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βS^*- COMPACTNESS IN L-FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, the notion of βS^*-compactness is introduced in $L-$fuzzy topological spaces based on S^*-compactness. *A* βS^*-compactness L-fuzzy set is S^*-compactness and also $\beta-$compactness. Some of its properties are discussed. We give some characterizations of βS^*-compactness in terms of pre-open, regular open and semi-open $L-$fuzzy set. It is proved that βS^*-compactness is a good extension of $\beta-$compactness in general topology. Also, we investigated the preservation theorems of βS^*-compactness under some types of continuity.

1. Introduction

It is known that compactness and its stronger and weaker forms play very important roles in topology. The concepts of compactness in $[0, 1]-$fuzzy set theory was first introduced by C.L. Chang in terms of open covers [4]. Göguen pointed out a deficiency in Chang's compactness theory by showing that the Tychonoff Theorem is false [8]. Since Chang's compactness has some limitations, Gantner, Steinlage and Warren introduced $\alpha-$compactness [6], Lowen introduced fuzzy compactness, strong fuzzy compactness and ultra-fuzzy compactness [11, 12] and Wang and Zhao introduced $N-$compactness [18, 19]. Recently Shi introduced S^*-compactness [15] in $L-$fuzzy topological spaces.

The notion of $\beta-$compactness is one of the good strong forms of compactness in topology. It was generalized and studied by many authors in fuzzy topological spaces (see [1, 3 , 9]).

In this paper, following the lines of Shi [15] we shall introduce a new notion of $\beta-$compactness in $L-$fuzzy topological spaces named βS^*-compactness. A

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characterizations and properties of βS^*-compactness is of interest. Also, we show that the $\beta-$continuous ($M\beta-$continuous) image of a βS^*-compact $L-$fuzzy topological space is S^*-compact (βS^*-compact). Moreover, we introduce a good definition of local S^*-compactness (local βS^*-compactness) in $L-$fts's.

2. Preliminaries

Throughout this paper, $(L, \vee, \wedge, ')$ is a completely distributive de Morgan algebra, and X a nonempty set. L^X is the set of all $L-$fuzzy sets on X. An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in L is called a co-prime element if a' is a prime element [7]. The set of nonunite prime elements in L is denoted by $P(L)$. The set of nonzero co-prime elements in L is denoted by $M(L)$. The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ iff for every subsets $D \subseteq L$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$. In a completely distributive de Morgan algebra L , each element b is a *sup* of $\{a \in L : a \prec b\}$. In the sense of [10, 17], $\{a \in L : a \prec b\}$ is the greatest minimal family of b, in symbol $\beta(b)$. Moreover for $b \in L$, define $\beta^*(b) = \beta(b) \cap M(L)$, $\alpha(b) = \{a \in L : a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$. For $a \in L$ and $G \in L^X$, we denote $G^{(a)} = \{x \in X : G(x) \nleq a\}$ and $G_{(a)} = \{x \in X : a \in \beta(G(x))\}$ [15, 16].

An $L-$fuzzy topological space ($L-$fts , for short) is a pair $(X, \Im)$, where $\Im$ is a subfamily of L^X which contains $\longrightarrow_0$, $\longrightarrow_1$ and is closed with respect to suprema and finite infima. $\Im$ is called an $L-$fuzzy topology on X. Each member of $\Im$ is called an open $L-$fuzzy set and its complement is called a closed $L-$fuzzy set.

Definition 2.1 [10, 17]. For a topological space (X, τ), let $w_L(\tau)$ denote the family of all lower semicontinuous functions from (X, τ) to L, i.e., $w_L(\tau) = \{G \in L^X : G^{(a)} \in \tau, a \in L\}$. Then $w_L(\dot{\tau})$ is an $L-$topology on X, in this case, $(X, w_L(\tau))$ is called topologically generated by (X, τ).

Definition 2.2. An $L-$ fuzzy set G in an $L-$ fts $(X, \Im)$ is said to be:

(i) $\alpha-open$ (resp. $\alpha-closed$) if $G \leq int\ cl\ int\ G$ (resp.$G \geq cl\ int\ cl\ G$),[5] ;

(ii) *semiopen* (resp. *semiclosed*) if $G \leq cl\ int\ G$ (resp. $G \geq int\ cl\ G$), [2] ;

(iii) *preopen* (resp. *preclosed*) if $G \leq int\, cl\ G$ (resp. $G \geq cl\ int\ G$), [5] ;

(iv) $\beta-open$ (resp.$\beta-closed$) if $G \leq cl\ int\ cl\ G$ (resp. $G \geq int\ cl\ int\ G$),[5] ;

(v) *regular open* (resp. *regular closed*) if $G = int\ cl\ G$ (resp. $G = cl\ int\ G$),[2];

(vi) *regular semiopen* (resp.*regular semiclosed*) if there exists a *regular open* subset H of X such that $H \subseteq G \subseteq cl\ H$ (resp. if there exists a *regular closed* subset H of X such that $H \supseteq G \supseteq int\ H$), [14] .

It is obvious that each of *semiopen* and *preopen* $L-$fuzzy set implies $\beta-open$.

Definition 2.3. A function $f : X \to Y$ is said to be fuzzy $\beta-$continuous [5] (resp. $M\beta-$continuous [9]) if the inverse image of every *open* (resp. $\beta - open$) $L-$fuzzy set in Y is $\beta - open$ (resp. $\beta - open$) $L-$fuzzy set in X.

Definition 2.4 [15]. Let $(X, \Im)$ be an $L-$fts, $a \in M(L)$ and $G \in L^X$. A subfamily ξ of L^X is called a $\beta_a - cover$ of G if for any $x \in X$ with $a \notin \beta(G'(x))$, there exists an $A \in \xi$ such that $a \in \beta(A(x))$. A $\beta_a - cover$ ξ of G is called an *open* (resp. *regular open*, *preopen*, etc.) $\beta_a - cover$ of G if each member of ξ is open (resp. *regular open*, *preopen*, etc.) .

It is obvious that ξ is a $\beta_a - cover$ of G iff for any $x \in X$ it follows that $a \in \beta(G'(x) \vee \longrightarrow A \in \xi \vee A(x))$.

Definition 2.5 [15]. Let $(X, \Im)$ be an $L-$fts, $a \in M(L)$ and $G \in L^X$. A subfamily ξ of L^X is called a $Q_a - cover$ of G if for any $x \in X$ with $G(x) \nleq a'$, it follows that $\longrightarrow A \in \xi \vee A(x) \geq a$. A $Q_a - cover$ ξ of G is called an *open* (resp. *regular open*, *preopen*, etc.) $Q_a - cover$ of G if each member of ξ is *open* (resp. *regular open*, *preopen*, etc.) .

Definition 2.6 [15]. Let $(X, \Im)$ be an $L-$fts, $a \in M(L)$ and $G \in L^X$. G is called S^*-compact if for any $a \in M(L)$, each *open* $\beta_a - cover$ of G has a finite subfamily F which is an *open* $Q_a - cover$ of G. $(X, \Im)$ is said to be S^*-compact if $\longrightarrow_1$ is S^*-compact.

Definition 2.7 [14]. An $L-$fts $(X, \Im)$ is said to be extremely disconnected if $cl\ G \in \Im$ for every $G \in \Im$.

Definition 2.8 [13]. Let X be a set. A prefilterbase in X is a family $\Omega \subseteq L^X$ having the following two properties:

(i) for every $G \in \Omega$, $G \neq \varphi$.

(ii) for every $G, H \in \Omega$ there is a $W \in \Omega$ such that $W \leq G \wedge H$.

Moreover, Ω is said to be maximal iff for each $G \subseteq L^X$, one of the two $L-$fuzzy sets G, G' contains a member of Ω.

3. Characterizations and properties of βS^*-compactness in $L-$fts's

Definition 3.1. Let $(X, \Im)$ be an $L - fts$ and $G \in L^X$. Then G is called βS^*-compact if for any $a \in M(L)$, every $\beta - open\, \beta_a - cover$ of G has a finite subfamily F which is $\beta - open\, Q_a - cover$ of G . $(X, \Im)$ is said to be βS^*-compact if X is βS^*-compact.

It is clear that every βS^*-compactness is $\beta-$compactness [1].

Remark 3.2. Since every open $L-$fuzzy set is $\beta-open$ then every βS^*-compactness is S^*-compactness.

Example 3.3. Let $L=[0,1]$, X be an infinite set , $\Im=\{0,G,X\}$ be an $L-$fuzzy topology , where $G(x)=0.5$ for all $x\in X$. Then any $L-$fuzzy set in $(X,\Im)$ is $\beta-open$ and the set of all open $L-$fuzzy set in $(X,\Im)$ is $\Im$. In this case, we can easily obtain that $H(x)=0.7$ for all $x\in X$ is not βS^*-compact and any $L-$fuzzy set is S^*-compact.

Theorem 3.4. Let $(X,\Im)$ be an $L-fts$. If G and H are βS^*-compact $L-$fuzzy subsets of X, then so is $G\vee H$.

Proof. For any $a\in M(L)$, suppose that ξ is an $\beta-open\,\beta_a-cover$ of $G\vee H$. Then by

$$(G\vee H)'(x)\vee\longrightarrow A\in\xi\vee A(x)=(\,G'(x)\vee\longrightarrow A\in\xi\vee A(x))\wedge(\,H'(x)\vee\longrightarrow A\in\xi\vee A(x))$$

we obtain that for any $x\in X$, $a\in\beta(\,G'(x)\vee\longrightarrow A\in\xi\vee A(x))$ and $a\in\beta(\,H'(x)\vee\longrightarrow A\in\xi\vee A(x))$. This shows that ξ is an $\beta-open\,\beta_a-cover$ of G and H, we know that ξ has finite subfamily F_1 and F_2 such that F_1 and F_2 is a $\beta-open\,Q_a-cover$ of G and H respectively. Hence for any $x\in X$, $a\leq G'(x)\vee\longrightarrow A\in F_1\vee A(x)$ and $a\leq H'(x)\vee\longrightarrow A\in F_2\vee A(x)$. Take $W=F_1\cup F_2$ is a finite subfamily of ξ and it satisfies the following condition $a\leq G'(x)\vee\longrightarrow A\in W\vee A(x)$ and $a\leq H'(x)\vee\longrightarrow A\in W\vee A(x)$, hence $a\leq(G\vee H)'(x)\vee\longrightarrow A\in W\vee A(x)$. This shows that W is a $\beta-open\,Q_a-cover$ of $G\vee H$, therefore $G\vee H$ is βS^*-compact

Corollary 3.5. Let $(X,\Im)$ be an $L-fts$. Every $L-$fuzzy subset G with finite support is βS^*-compact relative to X.

Proof. Obvious.

Theorem 3.6. An $L-fts$ $(X,\Im)$ is βS^*-compact if every $\beta-closed$ fuzzy subset is βS^*-compact relative to X.

Proof. For any $a\in M(L)$, suppose that $\{\upsilon_j:j\in J\}$ be an $\beta-open\,\beta_a-cover$ of X. Let $j_0\in J$, then υ'_{j_0} is $\beta-closed$ and so by the hypothesis υ'_{j_0} is βS^*-compact. Now, $\xi=\{\upsilon_j:j\in J-(\,j_0\}\}$ is an $\beta-open\,\beta_a-cover$ of X. Since υ'_{j_0} is βS^*-compact there exists a finite subfamily ξ_0 of ξ such that ξ_0 is a $\beta-open\,Q_a-cover$ of X. Hence X is a βS^*-compact.

Corollary 3.7. An $L-fts$ X is βS^*-compact if every *semiclosed* ($\alpha-closed$, *preclosed*, *regular semiclosed*) $L-$fuzzy subset of X is βS^*-compact relative to X.

Proof. Clearly since each *semiclosed* ($\alpha-closed$, *preclosed*, *regular semiclosed*) $L-$fuzzy subset of X is $\beta-closed$.

Now, we characterize βS^*-compactness in the sense of *preopen*, *regular open* and *semiopen* $L-$fuzzy subsets.

Theorem 3.8. An extremely disconnected $L-fts$ X is βS^*-compact iff for any $a \in M(L)$, every *preopen* $\beta_a - cover$ of X has a finite subfamily F which is a *preopen* $Q_a - cover$ of X.

Proof. For any $a \in M(L)$, Let $\{\upsilon_j : j \in J\}$ be a *preopen* $\beta_a - cover$ of X. Then $\upsilon_j \leq int\ cl\, \upsilon_j$ for each $j \in J$ and so $\upsilon_j \leq cl\ \upsilon_j \leq cl\ int\ cl\, \upsilon_j$. Hence the family $\{\upsilon_j : j \in J\}$ is a $\beta - open\, \beta_a - cover$ of X. Thus, by the hypothesis, there exists a finite subset F of J which is a *preopen* $Q_a - cover$ of X.

Conversely, Let $\{\upsilon_j : j \in J\}$ be a $\beta - open\, \beta_a - cover$ of X. Then for each $j \in J$, $\upsilon_j \leq cl\ int\ cl\, \upsilon_j = int\ cl\ int\ cl\, \upsilon_j = int\ cl\, \upsilon_j$ from the extremely disconnected of X. Hence $\upsilon_j \leq int\ cl\, \upsilon_j$ for each $j \in J$ and so $\{\upsilon_j : j \in J\}$ is a *preopen* $\beta_a - cover$ of X. So there exists a finite subset F of J which is a $\beta - open\ Q_a - cover$ of X.

Theorem 3.9. Each extremely disconnected $L-fts$ X in which every $\beta-$ *open* $L-$fuzzy subset of X is semiclosed is βS^*-compact iff for any $a \in M(L)$, every *semiopen* $\beta_a - cover$ of X has a finite subfamily F which is a *semiopen* $Q_a - cover$ of X.

Proof. For any $a \in M(L)$, Let $\{\upsilon_j : j \in J\}$ be a *semiopen* $\beta_a - cover$ of X. Since every *semiopen* is $\beta - open$, then $\{\upsilon_j : j \in J\}$ is a $\beta - open\, \beta_a - cover$ of X. By the βS^*-compactness of X , there exists a finite subset F of J which is a *semiopen* $Q_a - cover$ of X.

Conversely, Let $\{\upsilon_j : j \in J\}$ be a $\beta - open\, \beta_a - cover$ of X. Since the closure of each $\beta - open$ is *semiopen* , the family $\{cl\ \upsilon_j : j \in J\}$ is a *semiopen* $\beta_a - cover$ of X . By the hypothesis, there exists a finite subset F of J which is a *semiopen* $Q_a - cover$ of X. But for each $j \in J$, we have $\upsilon_j \leq cl\ int\ cl\, \upsilon_j$ which implies that $cl\ \upsilon_j \leq cl\ int\ cl\, \upsilon_j = int\ cl\ int\ cl\, \upsilon_j = int\ cl\, \upsilon_j$ for each $j \in J$ and hence $(int\ cl\, \upsilon_j : j \in F\}$ is a *semiopen* $Q_a - cover$ of X. By the hypothesis each $\beta - open$ $L-$fuzzy subset of X is *semiclosed*, then $\upsilon_j \geq int\ cl\, \upsilon_j$ for each $j \in F$. Hence $(\ \upsilon_j : j \in F\}$ is a $\beta - open\ Q_a - cover$ of X.

Theorem 3.10. Each extremely disconnected $L-fts$ X in which every $\beta - open$ $L-$fuzzy subset of X is *semiclosed* is βS^*-compact iff for any $a \in M(L)$, every *regular open* $\beta_a - cover$ of X has a finite subfamily F which is a *regular open* $Q_a - cover$ of X.

Proof. Follows from the above theorem, since each *regular open* $L-$fuzzy subset of X is *semiopen*.

Definition 3.11. Let $(X, \Im)$ be an $L-fts$. A prefilterbase Ω on X is said to be $\beta-$converges ($S-$converges) to $a \in M(L)$ if for every $\beta - open$ (*semiopen*) $L-$fuzzy set G containing a there exists $H \in \Omega$ such that $H \leq cl\ G$.

Definition 3.12. Let $(X, \Im)$ be an $L-fts$. A prefilterbase Ω on X is said to be $\beta-$accumulates ($S-$accumulates) at $a \in M(L)$ if for every $\beta-open$ (*semiopen*) $L-$fuzzy set G containing a and for every $H \in \Omega$, we have $H \wedge cl\ G \neq \varphi$.

Proposition 3.13. Let Ω be a maximal prefilterbase in an $L-fts$ $(X, \Im)$, then the following statements are equivalent:

(i) Ω is $\beta-$accumulates ($S-$accumulates) at $a \in M(L)$.
(ii) Ω is $\beta-$converges ($S-$converges) to $a \in M(L)$.

Proof. $(i) \to (ii)$: To prove that Ω is $\beta-$converges ($S-$converges) to $a \in M(L)$, Let G be a $\beta-open$ (*semiopen*) $L-$ fuzzy set in X such that $a \in G$. Since Ω is $\beta-$accumulates ($S-$accumulates) at a, then for every $H \in \Omega$, $H \wedge cl\ G \neq \varphi$. Thus there exists a proper $L-$ fuzzy subset $C \leq H$ such that $C \leq cl\ G$. Since $C \neq \varphi$, then C is a member of some prefilterbase in X. But Ω is maximal, then C is a member of Ω. Thus for every $\beta-open$ (*semiopen*) $L-$ fuzzy set G containing a there exists $H = C \in \Omega$ such that $H \leq cl\ G$. Then Ω is $\beta-$converges ($S-$converges) to a.

$(ii) \to (i)$: *Let* G be a $\beta-open$ (*semiopen*) $L-$ fuzzy set in X such that $a \in G$. Since Ω is $\beta-$converges ($S-$converges) to a, then there exists $H \in \Omega$ such that $H \leq cl\ G$ and thus $H \wedge cl\ G$ is a member of some prefilterbase in X. But Ω is maximal, then $H \wedge cl\ G \in \Omega$, So for every $H_j \in \Omega$, $H_j \wedge (H \wedge cl\ G)$ contains a member of Ω, then $H_j \wedge cl\ G \neq \varphi$ for every $H_j \in \Omega$. Hence Ω is $\beta-$accumulates ($S-$accumulates) at a.

The following result shows that the notion of $\beta-$converges (resp. $\beta-$accumulates) and $s-$converges (resp. $s-$accumulates) are equivalent for any prefilterbase.

Proposition 3.14. Let $(X, \Im)$ be an $L-fts$. A prefilterbase Ω on X is $\beta-$converges (resp. $\beta-$accumulates) to $a \in M(L)$ iff Ω is $s-$converges (resp. $s-$accumulates) to $a \in M(L)$.

Proof. Since any *semiopen* $L-$fuzzy set containing a is $\beta-open$ $L-$fuzzy set containing a, The necessity is obvious. The sufficiency follows from the fact that the closure of any $\beta-open$ $L-$fuzzy set containing a is a *semiopen* $L-$fuzzy set containing a.

Now, we give a characterization of $\beta S^{*}-$compact in the sense of convergent prefilterbasis and by means of finite intersection property.

Theorem 3.15. The following statements are equivalent for any $L-fts$ $(X, \Im)$:

(i) X is $\beta S^{*}-$compact.
(ii) Each maximal prefilterbase is $\beta-$converges.
(iii) Each prefilterbase is $\beta-$accumulates at an $L-$fuzzy point $a \in M(L)$.

Proof. $(i) \rightarrow (ii)$: Let $\Omega = \{G_j : j \in J\}$ be a maximal prefilterbase on X. Suppose that Ω does not $\beta-$converges, then Ω does not $\beta-$accumulate. Then for all $a \in M(L)$, there exists a $\beta - open$ $L-$fuzzy set G_a of X with $a \in G_a$ and $H_{j_a} \in \Omega$ such that $H_{j_a} \wedge cl\ G_a = \varphi$. Then the family $\{G_a : a \in X\}$ of $\beta - open$ $L-$fuzzy subsets is $\beta - open\, \beta_a - cover$ of X. Since X is βS^*-compact, there exists a finite subfamily $\{G_{a_1}, ..., G_{an}\}$ which is $\beta - open\, Q_a - cover$ of X. So $\{cl\ G_{a_1}, ..., cl\ G_{an}\}$ is $\beta - open\, Q_a - cover$ of X. Since Ω is a prefilterbase there exists $H_0 \in \Omega$ such that $H_0 \leq \stackrel{n}{\longrightarrow j} = 1\wedge\ H_{j_a}$ and $H_0 \wedge cl\ G_{aj} = \varphi$. So, $H_0 \wedge \stackrel{n}{\longrightarrow j} = 1\wedge\ cl\ G_{aj} = \varphi$. Hence $H_0 = \varphi$, which contradicts that Ω is a prefilterbase.

$(ii) \rightarrow (iii)$: Since each maximal prefilterbase Ω on X $\beta-$converges, Ω is $\beta-$accum- ulates. Since each prefilterbase is contained in a maximal prefilterbase which is $\beta-$accumulates, each prefilterbase $\beta-$accumulates.

$(iii) \rightarrow (i)$: obvious.

Now in the following, we shall prove that βS^*-compactness is a good extension of $\beta-$compactness in general topology.

Lemma 3.16: Let $(X, w_L(\tau))$ be generated topology by (X, τ), Then
(i) χ_G is a $\beta - open\, L-$fuzzy set in $(X, w_L(\tau))$ if G is a $\beta - open$ set in (X, τ).
$(ii)\, G^{(a)}$ is a $\beta - open$ set in (X, τ) for all $a \in L$ if G is a $\beta - open\, L-$fuzzy set in $(X, w_L(\tau))$.

Proof. (i) Since G is a $\beta-open$, then $G \leq cl\ int\ cl\ G$. Hence $\chi_G \leq \chi_{cl\ int\ cl\ G} = cl\ int\ cl\ \chi_G$ which implies that χ_G is a $\beta - open\, L-$fuzzy set in $(X, w_L(\tau))$.

(ii) Obvious.

Theorem 3.17. Let (X, τ) be a topological space. Then (X, τ) is $\beta-$compact iff $(X, w_L(\tau))$ is a βS^*-compact.

Proof. Let (X, τ) be a $\beta-$compact . For all $a \in M(L)$, let ξ be a $\beta - open\, \beta_a - cover$ of X in $(X, w_L(\tau))$. By Lemma 3.16 $\{G^{(a)} : G \in \nu\}$ is a $\beta - open$ of (X, τ). By $\beta-$compactness of (X, τ), there exists a finite subfamily F of ξ such that $\{G^{(a)} : G \in F\}$ is a cover of (X, τ). Hence F is a $\beta - open\, Q_a - cover$ of X. Therefore $(X, w_L(\tau))$ is a βS^*-compact.

Conversely, let $(X, w_L(\tau))$ be a βS^*-compact and μ be a $\beta - open\ - cover$ of (X, τ). Then for each $a \in \beta^*(1)$, $\{\chi_G : G \in \mu\}$ is a $\beta - open\, \beta_a - cover$ of X in $(X, w_L(\tau))$. By βS^*-compactness of $(X, w_L(\tau))$, we know that there exists a finite subfamily F of μ such that $\{\chi_G : G \in F\}$ is a $Q_a-cover$ of X in $(X, w_L(\tau))$. Hence F is a $\beta - open\ - cover$ of (X, τ). Therefore (X, τ) is $\beta-$compact.

4. Functions and βS^*-Compactness in $L-$fts's

Throughout, X and Y will be denote $L-fts$'s.

Theorem 4.1. Let $f : X \to Y$ be fuzzy $\beta-$continuous surjection. If X is a βS^*-compact $L-fts$ then Y is S^*-compact $L-fts$.

Proof. For all $b \in M(L)$, let $\{\upsilon_j : j \in J\}$ be a family of *open* $L-$fuzzy subsets of Y which is *open* β_b-*cover* of Y. Then $\{f^{-1}(\upsilon_j) : j \in J\}$ is a family of $\beta-$*open* $L-$fuzzy subsets of X which is $\beta-$*open* β_a-*cover* of X, for all $a \in M(L)$ where $f(a) = b$. From the βS^*-compactness of X there exists a finite subset F of J which is $\beta-$*open* Q_a-*cover* of X. Hence $f(\longrightarrow j \in F \vee f^{-1}(\upsilon_j)) =\longrightarrow j \in F \vee f(\longrightarrow j \in F \vee f^{-1}(\upsilon_j)) =\longrightarrow j \in F \vee \upsilon_j$ and so is *open* Q_a-*cover* of X which means that Y is S^*-compact .

Theorem 4.2. Let $f : X \to Y$ be fuzzy $M\beta-$continuous surjection. If X is a βS^*-compact $L-fts$ then Y is a βS^*-compact $L-fts$.

Proof. Similar to the above theorem.

Lemma 4.3. If $f : X \to Y$ is fuzzy open and fuzzy continuous function , then f is fuzzy $M\beta-$continuous.

Proof. Let H be a $\beta-$*open* $L-$fuzzy set in Y, then $H \le cl\, int\, cl\, H$. So $f^{-1}(H) \le f^{-1}(cl\, int\, cl\, H) \le cl\,(f^{-1}(int\, cl\, H))$. Since f is fuzzy continuous, then $f^{-1}(int\, cl\, H) = int\,(f^{-1}(\, cl\, H))$. Also , $f^{-1}(int\, cl\, H) = int\,(f^{-1}(int\, cl\, H)) \le int\,(f^{-1}(\, cl\, H)) \le int\, cl\ (f^{-1}(H))$. Thus $f^{-1}(H) \le cl\,(f^{-1}(int\, cl\, H)) \le cl\ int\, cl(f^{-1}(H))$. Hence the result.

Corollary 4.4. Let $f : X \to Y$ be fuzzy open and fuzzy continuous function and X is fuzzy βS^*-compact , then $f(X)$ is fuzzy βS^*-compact .

Proof. It is follows directly from Lemma 4.3 and Theorem 4.2.

Definition 4.5. A function $f : X \to Y$ is said to be fuzzy $M\beta-$*open* iff the image of every $\beta-$*open* $L-$fuzzy set in X is $\beta-$*open* $L-$fuzzy set in Y.

Theorem 4.6. Let $f : X \to Y$ be a fuzzy $M\beta-$*open* bijective function and Y is βS^*-compact, then X is βS^*-compact.

Proof. For all $a \in M(L)$, let $\{\upsilon_j : j \in J\}$ be a family of $\beta-$ *open* $L-$fuzzy subsets of X which is $\beta-$*open* β_a-*cover* of X. Then $\{f(\upsilon_j) : j \in J\}$ is a family of $\beta-$*open* $L-$fuzzy subsets of Y which is $\beta-$*open* β_b-*cover* of Y, for all $b \in M(L)$ where $f(a) = b$. From the βS^*-compactness of Y there exists a finite subset F of J which is $\beta-$*open* Q_b-*cover* of Y. But $X = f^{-1}(Y) = f^{-1}f(\longrightarrow j \in F \vee \upsilon_j) =\longrightarrow j \in F \vee \upsilon_j$ which is $\beta-$*open* Q_a-*cover* of X and therefore X is βS^*-compact.

5- Local S^*-compactness (Local βS^*-compactness) in $L-$fts's

In this section, we introduce a good definition of local S^*-compactness (local βS^*-compactness) in $L-$fts's. We show that local βS^*-compactness is preserved under $M\beta-$continuous *open* functions.

Definition 5.1. Let $(X, \Im)$ be an $L-$fts. An $L-$fuzzy set G is said to be *very* S^*-compact (*very* βS^*-compact) if for some $b \in L$ it is of the form

$H(x) = QDATOPD\{.b, \qquad if\ \ x \in D \subseteq X0, \qquad otherwise$

where $D = supp\, G$, and for all $a \in M(L)$ and every collection $\{\upsilon_j : j \in J\}$ of $open\, \beta_a - cover$ ($\beta - open\, \beta_a - cover$) of X for all $x \in D$, there is a finite subfamily F of J which is $open\ Q_a - cover$ ($\beta - open\, Q_a - cover$) of X for all $x \in D$.

It is simply required that χ_D be S^*-compact and also βS^*-compact.

By using the above Definition 5.1 , we have the following diagram:

$$\begin{array}{ccc} very\ \beta S^*\text{−compactness} & \Longrightarrow & \beta S^*\text{−compactness} \\ \Downarrow & & \Downarrow \\ very\ S^*\text{−compactness} & \Longrightarrow & S^*\text{−compactness.} \end{array}$$

Definition 5.2. Let $(X, \Im)$ be an $L-$fts. We say that $(X, \Im)$ is locally S^*-compact (locally βS^*-compact) if for all $x \in X$ and for all $a \in M(L)$ there exists a *very* S^*-compact (*very* βS^*-compact) $L-$fuzzy set H and $G \in \Im$ such that $H \geq G$ and $H(x) \nleq a$.

Remark 5.3. From the above Definition 5.2, it is clear that every locally βS^*-compact is locally S^*-compact.

Theorem 5.4. Let (X, τ) be a topological space. Then (X, τ) is locally compact (locally $\beta-$compact) if the $L-$fts $(X, w(\tau))$ is locally S^*-compact (locally βS^*-compact).

Proof. Let $x \in X$ and $a \in M(L)$. By the locally compact (locally $\beta-$compact) of (X, τ) there exist $U \in \tau$ and compact ($\beta-$compact) set C relative to (X, τ) such that $x \in U \subseteq C$. Then $\chi_U \in w(\tau)$, $\chi_U(x) = 1 \nleq a$ and $\chi_U \leq \chi_C$. We have by the goodness of S^*-compactness (βS^*-compactness) that χ_C is S^*-compact (βS^*-compact) in the $L-$fts $(X, w(\tau))$. Hence $(X, w(\tau))$ is is locally S^*-compact (locally βS^*-compact).

Conversely, Let $x_0 \in X$ and $a \in M(L)$. By the locally S^*-compact (locally βS^*-compact) of $(X, w(\tau))$ there exists $G \in w(\tau)$ and a *very* S^*-compact (*very* βS^*-compact) $L-$fuzzy set H, where

$H(x) = QDATOPD\{.b, \quad if\ x \in D \subseteq X0, \quad otherwise$

such that $G \leq H$ and $H(x_0) \nleq a$. Since $G \in w(\tau)$ there is a basic open $L-$fuzzy set λ, where

$\lambda(x) = QDATOPD\{.d, \quad if\ x \in V \in \tau 0, \quad otherwise$

such that $\lambda \leq G \leq H$ and $\lambda(x_0) \nleq a$. Then $V \subseteq D$ and so $x_0 \in V \in \tau$. We also have D is compact ($\beta-$compact) in (X, τ). Hence (X, τ) is locally compact (locally $\beta-$compact).

Theorem 5.5. Let $f : X \to Y$ be fuzzy $\beta-$continuous (fuzzy $M\beta-$continuous) open surjection. If X is locally βS^*-compact then Y is locally S^*-compact (locally βS^*-compact).

Proof. Let $y \in Y$ and $a \in M(L)$. Then for each $x \in f^{-1}(\{y\})$, there exists a *very* βS^*-compact $L-$fuzzy set H in X and $G \in \Im_X$ such that $H \geq G$ and $G(x) \nleq a$. By Theorems 4.1, 4.2 , we have $f(H)$ is a *very* S^*-compact (βS^*-compact) $L-$fuzzy subset of Y satisfy that $f(H) \geq f(G)$, $(f(G))(y) \nleq a$ where $f(G) \in \Im_Y$. Hence Y is locally S^*-compact (locally βS^*-compact).

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