J. Nonlinear Sci. Appl. 3 (2010), no. 2, 87–95

THE JOURNAL OF NONLINEAR SCIENCE AND APPLICATIONS
http://www.tjnsa.com

SOME FIXED POINT THEOREMS WITH APPLICATIONS TO BEST SIMULTANEOUS APPROXIMATIONS

T.D. NARANG[1]* AND SUMIT CHANDOK[2]

Communicated by Renu Chugh

ABSTRACT. For a subset K of a metric space (X, d) and $x \in X$, the set $P_K(x) = \{y \in K : d(x, y) = d(x, K) \equiv \inf\{d(x, k) : k \in K\}\}$ is called the set of best K-approximant to x. An element $g_\circ \in K$ is said to be a best simultaneous approximation of the pair $y_1, y_2 \in X$ if

$$\max\{d(y_1, g_\circ), d(y_2, g_\circ)\} = \inf_{g\in K} \max\{d(y_1, g), d(y_2, g)\}.$$

Some results on T-invariant points for a set of best simultaneous approximation to a pair of points y_1, y_2 in a convex metric space (X, d) have been proved by imposing conditions on K and the self mapping T on K. For self mappings T and S on K, results are also proved on both T- and S- invariant points for a set of best simultaneous approximation. The results proved in the paper generalize and extend some of the results of P. Vijayaraju [Indian J. Pure Appl. Math. 24(1993) 21-26]. Some results on best K-approximant are also deduced.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be (s.t.b.) a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** [9].

Date: Received: 25 July 2009.
* Corresponding author.
2000 *Mathematics Subject Classification.* 47H10; 54H25.
Key words and phrases. Best approximation, fixed point, nonexpansive, R-weakly commuting, R-subweakly commuting, asymptotically nonexpansive and uniformly asymptotically regular maps.

A convex metric space (X, d) is said to satisfy **Property (I)** [2] if for all $x, y, p \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [9]). Property (I) is always satisfied in a normed linear space.

A subset K of a convex metric space (X, d) is s.t.b.

$i)$ a **convex set** [9] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$;

$ii)$ **starshaped** or p**-starshaped** [3] if there exists $p \in K$ such that $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$.

Clearly, each convex set is starshaped but not conversely.

A self map T on a metric space (X, d) is s.t.b.

$i)$ **nonexpansive** if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$;

$ii)$ **contraction** if there exists an α, $0 \leq \alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.

For a nonempty subset K of a metric space (X, d), a mapping $T : K \to K$ is s.t.b.

$i)$ **demicompact** if every bounded sequence $< x_n >$ in K satisfying $d(x_n, Tx_n) \to 0$ has a convergent subsequence;

$ii)$ **asymptotically nonexpansive** [1] if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(x, y)$, for all $x, y \in K$.

Let $T, S : K \to K$. Then T is s.t.b.

$i)$ S**-asymptotically nonexpansive** if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy)$, for all $x, y \in K$;

$ii)$ **uniformly asymptotically regular** on K if, for each $\epsilon > 0$, there exists a positive integer N such that $d(T^n(x), T^n(y)) < \epsilon$ for all $n \geq N$ and for all $x, y \in K$.

Let M a nonempty subset of a metric space (X, d), then mappings $T, S : M \to M$ are s.t.b.

$i)$ **commuting** on M if $STx = TSx$ for all $x \in M$;

$ii)$ R**-weakly commuting** [5] on M if there exists $R > 0$ such that $d(TSx, STx) \leq Rd(Tx, Sx)$ for all $x \in M$.

Suppose (X, d) is a convex metric space, M a q-starshaped subset with $q \in F(S) \cap M$ and is both T- and S-invariant. Then T and S are called

$i)$ R**-subcommuting** [8] on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq (R/k) dist(Sx, W(Tx, q, k))$, $k \in [0, 1)$;

$ii)$ R**-subweakly commuting** [7] on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R dist(Sx, W(Tx, q, k))$, $k \in [0, 1]$;

iii) **uniformly R-subweakly commuting** on M if for all $x \in M$, there exists a real number $R > 0$ such that $d(T^n Sx, ST^n x) \leq R dist(Sx, W(T^n x, q, k))$, $k \in [0, 1]$.

It is well known that commuting maps are R-subweakly commuting maps and R-subweakly commuting maps are R-weakly commuting but not conversely (see [7]).

In this paper we prove some results on T-invariant points for a set of best simultaneous approximation to a pair of points y_1, y_2 in a convex metric space (X, d) by imposing conditions on K and the self mapping T on K. For self mappings T and S on K, results are also proved on both T- and S- invariant points for a set of best simultaneous approximation. The results proved in the paper generalize and extend some of the results of Vijayaraju [10]. Some results on best K-approximant are also deduced.

Throughout, we shall write $F(S)$ for set of fixed points of a mapping S and $F(T, S)$ for set of fixed points of both mappings T and S.

2. Main results

Theorem 2.1. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I). Suppose that $y_1, y_2 \in X$. Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. If the set D of best simultaneous approximation to y_1 and y_2 is nonempty, complete, bounded, starshaped and is invariant under T, then it contains a T-invariant point provided that T is continuous and demicompact.*

Proof. Since T is asymptotically nonexpansive, there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(x, y)$, for all $x, y \in K$. Suppose that z is a star-center of D. Define T_n as $T_n(x) = W(T^n x, z, a_n)$ for all $x \in D$ where $a_n = (1 - 1/n)/k_n$. Since z is a star-center of D and $T(D) \subseteq D$, T_n is a self map of D for each n. Consider

$$\begin{aligned}
d(T_n x, T_n y) &= d(W(T^n x, z, a_n), W(T^n y, z, a_n)) \\
&\leq a_n d(T^n x, T^n y) \\
&\leq a_n k_n d(x, y) \\
&= ((1 - (1/n))/k_n) k_n d(x, y) \\
&= (1 - (1/n)) d(x, y).
\end{aligned}$$

Therefore each T_n is a contraction on D. So, by Banach's contraction principle, T_n has a unique fixed point, say, u_n in D. As D is bounded and $a_n \to 1$, we have

$$\begin{aligned}
d(u_n, T^n u_n) &= d(T_n u_n, T^n u_n) \\
&= d(W(T^n u_n, z, a_n), T^n u_n) \\
&\leq a_n d(T^n u_n, T^n u_n) + (1 - a_n) d(z, T^n u_n) \\
&\to 0.
\end{aligned}$$

Since T is uniformly asymptotically regular and symptotically nonexpansive on K, it follows that

$$\begin{aligned} d(u_n, Tu_n) &\leq d(u_n, T^n u_n) + d(T^n u_n, T^{n+1} u_n) + d(T^{n+1} u_n, Tu_n) \\ &\leq d(u_n, T^n u_n) + d(T^n u_n, T^{n+1} u_n) + k_1 d(T^n u_n, u_n) \\ &\to 0. \end{aligned}$$

Since T is demicompact, u_n has a subsequence u_{n_i} such that $u_{n_i} \to u \in D$. Since T is continuous, $T(u_{n_i}) \to Tu$. Therefore

$$\begin{aligned} d(u, Tu) &\leq d(u, u_{n_i}) + d(u_{n_i}, Tu_{n_i}) + d(Tu_{n_i}, Tu) \\ &\to 0. \end{aligned}$$

and hence $Tu = u$. □

Corollary 2.2. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I). Suppose that $y_1, y_2 \in X$. Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.1}$$

for all $x \in X$ and $i = 1, 2$. If the set D of best simultaneous approximation to y_1 and y_2 is nonempty, complete, bounded and starshaped, then it contains a T-invariant point provided that T is continuous and demicompact.

Proof. Since D is the set of best simultaneous approximation to y_1 and y_2, T maps D into itself. Indeed, if $x \in D$ we have $d(Tx, y_i) \leq d(x, y_i)$ for all $x \in X$ and $i = 1, 2$, so Tx is in D. Hence the result follows from Theorem 2.1. □

If $y_1 = y_2 = x$, we have

Corollary 2.3. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I). Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. If the set D of best K-approximants to $x \in X$ is nonempty, complete, bounded, starshaped and is invariant under T, then it contains a T-invariant point provided that T is continuous and demicompact.*

Theorem 2.4. *Let K be a nonempty subset of a convex metric linear space (X, d) with Property (I). Suppose that $y_1, y_2 \in X$. Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. If the set D of best simultaneous approximation to y_1 and y_2 is nonempty, complete, bounded, starshaped and is invariant under T, then it contains a T-invariant point provided that $(I - T)(D)$ is closed where I denotes the identity mapping.*

Proof. Defining T_n as in Theorem 2.1 and proceeding we see that each T_n is a contraction on D and $d(u_n, T^n u_n) \to 0$ where u_n is the unique fixed point of T_n in D.

Consider $u_n - Tu_n = (I-T)u_n \in (I-T)D$. Since T is uniformly asymptotically regular and symptotically nonexpansive on K, we have

$$\begin{aligned} d((I-T)u_n, 0) &= d(u_n - Tu_n, 0) \\ &= d(u_n, Tu_n) \\ &\leq d(u_n, T^n u_n) + d(T^n u_n, T^{n+1}u_n) + d(T^{n+1}u_n, Tu_n) \\ &\leq d(u_n, T^n u_n) + d(T^n u_n, T^{n+1}u_n) + k_1 d(T^n u_n, u_n) \\ &\to 0. \end{aligned}$$

i.e., $(I-T)u_n \to 0$. Since $(I-T)(D)$ is closed, $0 \in (I-T)D$ and so $0 = (I-T)u$ for some $u \in D$. Hence $Tu = u$. □

Corollary 2.5. [10] *Let K be a nonempty subset of a normed linear space X. Suppose that $y_1, y_2 \in X$. Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. If the set D of best simultaneous K-approximants to y_1 and y_2 is nonempty, complete, bounded and starshaped which is invariant under T, then it contains a T-invariant point provided that $(I-T)(D)$ is closed where I denotes the identity mapping.*

Corollary 2.6. *Let K be a nonempty subset of a convex metric linear space (X,d) with Property (I). Suppose that $y_1, y_2 \in X$. Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.2}$$

for all $x \in X$ and $i = 1, 2$. If the set D of best simultaneous approximation to y_1 and y_2 is nonempty, complete, bounded and starshaped, then it contains a T-invariant point provided that $(I-T)(D)$ is closed where I denotes the identity mapping.

Proof. Proceeding as in Corollary 2.2, the result follows from Theorem 2.4. □

Corollary 2.7. [10] *Let K be a nonempty subset of a normed linear space X. Suppose that $y_1, y_2 \in X$. Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.3}$$

for all $x \in X$ and $i = 1, 2$. If the set D of best simultaneous approximation to y_1 and y_2 is nonempty, complete, bounded and starshaped, then it contains a T-invariant point provided that $(I-T)(D)$ is closed where I denotes the identity mapping.

If $y_1 = y_2 = x$, we have

Corollary 2.8. *Let K be a nonempty subset of a convex metric linear space (X,d) with Property (I). Let T be an asymptotically nonexpansive, uniformly asymptotically regular self mapping of K. If the set D of best K-approximants to $x \in X$ is nonempty, complete, bounded, starshaped and is invariant under T, then it contains a T-invariant point provided that $(I-T)(D)$ is closed where I denotes the identity mapping.*

We need the following lemma of Shahzad [8] for our next theorem.

Lemma 2.9. [8] *Let D be a closed subset of a metric space (X, d), and S, T are R-weakly commuting self maps of D such that $T(D) \subseteq S(D)$. Suppose T is S-contraction. If $\overline{T(D)}$ is complete and T is continuous, then $F(T) \cap F(S)$ is singleton.*

Theorem 2.10. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S)$, and D is invariant under T. If T and S are uniformly R-subweakly commuting on D, T is uniformly asymptotically regular on D and S is affine on D such that $S(D) = D$, then D contains T- and S- invariant point.*

Proof. Define T_n as in Theorem 2.1, we observe that for each n, T_n is a self map on D. Consider

$$\begin{aligned} d(T_nSx, ST_nx) &= d(W(T^nSx, z, a_n), SW(T^nx, z, a_n)) \\ &= d(W(T^nSx, z, a_n), W(ST^nx, z, a_n)) \\ &\leq a_n d(T^nSx, ST^nx) \\ &\leq a_n R dist(Sx, W(T^nx, z, a_n)) \\ &\leq a_n R d(Sx, T_nx) \end{aligned}$$

for all $x \in D$. Therefore T_n and S are R-weakly commuting for each n. Since $T_n(D) \subseteq D$ and $S(D) = D$, $T_n(D) \subseteq S(D)$. Since T is S-asymptotically nonexpansive, we have

$$\begin{aligned} d(T_nx, T_ny) &= d(W(T^nx, z, a_n), W(T^ny, z, a_n)) \\ &\leq a_n d(T^nx, T^ny) \\ &\leq a_n k_n d(Sx, Sy) \\ &= ((1 - (1/n))/k_n) k_n d(Sx, Sy) \\ &= (1 - (1/n)) d(Sx, Sy). \end{aligned}$$

Therefore each T_n is a S-contraction on D. Also, D is compact and T is continuous on D and so by Lemma 2.9, there is a point x_n in D such that $x_n = T_nx_n = Sx_n$. Therefore

$$\begin{aligned} d(x_n, T^nx_n) &= d(T_nx_n, T^nx_n) \\ &= d(W(T^nx_n, z, a_n), T^nx_n) \\ &\leq a_n d(T^nx_n, T^nx_n) + (1 - a_n) d(z, T^nx_n) \\ &\to 0. \end{aligned}$$

Since T is uniformly asymptotically regular and S-asymptotically nonexpansive on D, S is affine on D and $x_n = T_n x_n = S x_n$, it follows that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(x_n)) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(T_n x_n)) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(W(T^n x_n, z, a_n)) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), W(ST^n x_n, z, a_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) \\ &\quad + k_1 (a_n d(ST^n x_n, ST^n x_n) + (1 - a_n) d(ST^n x_n, z)) \\ &\to 0. \end{aligned}$$

Since D is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x \in D$. Since T is continuous, $T(x_{n_i}) \to T(x)$, and so

$$d(x, Tx) \leq d(x, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx) \to 0,$$

which gives $Tx = x$. Since S is continuous and $x_{n_i} = S(x_{n_i})$, it follows that $Sx = x$. Hence $x \in F(T, S)$.

□

We need the following lemma of Jungck [4] for our next theorem.

Lemma 2.11. [4] *Let (X, d) be a compact metric space. Suppose that T and S are commuting mappings of X into itself such that $T(X) \subseteq S(X)$, S is continuous and $d(Tx, Ty) < d(Sx, Sy)$ for all $x, y \in X$ whenever $Sx \neq Sy$. Then T and S have a unique common fixed point in X.*

Theorem 2.12. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S)$, and D is invariant under T. If T and S commute on D, T is uniformly asymptotically regular on D and S is affine on D such that $S(D) = D$, then D contains T- and S- invariant point.*

Proof. Define T_n as in Theorem 2.1, we observe that for each n, T_n is a self map on D. Consider

$$T_n(Sx) = W(T^n(Sx), Sz, a_n) = W(S(T^n x), Sz, a_n) = SW(T^n x, z, a_n) = S(T_n x).$$

Therefore T_n and S commute for each n. Since $T_n(D) \subseteq D$ and $S(D) = D$, so $T_n(D) \subseteq S(D)$. Suppose $x, y \in D$ and $Sx \neq Sy$. Then we have

$$\begin{aligned} d(T_n x, T_n y) &= d(W(T^n x, z, a_n), W(T^n y, z, a_n)) \\ &\leq a_n d(T^n x, T^n y) \\ &\leq a_n k_n d(Sx, Sy) \\ &= ((1 - (1/n))/k_n) k_n d(Sx, Sy) \\ &= (1 - (1/n)) d(Sx, Sy). \end{aligned}$$

Also, D is compact and S is continuous on D and so by Lemma 2.11, there is a point x_n in D such that $x_n = T_n x_n = S x_n$. Therefore

$$\begin{aligned} d(x_n, T^n x_n) &= d(T_n x_n, T^n x_n) \\ &= d(W(T^n x_n, z, a_n), T^n x_n) \\ &\leq a_n d(T^n x_n, T^n x_n) + (1 - a_n) d(z, T^n x_n) \\ &\to 0. \end{aligned}$$

Since T is uniformly asymptotically regular and S-asymptotically nonexpansive on D, S commutes with T^n and $x_n = S x_n$, it follows that

$$\begin{aligned} d(x_n, T x_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, T x_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(x_n)) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n) \\ &\to 0. \end{aligned}$$

Since D is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x \in D$. Since T is continuous, $T(x_{n_i}) \to T(x)$, it follows that

$$d(x, Tx) \leq d(x, x_{n_i}) + d(x_{n_i}, T x_{n_i}) + d(T x_{n_i}, Tx) \to 0,$$

which gives $Tx = x$. Since S is continuous and $x_{n_i} = S(x_{n_i})$, it follows that $Sx = x$. Hence $x \in F(T, S)$. □

Corollary 2.13. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.4}$$

for all $x \in X$ and $i = 1, 2$. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S)$, and D is invariant under T. If T and S is commuting on D, T is uniformly asymptotically regular on D and S is affine on D such that $S(D) = D$, then D contains T- and S- invariant point.

Corollary 2.14. [10] *Let K be a nonempty subset of a normed linear space X, T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S)$, and D is invariant under T. If T and S is commuting on D, T is uniformly asymptotically regular on D and S is affine on D such that $S(D) = D$, then D contains T- and S- invariant point.*

Corollary 2.15. [10] *Let K be a nonempty subset of a normed linear space X, T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2.5}$$

for all $x \in X$ and $i = 1, 2$. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped *with respect to $z \in F(S)$, and D is invariant under T. If T and S is commuting on D, T is uniformly asymptotically regular on D and S is affine on D such that $S(D) = D$, then D contains T- and S- invariant point.*

If $y_1 = y_2 = x$, we have

Corollary 2.16. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S are continuous self-mappings of K such that T is S-asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that the set D of best K-approximants is nonempty, compact and starshaped with respect to $z \in F(S)$, and D is invariant under T. If T and S is commuting on D, T is uniformly asymptotically regular on D and S is affine on D such that $S(D) = D$, then D contains T- and S- invariant point.*

Remark 2.17. It is not necessary that S is linear in Theorem 3 of Sahab et al. [6]. The result is also true for an affine mapping S.

References

1. K. Goebel and W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc.Amer.Math.Soc. 35 (1972), 171-174.
2. M.D. Guay, K.L. Singh and J.H.M. Whitfield, Fixed point theorems for nonexpansive mappings in convex metric spaces, Proc. Conference on nonlinear analysis (Ed. S.P. Singh and J.H. Bury) Marcel Dekker 80 (1982), 179-189.
3. Sh. Itoh, Some fixed point theorems in metric spaces, Fundamenta Mathematicae 52 (1979), 109-117.
4. G. Jungck, An iff fixed point criterion, Math. Mag. 49(1976) 32-34.
5. R.P. Pant, Common fixed points of noncommuting mappings, J. Math. Anal. Appl., 188 (1994), 436-440.
6. S.A. Sahab, M.S. Khan and S. Sessa, A result in best approximation theory, J. Approx. Theory 55 (1988), 349-351.
7. N. Shahzad, Invariant approximations and R-subweakly commuting maps, J. Math. Anal. Appl. 257 (2001), No. 1, 39-45.
8. N. Shahzad, Noncommuting maps and best approximations, Radovi Mat. 10 (2001), 77-83.
9. W. Takahashi, A convexity in metric space and nonexpansive mappings I, Kodai Math. Sem. Rep., 22 (1970), 142-149.
10. P. Vijayaraju, Applications of fixed point theorems to best simultaneous approximations, Indian J. Pure Appl. Math. 24 (1993) 21-26.

[1] Department of Mathematics, Guru Nanak Dev University, Amritsar-143001, India.

E-mail address: tdnarang1948@yahoo.co.in

[2] School of Mathematics and Computer Applications, Thapar University, Patiala-147004, India.

E-mail address: chansok.s@gmail.com; sumit.chandok@thapar.edu