) - 2[f(x) + f(y)]|| \le c_2||x||^a||y||^b, \tag{1.4}$$

for all $x, y \in X$, then there exists a unique non-linear mapping $N : X \to Y$ such that

$$||f(x) - N(x)|| \le c||x||^{a+b} \tag{1.5}$$

and

$$N(x+y) + N(x-y) = 2[N(x) + N(y)] \tag{1.6}$$

for all $x, y \in X$, where $c = c_2/(4 - 2^{a+b})$.
Note that a mapping $N : X \to Y$ satisfying (1.6) is called Euler-Lagrange mapping, and a mapping $f : X \to Y$ satisfying (1.4) is approximately Euler-Lagrange mapping.

In 2008, the JMRassias "product-sum" stability was investigated for the first time ([1],[2],[7],[8]).

For the theorem that follows, let $(E, \perp)$ denote an orthogonality normed space with norm $||.||_E$ and $(F, ||.||_F)$ is a Banach space.

Theorem 1.3. *(K. Ravi, M. Arunkumar and J. M. Rassias, 2008: [7]) Let $f : E \to F$ be a mapping which satisfies the inequality*

$$\begin{aligned} &||f(mx+y) + f(mx-y) - 2f(x+y) - 2f(x-y) - 2(m^2-2)f(x) + 2f(y)||_F \\ &\qquad \le \varepsilon\big\{||x||_E^p||y||_E^p + \big(||x||_E^{2p}\big) + ||x||_E^{2p}\big)\big\} \end{aligned} \tag{1.7}$$

for all $x, y \in E$ with $x \perp y$, where ε and p are constants with $\varepsilon, p > 0$ and either $m > 1$; $p > 1$ or $m < 1$; $p > 1$ with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ and $-1 \neq |m|^{p-1} < 1$.
Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^{2n}}$$

exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$||f(x) - Q(x)||_F \leq \frac{\varepsilon}{2|m^2 - m^{2p}|}||x||_E^{2p} \tag{1.8}$$

for all $x \in E$.

Note that the mixed type product-sum function

$$(x, y) \to \varepsilon\big[||x||_E^p||y||_E^p + \big(||x||_E^{2p} + ||y||_E^{2p}\big)\big]$$

was introduced by J. M. Rassias ([1],[2],[7],[8]).

In this paper we introduce an Euler-Lagrange type quadratic functional equation and investigate the JMRassias "product-sum" stability of this equation. The stability results have applications in Mathematical Statistics, Stochastic Analysis and Psychology.

2. JMRassias product-sum stability of an Euler-Lagrange type functional equation

Let X be a real normed linear space and Y a real Banach space.

Definition 2.1. A mapping $f : X \to Y$ is called approximately Euler-Lagrange type quadratic, if the approximately Euler-Lagrange quadratic functional inequality

$$||f(x+y)+\frac{1}{2}[f(x-y)+f(y-x)]-2[f(x)+f(y)]|| \leq \varepsilon\big(||x||^{\frac{\alpha}{2}}||y||^{\frac{\alpha}{2}}+||x||^{\alpha}+||y||^{\alpha}\big) \tag{2.1}$$

holds for every $x, y \in X$ with $\varepsilon \geq 0$ and $\alpha \neq 2$.

Lemma 2.2. *Mapping $Q : X \to Y$ satisfies the Euler-Lagrange type quadratic equation*

$$Q(x + y) + \frac{1}{2}[Q(x - y) + Q(y - x)] = 2[Q(x) + Q(y)]$$

for all $x, y \in X$ if and only if there exists a mapping $T : X \to Y$ satisfying the Euler-Lagrange quadratic equation

$$T(x + y) + T(x - y) = 2[T(x) + T(y)]$$

for all $x, y \in X$ such that $Q(x) = T(x)$ for all $x \in X$.

Proof. ($\Rightarrow$) Let mapping $Q : X \to Y$ satisfy the Euler-Lagrange type quadratic equation

$$Q(x + y) + \frac{1}{2}[Q(x - y) + Q(y - x)] = 2[Q(x) + Q(y)] \tag{2.2}$$

for all $x, y \in X$. Assume that there exists a mapping $T : X \to Y$ such that $Q(x) = T(x)$ for all $x \in X$. Observe that for $x = y = 0$ and $x = x$, $y = 0$ from (2.2) we obtain respectively

$$T(0) = Q(0) = 0 \tag{2.3}$$

and

$$T(-x) = Q(-x) = Q(x) = T(x), \quad \text{for} \quad x \in X. \tag{2.4}$$

From (2.2) and (2.4) it is obvious that

$$\begin{aligned} T(x+y)+\frac{1}{2}[T(x-y)+T(y-x)] &= 2[T(x)+T(y)], \text{or} \\ T(x+y)+\frac{1}{2}[T(x-y)+T(-(x-y))] &= 2[T(x)+T(y)], \text{or} \\ T(x+y)+T(x-y) &= 2[T(x)+T(y)]. \end{aligned}$$

Hence, T satisfies the Euler-Lagrange quadratic equation.

($\Leftarrow$) Let mapping $T : X \to Y$ satisfy the Euler-Lagrange quadratic equation

$$T(x+y)+T(x-y) = 2[T(x)+T(y)] \tag{2.5}$$

for all $x, y \in X$. Assume that there exists a mapping $Q : X \to Y$ such that $Q(x) = T(x)$ for all $x \in X$. Observe that for $x = y = 0$ and $x = 0, y = x$ from (2.5) we obtain

$$Q(0) = T(0) = 0 \tag{2.6}$$

and

$$Q(x) = T(x) = T(-x) = Q(-x), \quad \text{for} \quad x \in X. \tag{2.7}$$

Thus, from (2.5) - (2.7) one gets

$$\begin{aligned} 2[Q(x)+Q(y)] &= 2[T(x)+T(y)] = T(x+y)+T(x-y) \\ &= T(x+y)+\frac{1}{2}T(x-y)+\frac{1}{2}T(-(y-x)) \\ &= Q(x+y)+\frac{1}{2}[Q(x-y)+Q(y-x)]. \end{aligned}$$

Hence, Q satisfies the Euler-Lagrange type quadratic equation.

Thus the proof of Lemma 2.2 is complete.

Theorem 2.3. *Assume that $f : X \to Y$ is an approximately Euler-Lagrange type additive mapping satisfying (2.1).*
Then, there exists a unique Euler-Lagrange type quadratic mapping $Q : X \to Y$ which satisfies the formula

$$Q(x) = \lim_{n\to\infty} f_n(x), \tag{2.8}$$

where

$$f_n(x) = \begin{cases} 2^{-2n} f(2^n x), & -\infty<\alpha<2 \\ 2^{2n} f(2^{-n} x), & \alpha>2 \end{cases}$$

for all $x \in X$ and $n \in N = \{0, 1, 2, ...\}$, which is the set of natural numbers and

$$||f(x) - Q(x)|| \le \frac{3\varepsilon}{|2^\alpha - 4|}||x||^\alpha \tag{2.9}$$

for some fixed $\varepsilon > 0$, $\alpha \neq 2$ and all $x \in X$.
$Q : X \to Y$ is a unique Euler-Lagrange type quadratic mapping satisfying equation

$$Q(x+y)+\frac{1}{2}[Q(x-y)+Q(y-x)] = 2[Q(x)+Q(y)]. \tag{2.10}$$

Proof. We start our proof considering: $-\infty < \alpha < 2$.
Step 1. By substituting $x = y$ in (2.1), we can observe that

$$||f(2x) + f(0) - 4f(x)|| \leq 3\varepsilon||x||^{\alpha},$$

from which for $x = 0$ it occurs that

$$f(0) = 0 \tag{2.11}$$

and in extension

$$||f(x) - 2^{-2}f(2x)|| \leq \frac{3}{4}\varepsilon||x||^{\alpha}. \tag{2.12}$$

Hence, for $n \in N - \{0\}$

$$\begin{aligned} ||f(x) - 2^{-2n}f(2^n x)|| &\leq ||f(x) - 2^{-2}f(2x)|| + ||2^{-2}f(2x) - 2^{-4}f(2^2 x)|| + ... \\ &+ ||2^{-2(n-1)}f(2^{n-1}x) - 2^{-2n}f(2^n x)|| \\ &\leq \frac{3}{4}(1 + 2^{\alpha-2} + ... + 2^{(n-1)(\alpha-2)})\varepsilon||x||^{\alpha} \\ &= \frac{3}{4-2^{\alpha}}(1 - 2^{n(\alpha-2)})\varepsilon||x||^{\alpha}. \end{aligned}$$

Thus,

$$||f(x) - 2^{-2n}f(2^n x)|| \leq \frac{3}{4-2^{\alpha}}(1 - 2^{n(\alpha-2)})\varepsilon||x||^{\alpha}, \tag{2.13}$$

for $n \in N - \{0\}$ and $-\infty < \alpha < 2$.

Step 2. Following, we need to show that if there is a sequence $\{f_n\} : f_n(x) = 2^{-2n}f(2^n x)$, then $\{f_n\}$ converges.

For every $n > m > 0$, we can obtain

$$\begin{aligned} ||f_n(x) - f_m(x)|| &= ||2^{-2n}f(2^n x) - 2^{-2m}f(2^m x)|| \\ &= 2^{-2m}||f(2^m x) - 2^{-2(n-m)}f(2^{(n-m)}2^m x)|| \\ &\leq 2^{m(\alpha-2)}\frac{3\varepsilon}{4-2^{\alpha}}(1 - 2^{(n-m)(\alpha-2)})||x||^{\alpha} \\ &< 2^{m(\alpha-2)}\frac{3\varepsilon}{4-2^{\alpha}}||x||^{\alpha} \to 0, \end{aligned}$$

for $m \to \infty$, as $\alpha < 2$. Therefore, $\{f_n\}$ is a Cauchy sequence. Since Y is *complete* we can conclude that $\{f_n\}$ is convergent. Thus, there is a well-defined $Q : X \to Y$ such that $Q(x) = \lim_{n\to\infty} 2^{-2n}f(2^n x)$, for $\alpha < 2$.

Step 3. Observe that

$$||f(x) - f_n(x)|| = ||f(x) - 2^{-2n}f(2^n x)|| \leq \frac{3\varepsilon}{4-2^{\alpha}}(1 - 2^{n(\alpha-2)})||x||^{\alpha},$$

from which by letting $n \to \infty$ we obtain

$$||f(x) - Q(x)|| \leq \frac{3\varepsilon}{4-2^{\alpha}}||x||^{\alpha}. \tag{2.14}$$

Step 4. Claim that mapping $Q : X \to Y$ satisfies (2.10). In fact, by letting $x \to 2^n x$ and $y \to 2^n y$, from (2.1), we have:

$$||f\big(2^n(x+y)\big) + \frac{1}{2}[f\big(2^n(x-y)\big) + f\big(2^n(y-x)\big)] - 2[f\big(2^n x\big) + f\big(2^n y\big)]|| \\ \leq \varepsilon\big(||2^n x||^{\frac{\alpha}{2}}||2^n y||^{\frac{\alpha}{2}} + ||2^n x||^{\alpha} + ||2^n y||^{\alpha}\big).$$

Next, by multiplying with 2^{-2n} we obtain

$$\begin{aligned} 0 \leq ||2^{-2n} f\big(2^n(x+y)\big) &+ \frac{1}{2}[2^{-2n} f\big(2^n(x-y)\big) + 2^{-2n} f\big(2^n(y-x)\big)] \\ &- 2[2^{-2n} f\big(2^n x\big) + 2^{-2n} f\big(2^n y\big)]|| \\ &\leq 2^{n(\alpha-2)}\varepsilon\big(||x||^{\frac{\alpha}{2}}||y||^{\frac{\alpha}{2}} + ||x||^{\alpha} + ||y||^{\alpha}\big) \end{aligned}$$

and by letting $n \to \infty$, for $\alpha < 2$ we can conclude that an $Q : X \to Y$ truly exists such that: $Q(x) = \lim_{n\to\infty} 2^{-2n} f(2^n x)$ satisfies the *Euler-Lagrange type quadratic property*

$$Q(x+y) + \frac{1}{2}[Q(x-y) + Q(y-x)] = 2[Q(x) + Q(y)]. \tag{2.15}$$

Therefore, existence of Theorem holds.

Step 5. We need to prove that Q is *unique.*

Observe, from (2.15), that for *a)* $x = y = 0$, *b)* $x = x, y = 0$ and *c)* $x = y$, we obtain:

$$a)\, Q(0) = 0, \quad b)\, Q(-x) = Q(x) \quad \text{and} \quad c)\, Q(2x) = 2^2 Q(x),$$

respectively. Therefore, by *induction*, by claiming that $Q(2^{n-1}x) = 2^{2(n-1)}Q(x)$, we can show that

$$Q(2^n x) = 2^2 Q(2^{n-1}x) = 2^{2n}Q(x)$$

or equivalently

$$Q(x) = 2^{-2n}Q(2^n x). \tag{2.16}$$

Assume, now, the existence of another $Q' : X \to Y$, such that $Q'(x) = 2^{-2n}Q'(2^n x)$. With the aid of the (2.14)-(2.16) and the triangular inequality, one gets

$$\begin{aligned} 0 \leq ||Q(x) - Q'(x)|| &= ||2^{-2n}Q(2^n x) - 2^{-2n}Q'(2^n x)|| \\ &\leq ||2^{-2n}Q(2^n x) - 2^{-2n} f(2^n x)|| + ||2^{-2n} f(2^n x) - 2^{-2n}Q'(2^n x)|| \\ &\leq 2^{n(\alpha-2)}\frac{6\varepsilon}{4 - 2^{\alpha}}||x||^{\alpha} \\ &\to 0, \end{aligned}$$

as $n \to \infty$, $(\alpha < 2)$. Thus, the *uniqueness* of Q is proved and the stability of *Euler-Lagrange type quadratic mapping* $Q : X \to Y$ is established.

The proof for the case of $\alpha > 2$ is similar to the proof for $-\infty < \alpha < 2$. In fact, we can find the general inequality

$$||f(x) - 2^{2n} f(2^{-n}x)|| \leq \frac{3\varepsilon}{2^{\alpha} - 4}(1 - 2^{n(2-\alpha)})||x||^{\alpha}, \tag{2.17}$$

for all $n \in N - \{0\}$. Thus from this inequality (2.17) and the formula

$$Q(x) = \lim_{n\to\infty} 2^{2n} f(2^{-n}x),$$

for $n \to \infty$, we get the inequality

$$||f(x) - Q(x)|| \leq \frac{3\varepsilon}{2^\alpha - 4}||x||^\alpha, \quad \text{for} \quad \alpha > 2.$$

The rest of the proof for $\alpha > 2$ is omitted as similar to the above mentioned proof for $-\infty < \alpha < 2$. □

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