J. Nonlinear Sci. Appl. 2 (2009), no. 2, 71–77

The Journal of Nonlinear Science and Applications
http://www.tjnsa.com

SOME PROPERTIES OF B-CONVEXITY

HONGMIN SUO[1,2,*]

Communicated by S. S. Chang

Abstract. In this paper, we give a characteristic of B-convexity structures of finite dimensional B-spaces: if a finite dimensional B-space has the weak selection property then its B-convexity structure satisfies H-condition. We also get some relationships among B-convexity structures, selection property and fixed point property. We show that in a compact convex subset of a finite dimensional B-space satisfying H-condition the weak selection property implies the fixed point property.

1. Introduction and preliminaries

The convexity of space plays a very important role in fixed point theory and continuous selection theory. There were many works deal with various kinds of generalized, topological, or axiomatically defined convexities[1, 2, 3, 4]. Most of them were to establish various fixed point theorems and selection theorems in topological space without linear structure such as some generalizations of Brouwer fixed point theorem, Fan-Browder fixed point theorem and Michael selection theorem[2, 5, 6, 7, 8]. Recently, Briec[2] introduced the B-convexity by algebra borrows from topological ordered vector spaces and semilattice both. Briec proved that all the basic results related to fixed point theorems available in B-convexity[2].

The aim of this paper is to give some relationships among B-convexity structure, selection property and fixed point theorems. We prove that if X is a B-space

Date: Received: 2 January 2009; Revised: 5 March 2009.

*Corresponding author. Supported by the Scientific Research Foundation of GuiZhou (No.2008(2050)).

2000 *Mathematics Subject Classification.* Primary 46A55; Secondary 47H10, 54H25.

Key words and phrases. B-Convexity, continuous selection, fixed point, KKM-maping.

with B-convexity and of weak selection property with respect to any standard simplex Δ_N then X satisfies H-condition, and we show that in a compact convex subset of a B-space with B-convexity structure the weak selection property implies the fixed point property.

A B-convex set can be seen as an abstract cone, in as much as we have a partial order and a multiplication by positive reals compatible with that partial order, we will remain in the finite dimensional setting of R^n with its natural partial order. Let n_1 and n_2 be two positive integers whose sum is n and

$$R_-^{n_1} = \{(x_1, \cdots, x_{n_1)} \in R^{n_1} \quad max\{x_i\} \leq 0\},$$
$$R_+^{n_2} = \{(x_1, \cdots, x_{n_2)} \in R^{n_2} \quad max\{x_i\} \leq 0\}.$$

We identify $R_-^{n_1} \times R_+^{n_2}$ with an octant of R^n . For $t \in R_+$ and $x \in R_-^{n_1} \times R_+^{n_2}$, tx is usual multiplication by a scalar, for x and y in $R_-^{n_1} \times R_+^{n_2}$, we let $x \vee y$ be the element of $R_-^{n_1} \times R_+^{n_2}$ defined in the following way:

$$(x \vee y)_j = \begin{cases} min\{x_j, y_j\} & if \quad j \leq n_1 \\ max\{x_j, y_j\} & if \quad j > n_1. \end{cases} \tag{1.1}$$

Then one can easily see that:

(A) $(x, y) \to x \vee y$ is associative, commutative, and idempotent , and also continuous ,and $x \vee 0 = x$ for all $R_-^{n_1} \times R_+^{n_2}$.

(B) For $t \in R_+$, the map $t \to tx$ is continuous and order preserving, and for all t_1, t_2 in R_+ and for all x and y in $R_-^{n_1} \times R_+^{n_2}, (t_1 t_2)x = t_1(t_2 x)$ and $t(x \vee y) = (tx) \vee (ty)$.

A finite dimensional B-space (of type(n_1, n_2))) is, by definition, a subset X of $R_-^{n_1} \times R_+^{n_2}$ such that:

(BS) $0 \in X, \forall t \geq 0$ and $\forall x \in X, tx \in X$ and $\forall x, y \in X,\ x \vee y \in X$.

For a subset B of X the following properties are equivalent[1]:

(B1)$\forall x, y \in B, tx \vee y \in B \quad \forall t \in [0, 1]$,
(B2)$\forall x_1, \cdots, x_m \in B$, and $\forall t_1, \cdots, t_m \in [0, 1]$ such that

$$max_{1 \leq i \leq m}\{t_j\} = 1, \quad t_1 x_1 \vee \cdots \vee t_m x_m = \vee t_i x_i \in B.$$

Definition 1.1. A subset of X for which (B1) or (B2) holds is called B-convex[1].

For example (B1) holds for increasing set (S is increasing if $x \leq y$ and $x \in S$ implies $y \in S$). Sets of the form $\prod_{i=1}^{m}[a_i, b_i]$ are B-convex in R_+^n.

Since an arbitrary intersection of B-convex sets is B-convex, and arbitrary set $S \subset X$ is always contained in a smallest B-convex subset of X, we call that set the B-convex hull of S, it is denoted by $[S]$. From (B2) one has the following characterization:

The B-convex hull of S it is the set of all elements of the form $t_1x_1 \vee \cdots \vee t_mx_m$ with $x_i \in S$ and $max_{1\leq i\leq m}\{t_j\} = 1,\ t_i \in [0,1]$.

B-convex sets also are contractible[2]. We recall that a set A is contractible if there exists a continuous map $h : A \times [0,1] \to A$ such that the map $a \to h(a,0)$ is constant and $a \to h(a,1)$ is the identity map of A.

For finite dimensional B-space X we define a map as follows:

$$(K(x,y,t) = \begin{cases} x \vee 2ty & if \quad 0 \leq t \leq 1/2 \\ (2-2t)x \vee y & if \quad 1/2 < t \leq 1. \end{cases} \tag{1.2}$$

To see that a B-convex set B is contractible one fixes $x_0 \in B$ and take $h(x,t) = K(x_0,x,t)$.

Other properties of B-convex and foxed points theorem and related matters in the framework of B-convexity see [2].

A topological space X with a convexity structure C (e.g. B-convexity) is said to be of weak selection property with respect to S if every multivalued mapping $F : S \to 2^X$ admits a singlevalued continuous selection whenever F is lower semicontinuous and nonempty closed convex valued. (X,C) is said to be of weak selection property with respect to S if $F : S \to 2^X$ admits a singlevalued continuous selection whenever F is multivalued mapping with nonempty convex images and preimages relatively open in X (i.e.,$F(x)$ is convex for each $x \in S$ and F^{-1} is open in S). X is said to be of fixed point property if every continuous selfmap F on X has a fixed point in X.

Let $N = \{0,1,2,\cdots,n\},\ \triangle_N = e^0e^1\cdots e^n$ be the standard simplex of dimension n, where $\{e^0e^1\cdots e^n\}$ is the canonical basis of R^{n+1}, and for $J \subset N$, and $\triangle_N = co\{e^j : j \in J\}$ be a face of $\triangle_N$. For each $x \in e^0e^1\cdots e^n$, there is a unique set of numbers $t_0,\cdots,t_n$ with, $\sum_{t=0}^n t_i = 1,\ t_i \geq 0, i \in N$ such that $x = \sum_{i=0}^n t_ie^i$. The coefficients $t_0,\cdots,t_n$ are called the barycentric coordinates of x. Let

$$\chi(v) = \{i : v = \sum_{i=0}^n t_ie^i, t_i \geq 0\}.$$

Definition 1.2. Let $\{T_i : i \in I\}$ be some simplicial subdivision of standard simplex $\triangle_N = e^0e^1\cdots e^n$, ν denote the collection of all vertices of all subsimplexes in in the subdivision. A function $\lambda : \nu \to \{0,1,\cdots,n\}$ satisfying

$$\lambda(v) \in \chi(v), \forall v \in \nu,$$

is called a normal labeling of this subdivision. Moreover, T_i is called a completely labeled subsimplex or completely labeled lattice if T_i must have vertices with the completes set of labels: $0, 1, \cdots n$.

Theorem 1.3. *Let $\{T_i : i \in I\}$ be any simplicial subdivision of $\triangle_N$ and normally labeled by a function λ. Then there exist odd numbers of completely labeled subsimplexes of lattices in the subdivision with respect to the labeling function λ.*

Last theorem is famous Sperner's lemma[3].

Theorem 1.4. *Let Y be a topological space. For each $J \subset N$, let Γ_J be a nonempty contractible subset of Y. If $\emptyset \neq J \subset J^{'} \subset N$ implies $\Gamma_J \subset \Gamma_{J'}$, then there exists a continuous mapping f such that $F(\triangle_J) \subset \Gamma_J$ for each nonempty subset $J \subset N$.*

This is Horvath' lemma[6, 7].

2. Main results

According to Horvath's lemma, we call that a finite dimensional B-space satisfies H-condition if the B-convexity has the following property:

(H) For each finite subset $\{y_0, y_1, \cdots, y_n\} \subset Y$, there exists a continuous mapping $f : \triangle_N \to [\{y_0, y_1, \cdots, y_n\}]$ such that $f(\triangle_J) \subset [y_j : j \in J]$ for each nonempty subset $J \subset N$.

Now, we first prove the crucial result of this section as below.

Theorem 2.1. *If a finite dimensional B-space Y with B-convexity is of weak selection property with respect to any standard simplex, then a finite dimensional B-space Y satisfies H-condition.*

Proof. . Let $A = \{y_0, y_1, \cdots, y_n\}$ be any finite subset of Y, $\triangle_N = e^0e^1 \cdots e^n$ the standard simplex of dimension n. For each $J \subset N$ and each face $\triangle_J$ of $\triangle_N$, denote the interior of $\triangle_J$ by

$$\triangle_J^0 = \{v \in \triangle_J : \chi(v) = J\}.$$

Define $T : \triangle_N \to 2^Y$ as follows:

$$T(x) = [\{y_j : j \in \chi(x)\}],\ x \in \triangle_N.$$

It is routinely to check that T is with nonempty convex images and preimages relatively open in $\triangle_N$. In fact, for each $y \in Y$ and each $x \in T^{-1}(y)$, there is only one face $\triangle_J$, $J = \chi(x)$ such that $x \in \triangle_J^0$. So $x \notin \triangle_{J'}$ for any face $\triangle_{J'}$ not containing $\triangle_J$. For any $\triangle_{J'} \supset \triangle_J$, there exists a neighborhood $O(x) \subset \triangle_N$ of x such that $O(x) \bigcap \triangle_{J'} = \emptyset$ as every face $\triangle_{J'}$ is closed and the number of faces $\triangle_N$ of is finite. Therefore, for any $z \in O(x)$, any face $\triangle_{J'}$ contains z only if $\triangle_J \subset \triangle_{J'}$. Then for each $z \in O(x)$, $z \in \triangle_{\chi(z)}$ implies $\triangle_{\chi(z)} \supset \triangle_J$, So that $\chi(z) \supset J = \chi(x)$. It follow that $T(z) \supset T(x)$ for all $z \in O(x)$, and so $y \in T(x) \subset T(z)$, *i.e.*, $z \in T^{-1}(y)$ for all $z \in O(x)$. . Hence $T^{-1}(y)$ is relatively open in $\triangle_N$.

In addition, it is obvious that T is nonempty closed and convex. Since Y is of selection property with respect to any standard simplex, there exists a single-valued continuous mapping $f : \triangle_N \to Y$ such that $f(x) \in T(x)$ for all $x \in \triangle_N$. The definition of T implies that $f(\triangle_J) \subset [\{y_j : j \in J\}]$ for each nonempty subset $J \subset N$, which complete the proof. $\square$

Corollary 2.2. . *If a finite dimensional B-space Y with B-convexity is of weak selection property with respect to any compact Hausdorff space, then a finite dimensional B-space Y satisfies H-condition.*

Proof. . It is immediate from Theorem 2.1. $\square$

Let X be a subset of a finite dimensional B-space Y. A multivalued mapping $F : X \to 2^Y$ is called a KKM-mapping if $[A] \subset \bigcup_{x\in A} F(x)$ for each finite subset $A \subset X$.

Theorem 2.3. *Let X is subset of a finite dimensional Y B-space satisfying H-condition and $F : Y \to 2^X$ is a KKM-mapping. If F is closed-valued, then family $\{F(y) : y \in Y\}$ has the finite intersection property.*

Proof. . Let $\{y_0, y_1, \cdots, y_n\}$ be arbitrary finite subset of X. Since Y satisfies H-condition, there exists a singlevalued continuous mapping $f : \triangle_N \to [\{y_0, \cdots, y_n\}]$ such that $f(\triangle_J) \subset [\{y_j : j \in J\}]$ for each nonempty subset $j \subset N$.

For each $k \in \{1, 2, \cdots\}$ and each $\varepsilon_k = 1/k \geq 0$, let $\{T_i^k : i \in I_k\}$ be some simplicial subdivision of $\triangle_N$ such that the mesh of the subdivision less than $1/2^k$. And let ν^k be the set of vertices of all subsimplexes in this subdivision.

For each $\upsilon \in \nu^k$, let

$$\lambda^k(\upsilon) = min\{j \in \chi(\upsilon) : f(\upsilon) \in F(y_j)\}.$$

Then $\lambda^k(\upsilon)$ is nonempty, since $\upsilon \in conv\{e^j : j \in \chi(\upsilon)\}$ and

$$f(\upsilon) \in f([\{e^j : j \in \chi(\upsilon)\}]) \subset [\{y_j : j \in \chi(\upsilon)\}] \subset \bigcup_{j\in\chi(\upsilon)} F(y_j).$$

By the hypothesis, it is easy to see that λ^k is a normal label function of the subdivision.

So for each $k = 1, 2, \cdots$, there must exist a subsimplex T_{i_k} with complete labels by Sperner's Lemma. Let $z_0^k, \cdots, z_n^k$ be all vertices of subsimplex T_{i_k} , and

$$\lambda(z_0^k) = 0,\ \lambda(z_1^k) = 1, \cdots, \lambda(z_n^k) = n.$$

By the definition of λ, we have

$$f(z_0^k) \in F(y_0), f(z_1^k) \in F(y_1), \cdots, f(z_n^k) \in F(y_n).$$

Note that $z_0^k, \cdots, z_n^k$ are some vertices of subsimplex T_{i_k},so that $d(z_i^k, z_j^k) \leq 1/2^k,\ i, j \in \{0, 1, \cdots, n\}$. Since $\triangle_N$ is compact, we may assume that there is $y^* \in \triangle_N$ such that $z_i^k \to y^*,\ i = 0, 1, \cdots, n$. Then $f(z_i^k) \to f(y^*)$. It follows from the closeness of each $F(y_i)$ that $f(y^*) \in F(y_i),\ i = 0, 1, \cdots, n$, and $\bigcap_{i\in N} F(y_i) \neq \emptyset$. This completes the proof. □

Theorem 2.4. . *Let a finite dimensional B-space Y satisfying H-condition, X is a convex compact subset of Y, and $F : X \to 2^X$ a multivalued mapping with nonempty convex images and preimages relatively open in X. Then F has a fixed point.*

Proof. . Since X is compact and $X = \bigcup_{x\in X} F^{-1}(x)$, there exists a finite subset $\{x_0, x_1, \cdots, x_n\}$ of X such that $X = \bigcup_{i=0}^{n} F^{-1}(x_i)$. Then $\bigcap_{i=0}^{n}[X \setminus F^{-1}(x_i)] = \emptyset$. Let

$$G(x) = [X \setminus F^{-1}(x)], \quad \forall x \in X.$$

With Theorem 2.3, we know that G is not a KKM-mapping, so that there exists a finite subset $\{y_0, y_1, \cdots, y_n\}$ such that

$$[\{y_0, y_1, \cdots, y_n\}] \not\subset \bigcup_{i=0}^{m} G(y_i).$$

Then there is some $y^* \in [\{y_0, y_1, \cdots, y_n\}]$ such that $y^* \notin G(y_i)$ for all $i = 0, 1, \cdots m$, that is

$$y^* \in F^{-1}(y_i), \quad \forall i = 0, 1, \cdots, m.$$

Consequently

$$y^i \in F^*(y), \quad \forall i = 0, 1, \cdots, m.$$

Therefore

$$y^* \in [\{y_0, y_1, \cdots, y_m\}] \subset F(y^*).$$

Which complete the proof. □

Theorem 2.5. . *Let X be a compact topological space, a finite dimensional B-space Y satisfying H-condition, and $F : X \to 2^Y$ a multivalued mapping with nonempty convex images and preimages relatively open in X. Then F has a continuous selection.*

Proof. . Since X is compact and $X = \bigcup_{y\in Y} F^{-1}(y)$, there exists a finite subset $\{y_0, y_1, \cdots, y_m\}$ of X such that $X = \bigcup_{i=0}^{n} F^{-1}(y_i)$. Now let $\{p_i : i = 0, 1, \cdots, n\}$ be a partition of unity subordinate to the finite covering $\{F^{-1}(y_i) : i = 0, 1, \cdots, n\}$. Define a mapping $\phi : X \to \triangle_N$ by

$$\phi(x) = \sum_{i=0}^{n} p_i(x)e^i, \quad \forall x \in X.$$

On the other hand, since Y satisfies H-condition, there exists a singlevalued continuous mapping $f : \triangle_N \to [\{y_0, y_1, \cdots, y_n\}]$ such that $s(\triangle_J) \subset [y_j : j \in J]$ for each nonempty subset $J \subset N$.

Now our desired mapping g is given by

$$g = f \circ \phi.$$

In fact, it is easy to verify that $\phi(x) \in \triangle_{J(x)}$ for each $x \in X$, where $J(x) = \{i \in N : p_i(x) \neq 0\}$. By the convexity of $F(x)$, we do have that $\{y_j : J(x)\} \subset F(x)$ and thus

$$g(x) = f(\phi(x)) \subset f(\triangle_{J(x)}) \subset [y_j : j \in J] \subset [y_j, p_j(x) \neq 0] \subset [y_j : y_j \in F(x)] \subset F(x).$$

This complete the proof. □

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1, 1

[1]School of Mathematics and Computer Science, GuiZhou University for Nationalities , 550025, Guiyang, Guizhou, China.

[2]School of Mathematics and Statistics, Southwest University, 400715, Chongqing, China.

E-mail address: gzmysxx88@sina.com