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GENERALIZATION SOME FUZZY SEPARATION AXIOMS TO DITOPOLOGICAL TEXTURE SPACES

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Abstract. The authors characterize the notion of quasi coincident in texture spaces and study the generalization of fuzzy quasi separation axioms defined by [12] to the ditopological texture spaces.

1. Introduction

Ditopological Texture Spaces: The notion of texture space was firstly introduced by L.M. Brown in [1, 2] under the name of fuzzy structure, and then it was called as texture space by L. M. Brown and R. Ertürk in [5, 6]. Ditopological texture spaces were introduced by L. M. Brown as a natural extension of the work of first author on the representation of lattice-valued topologies by bitopologies in [4]. It is well known that the concept of ditopology is more general than general topology, fuzzy topology and bitopology. So, it will be more advantage to generalize some various general (fuzzy, bi)-topological concepts to the ditopological texture spaces. An adequate introduction to the theory of texture spaces and ditopological texture spaces, and the motivation for its study may be obtained from [7, 8, 9, 10, 11, 16].

Let S be a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*, or a *texture space*.

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In a texture, arbitrary joins need not be coincide with unions, and clearly this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions. In this case $(S, \mathcal{S})$ is said to be *plain texture*. It is known that, in a plain texture the cases of

(1) $s \notin Q_s$ for all $s \in S$,
(2) $P_s \not\subseteq Q_s$ for all $s \in S$,
(3) $A = \bigvee_i A_i = \bigcup A_i$ for all $A_i \in \mathcal{S}, i \in I$

are equivalent.

A texture space is called *coseparated* if $Q_s \subseteq Q_t \implies P_s \subseteq P_t$ for all $s, t \in S$ In general, a texturing of S need not be closed under set complementation, it will be that if there exists a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $A = \sigma(\sigma(A))$ for all $A \in \mathcal{S}$, and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a *complementation* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \sigma)$ is said to be a *complemented texture*.

We will call a complementation σ on $(S, \mathcal{S})$ *grounded* [16] if there is an involution $s \to s'$ on S so that $\sigma(P_s) = Q_{s'}$ and $\sigma(Q_s) = P_{s'}$ where $s' = \sigma(s')$ for all $s \in S$, and in this case the complemented texture space $(S, \mathcal{S}, \sigma)$ is called *complemented grounded texture space*. It is noted that a complemented plain texture is grounded [17].

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets*

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the *q-sets*,

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}.$$

Theorem 1.1. [7] *In any texture space $(S, \mathcal{S})$, we have the following statements:*

(1) $s \notin A \Longrightarrow A \subseteq Q_s \Longrightarrow s \notin A^\flat$ *for all* $s \in S, A \in \mathcal{S}$.
(2) $A^\flat = \{s \in S \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
(3) $(\bigvee_{i \in I} A_i)^\flat = \bigcup_{i \in I} A_i^\flat$ *for all* $s \in S, A \in \mathcal{S}$.
(4) A *is the smallest element of* $\mathcal{S}$ *containing* $A^\flat$
(5) *For* $A, B \in \mathcal{S}$, *if* $A \not\subseteq B$ *then there exists* $s \in S$ *with* $A \not\subseteq Q_s$ *and* $P_s \not\subseteq B$
(6) $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.
(7) $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

The followings are some basic examples of textures.

Examples 1.2. (1) If X is a set and $\mathcal{P}(X)$ the powerset of X, then $(X, \mathcal{P}(X))$ is the *discrete texture* on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.

(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{I} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the *unit interval texture* $(\mathbb{I}, \mathcal{I})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$.

(3) The texture $(L, \mathcal{L})$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in \mathbb{I}\}$. For $r \in L$, $P_r = (0, r] = Q_r$.

Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. For $A \in \mathcal{S}$ we define the *closure* $[A]$ and the *interior* $]A[$ of A under (τ, κ) by the equalities

$$[A] = \bigcap\{K \in \kappa \mid A \subseteq K\} \text{ and }]A[= \bigvee\{G \in \tau \mid G \subseteq A\}.$$

On the other hand if the texture space $(S, \mathcal{S})$ is complemented, then we say that (τ, κ) is a complemented ditopology on $(S, \mathcal{S}, \sigma)$ if τ and κ are related by $\kappa = \sigma[\tau]$. In this case we have $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[) = [\sigma(A)]$.

It is well known that; in classic theory, $A \cap B = \emptyset \iff A \subseteq X \setminus B$ for $A, B \subseteq X$, and in fuzzy set theory, $A \cap B = \emptyset \implies A \subseteq X \setminus B$ for $A, B \subseteq X$.

So it could be defined an alternative binary implication in fuzzy set theory such as: Let I^X be the family of fuzzy sets and $A, B \in I^X$, then we say that;

Definition 1.3. [15] (1) A is *quasi-coincident* with B (denoted by AqB) $\iff$ there exists an $x \in X$ such that $A(x) + B(x) > 1$,

(2) A is *not quasi-coincident* with B (denoted by $A \not{q} B$) $\iff A(x) + B(x) \leq 1$ for all $x \in X$.

2. Q-separation axioms

In [9], L. M. Brown, R. Ertürk and Ş. Dost have generalized the fuzzy separation axioms in the sense of B. Hutton and I. Reilly [14] to the ditopological texture spaces and they have obtained important results in separation axioms theory. But in applications, it was seen that T_0 axiom which is the one of generalized separation axioms is not so useful and it was given that many equivalent conditions that T_0 axiom. Some of them are followings:

Theorem 2.1. [9] *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, then the following statements are characteristic properties of T_0 axiom in a ditopological texture space:*

(1) $P_s \not\subseteq P_t \implies \exists C_j \in \tau \cup \kappa; j \in J$ *with* $P_t \subseteq \bigvee_{j \in J} C_j \subseteq Q_s$*, for all* $s, t \in S$
(2) $Q_s \not\subseteq Q_t \implies \exists C_j \in \tau \cup \kappa; j \in J$ *with* $P_t \subseteq \bigcap_{j \in J} C_j \subseteq Q_s$*, for all* $s, t \in S$
(3) *For* $A \in \mathcal{S}$ *there exists* $C_j \in \tau \cup \kappa, j \in J, i \in I_j$ *with* $A = \bigvee_{j \in J} \bigcap_{i \in I_j} C_i^j$
(4) $Q_s \not\subseteq Q_t \implies \exists C \in \tau \cup \kappa$ *with* $P_s \not\subseteq C \not\subseteq Q_t$
(5) $[P_s] \subseteq [P_t]$ *and* $]Q_s[\subseteq]Q_t[$ *implies* $Q_s \subseteq Q_t$

(6) *For all* $s \in S$ *we have* $Q_s = \bigvee_{j \in J} C_j$ *for* $C_j \in \tau \cup \kappa$

(7) *If* $(S, \mathcal{S})$*is coseparated, then the following condition also characterizes the* T_0 *property: For all* $s \in S$ *we have* $P_s = \bigcap_{j \in J} C_j$ *for* $C_j \in \tau \cup \kappa$.

In this study, the new generalization separation axioms will be defined. These generalization axioms obtained that the fuzzy quasi separation axioms, in the sense of M. H. Ghanim, O. A. Tantawy [12]. In applications, the new generalized T_0 is more useful than generalized T_0 axiom in the sense of [9]. To do this we will give some basic results obtained by authors in a complemented grounded ditopological texture space.

Definition 2.2. Let $(S, \mathcal{S}, \sigma)$ be a complemented grounded texture space. Then we have the following statements:

(1) $AqB \iff A \not\subseteq \sigma(B) \iff \exists s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq \sigma(B)$ by Theorem 1.1(5),

(2) $Aq\sigma(B) \iff A \not\subseteq B$,

(3) $A \subseteq B \iff A \not{q} \sigma(B)$.

As a result of the above definition we obtain the followings:

(1) AqB and if $A \subseteq C, B \subseteq D$ then CqD,

(2) $A \not{q} B$ and if $C \subseteq A, D \subseteq B$ then $C \not{q} D$.

(3) If $U, V \in \tau$ and if $UqV \Longrightarrow [U]qV$ and so $\Longrightarrow [U]q[V]$.

Now we want to give new definitions of quasi (Q-) separation axioms in a complemented grounded ditopological texture space $(S, \mathcal{S}, \tau, \kappa, \sigma)$ which are generalized fuzzy quasi separation axioms in the sense of [12].

Definition 2.3. The ditopological space (τ, κ) is called

(1) *Quasi-*T_0 (QT_0) *space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exists a $U \in \tau$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$ or there exists a $K \in \kappa$ such that $P_t \not{q} K$, $P_s \not\subseteq \sigma(K)$.

(2) *Strong quasi-*T_0 (SQT_0) *space* if P_s is a closed set for all $s \in S$.

(3) *Quasi-*T_1 (QT_1) *space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exist $U \in \tau, K \in \kappa$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$ and $P_t \not{q} K$, $P_s \not\subseteq \sigma(K)$.

(4) *Quasi-*T_2 (QT_2) *space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exist $U \in \tau, K \in \kappa$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$ and $P_t \not{q} K$, $P_s \not\subseteq \sigma(K)$ and $U \subseteq K$.

(5) *Quasi-*$T_{2(1/2)}$ $(QT_{2(1/2)})$*space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exist $U \in \tau, K \in \kappa$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$, $P_t \not{q} K$, $P_s \not\subseteq \sigma(K)$ and $[U] \subseteq \sigma([\sigma(K)])$.

(6) Quasi-regular(Q-regular) if $\forall s \in S$ $F \in \kappa$ with $P_sq\sigma(F)$, there exist $U \in \tau, K \in \kappa$ such that $P_sqU, F \not{q} K$ and $[\sigma(K)] \subseteq \sigma([U])$.

A Q-regular space which is strong QT_0 is called *quasi-*T_3 (QT_3) *space.*

(7) *Quasi-normal(Q-normal) space* if $\forall$ $F_1, F_2 \in \kappa$ with $F_1q\sigma(F_2)$ there exits $U \in \tau, K \in \kappa$ such that $F_1qU, F_2 \not{q} K$ and $[\sigma(K)] \subseteq \sigma([U])$.

A Q-normal space which is strong QT_0 is called *quasi-*T_4 (QT_4) *space.*

Now, we will give some implications between these separation axioms as follows:

Corollary 2.4. *Let $(S, \mathcal{S}, \tau, \kappa, \sigma)$ be a complemented grounded texture space, then*

(1) $QT_{2(1/2)} \implies QT_2 \implies QT_1 \implies SQT_0 \implies QT_0$
(2) $QT_4 \implies QT_3 \implies QT_{2(1/2)}$

Proof. The first and the second implications of (1) are clear.

To show that the third implications of (1), ($QT_1 \implies SQT_0$), we must show $P_s = [P_s]$ for all $s \in S$. To do this, assume that there exists an $s \in S$ such that $[P_s] \not\subseteq P_s$. Then there exists $t \in S$ such that $[P_s] \not\subseteq Q_t$ and $P_t \not\subseteq P_s$. Since (τ, κ) is (QT_1), we have $U \in \tau, K \in \kappa$ such that $P_t qU$, $U \subseteq Q_{\sigma(s)}$ and $P_s \not{q}K$, $P_t \not\subseteq \sigma(K)$. Hence ($U \subseteq Q_{\sigma(s)} \implies P_s \subseteq \sigma(U) = F \in \kappa, \quad [P_s] \subseteq F$) and since $[P_s] \not\subseteq Q_t$, we have $F \not\subseteq Q_t$ that is $P_t \subseteq F$. On the other hand since $P_t qU \implies P_t \not\subseteq \sigma(U) = F$. These two case give a contradiction. That is $[P_s] \subseteq P_s$ and so $[P_s] = P_s$.

To show that the last implication of (1), ($SQT_0 \implies QT_0$), take $s, t \in S$ with $P_s \not\subseteq P_t$. Then there exists $r \in S$ with $P_s \not\subseteq Q_r$ and $P_r \not\subseteq P_t$. Since (τ, κ) is (SQT_0), we have $P_t \in \kappa$ for all $t \in S$; that is $P_r \not\subseteq P_t = \bigcap\{K \in \kappa \mid P_t \subseteq K\}$. Hence there exists $K \in \kappa$ such that $P_r \not\subseteq K$ and $P_t \subseteq K$; that is ($\exists K \in \kappa; P_s \not\subseteq K$ and $P_t \subseteq K \implies \sigma(K) \subseteq \sigma(P_t) = Q_{\sigma(t)}$ and $\sigma(K) \not\subseteq \sigma(P_s) = Q_{\sigma(s)}$). Now if we take $U = \sigma(K) \in \tau$, then we obtain that $U \subseteq Q_{\sigma(t)}$ and $P_s \not\subseteq K \iff P_s q\sigma(K)$. So we have $P_s qU$ and $U \subseteq Q_{\sigma(t)}$, that is $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is a QT_0 space.

Now we will show the implications in (2). Firstly to show that the second implication of (2), $QT_3 \implies QT_{2(1/2)}$, take $s, t \in S$ with $P_s \not\subseteq P_t = \sigma(Q_{\sigma(t)})$, that is $P_s qQ_{\sigma(t)} = \sigma(P_t)$. By SQT_0 of ditopological space (τ, κ), it is known that $F = P_t \in \kappa$ and so we have $P_s q\sigma(F)$. Now since (τ, κ) is quasi-regular, there exist $U \in \tau, K \in \kappa$ such that $P_s qU, \quad F = P_t \not{q}K$ and $[\sigma(K)] \subseteq \sigma([U])$. Then we have $[U] \subseteq \sigma([\sigma(K)]$. On the other hand since $[\sigma(K)] \subseteq \sigma([U])$, we have $P_s qU \implies P_s \not\subseteq \sigma(U) \implies P_s \not\subseteq \sigma([U]) \implies P_s \not\subseteq \sigma(K)$ and $P_t \not{q}K \implies P_t \subseteq \sigma(K) \subseteq [\sigma(K)] \subseteq \sigma([U]) \implies P_t \subseteq \sigma([U]) \implies U \subseteq \sigma(P_t) = Q_{\sigma(t)}$ That is $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is a $QT_{2(1/2)}$ space.

The first implication of (2), $QT_4 \implies QT_3$ is clear. □

Corollary 2.5. *If $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is QT_0 then it is T_0 in the sense of L. M. Brown, R. Ertürk and Ş. Dost* [9].

Proof. To show that this implication, we will use the "Theorem 2.1.(1). Let $s, t \in S$ be with $P_s \not\subseteq P_t$. Then there exists $U \in \tau$ such that $P_s qU$, $U \subseteq Q_{\sigma(t)}$ or there exists a $K \in \kappa$ such that $P_t \not{q}K$ and $P_s \not\subseteq \sigma(K)$ by QT_0 of (τ, κ). So we have $P_s \not\subseteq \sigma(U)$ and $P_t \subseteq \sigma(U)$. If it taken $\sigma(U) = F \in \kappa$, we obtain that $P_s \not\subseteq F$ and $P_t \subseteq F \implies P_t \subseteq F \subseteq Q_s$. That is $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is T_0.

If $K \in \kappa$, such that $P_t \not{q}K$ and $P_s \not\subseteq \sigma(K) \implies P_t \subseteq \sigma(K)$ and $\sigma(K) \subseteq Q_s$. If it taken $U = \sigma(K) \in \tau$, the we have $P_t \subseteq U \subseteq Q_s$. Hence $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is QT_0. □

The following example shows that the converse implication of above corollary is not true in generally.

Example 2.6. Let S=$\{a,b\}$, $\mathcal{S}=\mathcal{P}(S)$, $\tau=\{\emptyset,\{a\},S\}$, $\kappa=\{\emptyset,\{b\},S\}$. Then $(S,\mathcal{S},\sigma)$ is a grounded complemented texture space since it is plain and each plain texture is grounded. Hence $(S,\mathcal{S},\tau,\kappa,\sigma)$ is a complemented ditopological texture space for the complemented $\sigma(\{a\})$=$\{b\}$, $\sigma(\{b\})$=$\{a\}$. This ditoplogical texture space $(S,\mathcal{S},\tau,\kappa,\sigma)$ is T_0 in the sense of L. M. Brown, R. Ertürk, Ş. Dost but it is not QT_0.

Examples 2.7. (1) For $\mathbb{I}=[0,1]$ define $\mathcal{J}=\{[0,t]\mid t\in[0,1]\}\cup\{[0,t)\mid t\in[0,1]\}$, $\sigma([0,t])=[0,1-t)$ and $\sigma([0,t))=[0,1-t]$, $t\in[0,1]$ and $\tau=\{[0,t)\mid t\in[0,1]\}\cup\{\mathbb{I}\}$, $\kappa=\{[0,t]\mid t\in[0,1]\}\cup\{\emptyset\}$. Then $(\mathbb{I},\mathcal{J},\tau,\kappa,\sigma)$ is a grounded texture , which we will refer to as the *unit interval texture*. This time $(I,\mathcal{J})$ is a plain texture. Then the ditopological texture space $(\mathbb{I},\mathcal{J},\tau,\kappa,\sigma)$ is a QT_0 and T_0 space.

Definition 2.8. [5],[8] Let $(S_i,\mathcal{S}_i,\sigma_i)$ be textures $i\in I$, set $S=\prod_{i\in I}S_i$ and $A\subseteq\mathcal{S}_k$ for some $k\in I$. We write

$$E(k,A)=\prod_{i\in I}Y_i \text{ where } Y_i=\begin{cases}A, \text{ if } i=k\\ S_i, \text{ otherwise.}\end{cases}$$

Then the *product texturing* $\mathcal{S}=\bigotimes_{i\in I}\mathcal{S}_i$ on S consists of arbitrary intersections of elements of the set

$$\mathcal{E}=\left\{\bigcup_{j\in J}E(j,A_j)\mid J\subseteq I, A_j\in\mathcal{S}_j \text{ for } j\in J\right\}.$$

and $\sigma:\mathcal{S}\longrightarrow\mathcal{S}$, $\sigma(A)=\bigcap_{s\in A}\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))$ is a complementation on $\mathcal{S}$.

We have noted that by [8]; If $(S,\mathcal{S},\sigma)$ is a complemented product texture of $(S_i,\mathcal{S}_i,\sigma_i)$, then for $i\in I$ and $A_i\in\mathcal{S}_i$, $s=(s_i)$ it can be obtained the following equalities:

$$(a)\,P_s=\bigcap_{i\in I}E(i,P_{s_i})=\prod_{i\in I}P_{s_i},\quad (b)\,Q_s=\bigcup_{i\in I}E(i,Q_{s_i}),$$

$$(c)\,\sigma(E(i,A_i))=E(i,\sigma_i(A_i)),\quad (d)\,\sigma(P_s)=\bigcup_{i\in I}E(i,\sigma_i(P_{s_i}))$$

Corollary 2.9. *$(S_i,\mathcal{S}_i,\sigma_i)$ are complemented grounded textures for each $i\in I$ iff the product texture $(S,\mathcal{S},\sigma)$ of $(S_i,\mathcal{S}_i,\sigma_i)$ is complemented grounded.*

Definition 2.10. [5],[8] Let $(S_i,\mathcal{S}_i)$ $i\in I$ be textures with $S_i\cap S_j=\emptyset$ for $i\neq j$. Let $S=\bigcup_{i\in I}S_i$ and $\mathcal{S}=\{A\mid A\subseteq S, A\cap S_i\in\mathcal{S}_i,\forall i\in I\}$. Then $(S,\mathcal{S})$ is a texture which is called sum of disjoint textures $(S_i,\mathcal{S}_i)$ $i\in I$ and if $(S_i,\mathcal{S}_i,\sigma_i)$ are the complemented textures for all $i\in I$ then the complementation

$$\sigma:\mathcal{S}\longrightarrow\mathcal{S},\sigma(A)\cap S_i=\sigma_i(A\cap S_i),i\in I$$

makes $(S,\mathcal{S},\sigma)$ is a complemented texture.

We have noted that by [8]; If $(S,\mathcal{S},\sigma)$ is a complemented sum texture of $(S_i,\mathcal{S}_i,\sigma_i)$, then for $j\in I$ it can be obtained the following equalities:

$$(a)P_s=P_{s_j}\times\{j\},\quad (b)Q_s=(Q_{s_j}\times\{j\})\cup(\bigcup_{k\in I\setminus\{j\}}S_k\times\{k\})$$

Corollary 2.11. *$(S_i, \mathcal{S}_i, \sigma_i)$ $i \in I$ is complemented grounded texture for each $i \in I$ iff the sum texture $(S, \mathcal{S}, \sigma)$ of $(S_i, \mathcal{S}_i, \sigma_i)$ is complemented grounded.*

Theorem 2.12. *Let $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, $j \in J$, be non-empty complemented ditopolojical grounded texture spaces and $(S, \mathcal{S}, \sigma, \tau, \kappa)$ their product. Then $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is QT_0 if and only if $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$.*

Proof. ($\Longleftarrow$) Suppose that $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$ an take $s = (s_j), t = (t_j) \in S$ with $P_s \not\subseteq P_t$. Since $P_s \not\subseteq P_t = \bigcap_{j \in J} E(j, P_{t_j})$, there exists $j \in J$ with $P_s \not\subseteq E(j, P_{t_j})$ and so we have $E(j, P_{s_j}) \not\subseteq E(j, P_{t_j})$, and hence it will be $P_{s_j} \not\subseteq P_{t_j}$. By the QT_0 of $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, there exists $U_j \in \tau_j$ such that $P_{s_j} q U_j$ and $U_j \subseteq Q_{\sigma_j(t_j)}$ or there exists $F_j \in \kappa_j$ such that $P_{t_j} \not{q} F_j$ and $P_{s_j} \not\subseteq \sigma_j(F_j)$. Now if we choose $U = E(j, U_j)$, then $U \in \tau$. Since $P_{s_j} \not\subseteq \sigma_j(U_j)$, it can be obtained $P_s \not\subseteq \sigma(U) = E(j, \sigma_j(U_j))$ and this gives $P_s q U$. On the other hand since $U_j \subseteq Q_{\sigma_j(t_j)}$, we have $E(j, U_j) \subseteq \bigcup_{j \in J} E(j, Q_{\sigma_j(t_j)})$, and that is $U \subseteq Q_{\sigma(t)}$.
For the other case of "or", if we choose $F = E(j, F_j) \in \kappa$, it can be obtained the required properties similarly above case. So $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is QT_0.
($\Longrightarrow$) Let the complemented grounded product ditopological texture space $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be QT_0. Take any $j \in J$ and $s_j, t_j \in S_j$ with $P_{s_j} \not\subseteq P_{t_j}$. For $k \in J \setminus \{j\}$, choose $u_k \in S_k^\flat$, which is possible since $S_k \neq \emptyset$. Now let $s = (s_i), t = (t_i) \in S$ defined by

$$s_i = \begin{cases} s_j, \text{ if } i = j \\ u_i, \text{ if } i \neq j \end{cases} \quad t_i = \begin{cases} t_j, \text{ if } i = j \\ u_i, \text{ if } i \neq j \end{cases}$$

It is verify that $P_s \not\subseteq P_t$, since $P_{s_j} \not\subseteq P_{t_j}$. By the QT_0 of $(S, \mathcal{S}, \sigma, \tau, \kappa)$, there exists $B \in \tau$ such that $P_s q B$ and $B \subseteq Q_{\sigma(t)}$ or there exists $F \in \kappa$ such that $P_t \not{q} F$ and $P_s \not\subseteq \sigma(F)$. Firstly, suppose the case $B \in \tau$ with $P_s q B$ and $B \subseteq Q_{\sigma(t)}$, that is $\sigma(B) \in \kappa$ and $P_s \not\subseteq \sigma(B)$, $P_t \subseteq \sigma(B)$. By the definition of product cotopology [8], we have $j_1, j_2, \ldots, j_n \in J$ and $\sigma_{j_k}(B_{j_k}) \in \kappa_{k_j}$, $1 \leq k \leq n$, so that $\sigma(B) \subseteq \bigcup_{k=1}^n E(j_k, \sigma_{j_k}(B_{j_k}))$ and $P_s \not\subseteq \bigcup_{k=1}^n E(j_k, \sigma_{j_k}(B_{j_k}))$. Thus we have $P_s \not\subseteq E(j_k, \sigma_{j_k}(B_{j_k})$ for all k. On the other hand, $P_t \subseteq \bigcup_{k=1}^n E(j_k, \sigma_{j_k}(B_{j_k}))$, for some k, and hence it can be obtained $P_{s_j} \not\subseteq \sigma_{j_k}(B_{j_k})$ and $P_{t_j} \subseteq \sigma_{j_k}(B_{j_k})$. By the definition of s and t, $j_k = j$, we have $B_j \in \tau_j$ satisfying $P_{s_j} \not\subseteq \sigma_j(B_j)$ and $B_j \subseteq Q_{\sigma_j(t_j)}$.

The other case of "or", for the closed set case, can be shown similarly above. Thus, the complemented grounded ditopological texture space $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$.

□

Theorem 2.13. *Let $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, $j \in J$, be non-empty complemented ditopolojical grounded texture spaces and $(S, \mathcal{S}, \sigma, \tau, \kappa)$ their sum. Then $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is QT_0 if and only if $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$.*

Proof. ($\Longleftarrow$) Suppose that $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$ and take $s, t \in S$ with $P_s \not\subseteq P_t$. Let $j, k \in J$ be the unique indices satisfying $s \in S_j$, $t \in S_k$. If $j \neq k$ then $P_{s_j} \not\subseteq P_{t_k}$. Hence $P_{t_k} \not\subseteq S_j$. $S_j \in \tau_j$ has the required properties.

Indeed, $P_s \nsubseteq \sigma(S_j) = \emptyset$ and $S_j \subseteq Q_{t_k} = \sigma(P_t)$. If $j = k$ then $P_{s_k} \nsubseteq P_{t_k}$. By the QT_0 of $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, there exists $B_k \in \tau_k$ such that $P_{s_k} q B_k$ and $B_k \subseteq Q_{\sigma_k(t_k)}$ or there exists $F_k \in \kappa_k$ such that $P_{t_k} \not{q} F_k$ and $P_{s_k} \nsubseteq \sigma_k(F_k)$. Now, if we choose $B = B_k$, then $B \in \tau$ and $P_s = P_{s_k} q B_k$, $B = B_k \subseteq Q_{\sigma_k(t_k)}$. In the case of $F_k \in \kappa_k$ we have dual proof. So $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is QT_0.

$(\Longrightarrow)$ Let the complemented grounded sum ditopological texture space $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be QT_0. Take any $k \in J$ and $s_k, t_k \in S_k$ with $P_{s_k} \nsubseteq P_{t_k}$. Then for $s = (s_k, k), t = (t_k, k) \in S$ we have $P_s \nsubseteq P_t$. By the QT_0 of $(S, \mathcal{S}, \sigma, \tau, \kappa)$, there exists $B \in \tau$ such that $P_s q B$ and $B \subseteq Q_{\sigma(t)}$ or there exists $C \in \kappa$ such that $P_t \not{q} C$ and $P_s \nsubseteq \sigma(C)$. Firstly, suppose the case of $B \in \tau$ with $P_s q B$ and $B \subseteq Q_{\sigma(t)}$, that is $\sigma(B) \in \kappa$ and $P_s \nsubseteq \sigma(B)$, $P_t \subseteq \sigma(B)$. By the definiton of sum ditopological texture space there exists $B_j \in \tau_j, j \in J$ such that $\sigma_j(B_j) \in \kappa_j$ where, $\sigma(B) \subseteq \sigma_j(B_j) \times \{j\} \cup \bigcup_{j \in J \setminus \{j\}} (S_i \times \{i\})$ and $P_s \nsubseteq \sigma_j(B_j) \times \{j\} \cup \bigcup_{j \in J \setminus \{j\}} (S_i \times \{i\})$. If $j = k$ then $P_{s_k} \times \{k\} \nsubseteq \sigma_k(B_k) \times \{k\}$ and $P_{t_k} \times \{k\} \nsubseteq \sigma_k(B_k) \times \{k\}$ and so we have $P_{s_k} q B_k$ and $B_k \times \{k\} \subseteq \sigma_k(B_k) = Q_{\sigma(t)}$. If $j \neq k$ then $P_{s_k} \times \{k\} \nsubseteq S_j \times \{j\}$. It has the required properties.

The other case of "or", for the closed set case, can be shown similarly above. Thus, the complemented grounded ditopological texture spaces $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ are QT_0 for all $j \in J$.

□

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