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P-COMPACTNESS IN L-TOPOLOGICAL SPACES

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ABSTRACT. The concepts of P-compactness, countable P-compactness, the P-Lindelöf property are introduced in L-topological spaces by means of preopen L-sets and their inequalities when L is a complete DeMorgan algebra. These definitions do not rely on the structure of the basis lattice L and no distributivity in L is required. They can also be characterized by means of preclosed L-sets and their inequalities. Their properties are researched. Further when L is a completely distributive DeMorgan algebra, their many characterizations are presented.

1. INTRODUCTION

The notions of strong compactness, countable P-compactness and strongly Lindelöf property were introduced in general topology by means of preopen sets (see [5, 10, 16]). Nanda [11] generalized the notion of strong compactness in [5] to $[0, 1]$-topological spaces based on Chang's compactness [1] which is not a good extension. Kudri and Warner [6] introduced strong compact L-fuzzy subsets based on their compactness which is equivalent to the notion of strong fuzzy compactness in [7, 8, 17].

In [13, 15], a new definition of fuzzy compactness is presented in L-topological spaces by means of an inequality, which does not depend on the structure of L and no distributivity is required in L. When L is a completely distributive DeMorgan algebra, it is equivalent to the notion of fuzzy compactness in [7, 8, 17].

Lowen [9] introduced the notion of strong fuzzy compactness which is a generalization of the notion of compactness in general topology but different from the notion of strong

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compactness [5, 11]. In order to distinguish them, we call strong compactness in [5] as P-compactness and call strongly Lindelöf property in [10] as the P-Lindelöf property.

In this paper, our aim is to extend the notion of P-compactness to L-topology by means of preopen L-sets and their inequality. We also extend countable P-compactness [16] and the P-Lindelöf property to L-topology. These definitions do not rely on the structure of the basis lattice L and no distributivity in L is required.

2. Preliminaries

Throughout this paper $(L, \bigvee, \bigwedge, ')$ is a complete DeMorgan algebra. X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets for short) on X.

An element a in L is called prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called co-prime element if a' is a prime element [3]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [2]. In a completely distributive DeMorgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b in the sense of [7, 17], and denoted by $\beta(b)$. Moreover, for $b \in L$, we define $\beta^*(b) = \beta(b) \cap M(L)$, $\alpha(b) = \{a \in L \mid a' \prec b'\}$, and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations from [12].

$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}, \quad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}.$$

An L-topological space (or L-space for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its complement is called a closed L-set.

Definition 2.1 ([7, 17])**.** For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all the lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X \mid A^{(a)} \in \tau, a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X, in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ).

Definition 2.2 ([7, 17])**.** An L-space $(X, \mathcal{T})$ is called weakly induced if $\forall a \in L$, $\forall A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

It is obvious that $(X, \omega_L(\tau))$ is weakly induced.

For a subfamily $\Phi \subseteq L^X$, $2^{(\Phi)}$ denotes the set of all finite subfamilies of Φ. $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ.

Definition 2.3 ([13, 15])**.** Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called (countably) fuzzy compact if for every (countable) family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A(x) \right).$$

Definition 2.4 ([13, 15])**.** Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is said to have the Lindelöf property if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

Lemma 2.5 ([13, 15])**.** *Let L be a complete Heyting algebra, $f : X \to Y$ a map, and let $f_L^{\rightarrow} : L^X \to L^Y$ be the extension of f. Then for any family $\mathcal{P} \subseteq L^Y$, we have*

$$\bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y) \wedge \bigwedge_{B\in\mathcal{P}} B(y)\right) = \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\right).$$

Definition 2.6 ([13, 15])**.** Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{1\}$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be

(1) An a-shading of G if for any $x \in X$, it follows that $\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \nleq a$.

(2) A strong a-shading of G if $\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \nleq a$.

(3) An a-remote family of G if for any $x \in X$, it follows that $\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \ngeq a$.

(4) A strong a-remote family of G if $\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \ngeq a$.

Definition 2.7 ([13, 15])**.** Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)$. $\mathcal{U}$ is called a strong β_a-cover of G if $a \in \beta\left(\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right)\right)$.

Definition 2.8 ([13, 15])**.** Let $(X, \mathcal{T})$ be an L-space, $a \in L\backslash\{0\}$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a Q_a-cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \geq a$.

Definition 2.9 ([11])**.** An L-set G in an L-space $(X, \mathcal{T})$ is called preopen if $G \leq int(cl(A))$. G is called preclosed if G' is preopen.

Definition 2.10 ([11])**.** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called

(1) Precontinuous if $f_L^{\leftarrow}(G)$ is preopen in $(X, \mathcal{T}_1)$ for every open L-set G in $(Y, \mathcal{T}_2)$.
(2) M-pre-continuous (we shall call it P-irresolute) if $f_L^{\leftarrow}(G)$ is preopen in $(X, \mathcal{T}_1)$ for every preopen L-set G in $(Y, \mathcal{T}_2)$.

3. P-compactness

Lowen [9] introduced the notion of strong fuzzy compactness which is a generalization of the notion of compactness in general topology but different from the notion of strong compactness [5, 11]. In order to distinguish them, we call strong compactness in [5] as P-compactness and we extend it to L-topology. We also extend countable P-compactness [16] and strong Lindelöf property [10] (we call it the P-Lindelöf property) to L-topology.

Definition 3.1. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called (countably) P-compact if for every (countable) family $\mathcal{U}$ of preopen L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

Definition 3.2. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is said to have the P-Lindelöf property (or be a P-Lindelöf L-set) if for every family $\mathcal{U}$ of preopen L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \leq \bigvee_{\mathcal{V}\in 2^{[\mathcal{U}]}} \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A(x)\right).$$

Obviously we have the following theorem.

Theorem 3.3. *P-compactness implies countably P-compactness and the P-Lindelöf property. Moreover an L-set having the P-Lindelöf property is P-compact if and only if it is countably P-compact.*

Since an open L-set is preopen, we have the following theorem.

Theorem 3.4. *For an L-set in an L-space, the following conditions are true.*

(1) *P-compactness $\Rightarrow$ fuzzy compactness;*
(2) *Countably P-compactness $\Rightarrow$ countably fuzzy compactness;*
(3) *The P-Lindelöf property $\Rightarrow$ the Lindelöf property.*

From Definition 3.1 and Definition 3.2 we can obtain the following two theorems by using complement.

Theorem 3.5. *Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is (countably) P-compact if and only if for every (countable) family $\mathcal{B}$ of preclosed L-sets, it follows that*

$$\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{B}} B(x)\right) \geq \bigwedge_{\mathcal{F}\in 2^{(\mathcal{B})}} \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x)\right).$$

Theorem 3.6. *Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ has the P-Lindelöf property if and only if for every family $\mathcal{B}$ of preclosed L-sets, it follows that*

$$\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{B}} B(x)\right) \geq \bigwedge_{\mathcal{F}\in 2^{[\mathcal{B}]}} \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{B\in\mathcal{F}} B(x)\right).$$

Definition 3.7. Let $a \in L\backslash\{0\}$ and $G \in L^X$. A subfamily $\mathcal{A}$ of L^X is said to have a weak a-nonempty intersection in G if $\bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{A\in\mathcal{A}} A(x)\right) \geq a$. $\mathcal{A}$ is said to have the finite (countable) weak a-intersection property in G if every finite (countable) subfamily $\mathcal{F}$ of $\mathcal{A}$ has a weak a-nonempty intersection in G.

From Definition 3.1, Definition 3.2, Theorem 3.5 and Theorem 3.6 we immediately obtain the next two theorems.

Theorem 3.8. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G is (countably) P-compact.*
(2) *For any $a \in L\backslash\{1\}$, each (countable) preopen strong a-shading $\mathcal{U}$ of G has a finite subfamily which is a strong a-shading of G.*
(3) *For any $a \in L\backslash\{0\}$, each (countable) preclosed strong a-remote family $\mathcal{P}$ of G has a finite subfamily which is a strong a-remote family of G.*
(4) *For any $a \in L\backslash\{0\}$, each (countable) family of preclosed L-sets which has the finite weak a-intersection property in G has a weak a-nonempty intersection in G.*

Theorem 3.9. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G has the P-Lindelöf property.*
(2) *For any $a \in L\backslash\{1\}$, each preopen strong a-shading $\mathcal{U}$ of G has a countable subfamily which is a strong a-shading of G.*
(3) *For any $a \in L\backslash\{0\}$, each preclosed strong a-remote family $\mathcal{P}$ of G has a countable subfamily which is a strong a-remote family of G.*
(4) *For any $a \in L\backslash\{0\}$, each family of preclosed L-sets which has the countable weak a-intersection property in G has a weak a-nonempty intersection in G.*

4. Properties of P-compactness

Theorem 4.1. *Let L be a complete Heyting algebra. If both G and H are (countably) P-compact, then so is $G \vee H$.*

Proof. For any (countable) family $\mathcal{P}$ of preclosed L-sets, by Theorem 3.5 we have that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\\
=\ &\left\{\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{P}}B(x)\right)\right\}\\
\geq\ &\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\vee\left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left(H(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right)\right\}\\
=\ &\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge\bigwedge_{B\in\mathcal{F}}B(x)\right).
\end{aligned}$$

This shows that $G \vee H$ is (countably) P-compact. □

Analogously we have the following result.

Theorem 4.2. *Let L be a complete Heyting algebra. If both G and H have the P-Lindelöf property, then $G \vee H$ has the P-Lindelöf property.*

Theorem 4.3. *If G is (countably) P-compact and H is preclosed, then $G \wedge H$ is (countably) P-compact.*

Proof. For any (countable) family $\mathcal{P}$ of preclosed L-sets, by Theorem 3.5 we have that

$$\begin{aligned}
& \bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{P}} B(x)\right) \\
= & \bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}\bigcup\{H\}} B(x)\right) \\
\geq & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P}\cup\{H\})}} \bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right) \\
= & \left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right\} \wedge \left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right\} \\
= & \left\{\bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right)\right\} \\
= & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left((G\wedge H)(x)\wedge \bigwedge_{B\in\mathcal{F}} B(x)\right).
\end{aligned}$$

This shows that $G \wedge H$ is (countably) P-compact. □

Analogously we have the following result.

Theorem 4.4. *If G has the P-Lindelöf property and H is preclosed, then $G \wedge H$ has the P-Lindelöf property.*

Theorem 4.5. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a P-irresolute map. If G is a P-compact (countably P-compact or P-Lindelöf) L-set in $(X, \mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mathcal{T}_2)$.*

Proof. We need only prove that this result is true for P-compactness. Suppose that $\mathcal{P}$ is a family of preclosed L-sets, by Lemma 2.5 and P-compactness of G we have that

$$\begin{aligned}
& \bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y)\wedge \bigwedge_{B\in\mathcal{P}} B(y)\right) \\
= & \bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{P}} f_L^{\leftarrow}(B)(x)\right) \\
\geq & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{F}} f_L^{\leftarrow}(B)(x)\right) \\
= & \bigwedge_{\mathcal{F}\in 2^{(\mathcal{P})}} \bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y)\wedge \bigwedge_{B\in\mathcal{F}} B(y)\right).
\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is P-compact. □

Analogously we have the following result.

Theorem 4.6. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be an precontinuous map. If G is a P-compact (countably P-compact or P-Lindelöf) L-set in $(X, \mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is fuzzy compact (countably fuzzy compact or Lindelöf) in $(Y, \mathcal{T}_2)$.*

5. Further characterizations of P-compactness and goodness theorem

In this section, we assume that L is a completely distributive de Morgan algebra.

Analogous to the proof of Theorem 2.9 in [13] we can obtain the next theorem.

Theorem 5.1. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent.*

(1) *G is (countably) P-compact.*
(2) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$), each (countable) preclosed strong a-remote family $\mathcal{P}$ of G has a finite subfamily which is an (a strong) a-remote family of G.*
(3) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any (countable) preclosed strong a-remote family $\mathcal{P}$ of G, there exist a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ and $b \in \beta(a)$ (or $b \in \beta^*(a)$) such that $\mathcal{F}$ is a (strong) b-remote family of G.*
(4) *For any $a \in L\backslash\{1\}$ (or $a \in P(L)$), each (countable) preopen strong a-shading $\mathcal{U}$ of G has a finite subfamily which is an (a strong) a-shading of G.*
(5) *For any $a \in L\backslash\{1\}$ (or $a \in P(L)$) and any (countable) preopen strong a-shading $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in \alpha(a)$ (or $b \in \alpha^*(a)$) such that $\mathcal{V}$ is a (strong) b-shading of G.*
(6) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$), each (countable) preopen strong β_a-cover $\mathcal{U}$ of G has a finite subfamily which is a (strong) β_a-cover of G.*
(7) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any (countable) preopen strong β_a-cover $\mathcal{U}$ of G, there exist a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ and $b \in L$ (or $b \in M(L)$) with $a \in \beta(b)$ such that $\mathcal{V}$ is a (strong) β_b-cover of G.*
(8) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any $b \in \beta(a)\backslash\{0\}$, each (countable) preopen Q_a-cover of G has a finite subfamily which is a Q_b-cover of G.*
(9) *For any $a \in L\backslash\{0\}$ (or $a \in M(L)$) and any $b \in \beta(a)\backslash\{0\}$ (or $b \in \beta^*(a)$), each (countable) preopen Q_a-cover of G has a finite subfamily which is a β_b-cover of G.*

Remark 5.2. Analogous to Theorem 5.1, we can obtain characterizations of the P-lindelöf property.

Lemma 5.3. *Let $(X, \omega(\tau))$ be generated topologically by (X, τ). If A is a preopen L-set in (X, τ), then χ_A is a preopen set in $(X, \omega(\tau))$. If B is a preopen L-set in $(X, \omega(\tau))$, then $B_{(a)}$ is a preopen set in (X, τ) for every $a \in L$. In particular, if χ_A is a preopen set in $(X, \omega(\tau))$, then A is a preopen L-set in (X, τ).*

Proof. If A is a preopen set in (X, τ), then $A \subseteq int(cl(A)$. Thus we have

$$\chi_A \leq \chi_{int(cl(A))} = int(cl(\chi_A)).$$

This shows that χ_A is preopen.

If B is a preopen L-set in $(X, \omega(\tau))$, then $B \leq int(cl(B))$. This implies that $B_{(a)} \subseteq (int(cl(B)))_{(a)}$. From [15] we obtain

$$(int(cl(B)))_{(a)} \subseteq int((cl(B))_{(a)}) \subseteq int(cl(B_{(a)})).$$

Hence $B_{(a)}$ is a preopen set in (X, τ). □

The following two theorems show that P-compactness, countable P-compactness and the P-Lindelöf property are good extensions.

Theorem 5.4. *Let (X,τ) be a topological space and let $(X,\omega(\tau))$ be generated topologically by (X,τ). Then $(X,\omega(\tau))$ is (countably) P-compact if and only if (X,τ) is (countably) P-compact.*

Proof. Necessity. Let $\mathcal{A}$ be a (countable) preopen cover of (X,τ). Then $\{\chi_A \mid A \in \mathcal{A}\}$ is a family of preopen L-sets in $(X,\omega(\tau))$ with $\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{U}}\chi_A(x)\right) = 1$. From (countable) P-compactness of $(X,\omega(\tau))$ we know

$$\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}}\chi_A(x)\right) = \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}}\chi_A(x)\right) = 1.$$

This implies that there exists $\mathcal{V}\in 2^{(\mathcal{U})}$ such that $\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}}\chi_A(x)\right) = 1$. Hence $\mathcal{V}$ is a cover of (X,τ). Therefore (X,τ) is (countably) P-compact.

Sufficiency. Let $\mathcal{U}$ be a (countable) family of preopen L-sets in $(X,\omega(\tau))$ and $\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}}B(x)\right) = a$. If $a=0$, then we obviously have

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}}B(x)\right) \le \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{A\in\mathcal{V}}B(x)\right).$$

Now we suppose that $a\neq 0$. In this case, for any $b\in\beta(a)\backslash\{0\}$ we have that

$$b\in\beta\left(\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}}B(x)\right)\right)\subseteq\bigcap_{x\in X}\beta\left(\bigvee_{B\in\mathcal{U}}B(x)\right)=\bigcap_{x\in X}\bigcup_{B\in\mathcal{U}}\beta\left(B(x)\right).$$

By Lemma 5.1, this implies that $\{B_{(b)} \mid B\in\mathcal{U}\}$ is a preopen cover of (X,τ). From (countable) P-compactness of (X,τ) we know that there exists $\mathcal{V}\in 2^{(\mathcal{U})}$ such that $\{B_{(b)} \mid B\in\mathcal{V}\}$ is a cover of (X,τ). Hence $b\le\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}}B(x)\right)$. Further we have that

$$b\le\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}}B(x)\right)\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}}B(x)\right).$$

This implies that

$$\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{U}}B(x)\right) = a = \bigvee\{b \mid b\in\beta(a)\}\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(\bigvee_{B\in\mathcal{V}}B(x)\right).$$

Therefore $(X,\omega(\tau))$ is (countably) P-compact. □

Analogously we have the following result.

Theorem 5.5. *Let (X,τ) be a topological space and $(X,\omega(\tau))$ be generated topologically by (X,τ). Then $(X,\omega(\tau))$ has the P-Lindelöf property if and only if (X,τ) has the P-Lindelöf property.*

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