

LOCAL CONVERGENCE ANALYSIS OF INEXACT NEWTON-LIKE METHODS

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ABSTRACT. We provide a local convergence analysis of inexact Newton-like methods in a Banach space setting under flexible majorant conditions. By introducing center-Lipschitz-type condition, we provide (under the same computational cost) a convergence analysis with the following advantages over earlier work [9]: finer error bounds on the distances involved, and a larger radius of convergence.

Special cases and applications are also provided in this study.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of equation

$$F(x) = 0, \quad (1.1)$$

where F is a Fréchet-differentiable operator defined on a convex subset \mathcal{D} of a Banach space \mathcal{X} with values in a Banach space \mathcal{Y} .

A large number of problems in applied mathematics and also in engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, assume that a time-invariant system is driven by the equation $\dot{x} = Q(x)$, for some suitable operator Q , where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of

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discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

We use the inexact Newton-like method (INLM):

$$x_{n+1} = x_n + s_n, \quad \mathcal{B}(x_n) s_n = -F(x_n) + r_n \quad (n \geq 0), \quad (x_0 \in \mathcal{D}) \quad (1.2)$$

to generate a sequence $\{x_n\}$ ($n \geq 0$) approximating x^* . Here, $\mathcal{B}(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of bounded linear operators from \mathcal{X} into \mathcal{Y} , are approximations of the Fréchet-derivative $F'(x)$; the residuals r_n satisfy

$$\| \mathcal{P}_n r_n \| \leq \theta_n \| \mathcal{P}_n F(x_n) \| \quad (n \geq 0), \quad (1.3)$$

for a suitable forcing sequence $\{\theta_n\}$, and some invertible sequence $\{\mathcal{P}_n\}$ of preconditioners for the linear equation in (1.2).

If $\mathcal{P}_n = I$, and $\mathcal{B}_n = F'(x_n)$ ($n \geq 0$), we obtain the inexact Newton method (INM). A survey of convergence results under various Lipschitz-type conditions for (INM) and (INLM) can be found in [4], [5] (see also [1]–[3], [7]–[15]).

In this study, we are motivated by the elegant work in [9], and optimization considerations. We introduce center-Lipschitz-type conditions (see (2.2)), and use it to find upper bounds on the distances $\| F'(x)^{-1} F'(x^*) \|$ ($x \in \mathcal{D}$). This approach leads (under the same computational cost) to a local convergence analysis, with the following advantages: finer estimates for the distances $\| x_n - x^* \|$ ($n \geq 0$), and a larger convergence radius.

Special cases and applications are also provided in this study.

2. LOCAL CONVERGENCE ANALYSIS OF (INLM)

We provide the main local convergence theorem for (INLM):

Theorem 2.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$ be a continuously differentiable operator. Let $x^* \in \mathcal{D}$, $R > 0$, and set*

$$\kappa = \sup \{ t \in [0, R) : U(x^*, t) = \{ x \in \mathcal{X} : \| x - x^* \| < t \} \subseteq \mathcal{D} \}. \quad (2.1)$$

Assume:

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X});$$

there exist functions $f_0, f : [0, R) \rightarrow (-\infty, +\infty)$ continuously differentiable, such that:

$$\| F'(x^*)^{-1} [F'(x) - F'(x^*)] \| \leq f'_0(\| x - x^* \|) - f'_0(0), \quad (2.2)$$

$$\| F'(x^*)^{-1} [F'(x) - F'(x^* + \tau(x - x^*))] \| \leq f'(\| x - x^* \|) - f'(\tau \| x - x^* \|), \quad (2.3)$$

for all $x \in U(x^*, \kappa)$, and $\tau \in [0, 1]$;

$$f(0) = f_0(0), \quad \text{and} \quad f'(0) = f'_0(0) = -1; \quad (2.4)$$

$$f'_0, f' \text{ are convex and strictly increasing}; \quad (2.5)$$

Choose: $\theta \in [0, 1)$, w_1, w_2 , with $w_2 \in [0, w_1)$, such that:

$$w_1 \theta + w_2 < 1. \quad (2.6)$$

Set:

$$v = \sup \{t \in [0, R) : f'(t) < 0\}, \quad (2.7)$$

$$\rho_0 = \sup \{t \in [0, v) : w_1 (1 + \theta) \left(\frac{f(t)}{t f'_0(t)} - \frac{f'(t)}{f'_0(t)} \right) + w_1 \theta + w_2 < 1\}, \quad (2.8)$$

and

$$\sigma_0 = \min \{\kappa, \rho_0\}; \quad (2.9)$$

$\mathcal{B}(x_n)$ are invertible approximations of $F'(x_n)$ ($n \geq 0$), and

$$\| \mathcal{B}(x_n)^{-1} F'(x_n) \| \leq w_1, \quad (2.10)$$

$$\| \mathcal{B}(x_n)^{-1} F'(x_n) - I \| \leq w_2; \quad (2.11)$$

the residuals r_n ($n \geq 0$) satisfy

$$\| \mathcal{P}_n r_n \| \leq \theta_n \| \mathcal{P}_n F(x_n) \|, \quad (2.12)$$

for some invertible matrix sequence $\{\mathcal{P}_n\}$ ($n \geq 0$) preconditioners for the linear system in (INLM), and a forcing sequence $\{\theta_n\}$ ($n \geq 0$) of non-negative numbers satisfying

$$\theta_n \text{ cond}(\mathcal{P}_n F'(x_n)) \leq \theta \quad (n \geq 0). \quad (2.13)$$

Then, sequence $\{x_n\}$ generated by (INLM) is well defined, remains in $U(x^*, \sigma_0)$, for all $n \geq 0$, and converges to x^* .

Moreover, the following estimates hold for all $n \geq 0$:

$$\| x_{n+1} - x^* \| \leq \alpha_0 \| x_n - x^* \|, \quad (2.14)$$

where,

$$0 \leq \alpha_0 = w_1 (1 + \theta) \left(\frac{f(\| x_0 - x^* \|)}{\| x_0 - x^* \| f'_0(\| x_0 - x^* \|)} - \frac{f'(\| x_0 - x^* \|)}{f'_0(\| x_0 - x^* \|)} \right) + w_1 \theta + w_2 < 1. \quad (2.15)$$

Proof. By the definition of v , $x \in U(x^*, \sigma_0)$, hypotheses (2.5), we deduce:

$$f'_0(\|x - x^*\|) < 0.$$

Using (2.2), we obtain in turn:

$$\|F'(x^*)^{-1} (F'(x) - F'(x^*))\| \leq f'_0(\|x - x^*\|) - f'_0(0) < -f'_0(0) = 1 \quad (\text{by (2.4)}) \quad (2.16)$$

It follows from (2.16), and the Banach lemma on invertible operators [4], [5], [12] that $F'(x)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, and

$$\begin{aligned} \|F'(x)^{-1} F'(x^*)\| &\leq \left(1 - \|F'(x^*)^{-1} (F'(x) - F'(x^*))\|\right)^{-1} \\ &\leq \left(1 - (f'_0(\|x - x^*\|) - f'_0(0))\right)^{-1} = \frac{1}{|f'_0(\|x - x^*\|)|}, \end{aligned} \quad (2.17)$$

since $\sigma_0 \leq v$.

It follows that

$$x_+ = x + s, \quad \mathcal{B}(x)s = -F(x) + r \quad (2.18)$$

is well defined.

In view of (INLM), and (2.18), we obtain the identity:

$$\begin{aligned} x_+ - x^* &= \mathcal{B}(x)^{-1} (F(x^*) - (F(x) + F'(x)(x^* - x))) + \\ &\quad (\mathcal{B}(x)^{-1} F'(x) - I)(x^* - x) + \mathcal{B}(x)^{-1} r. \end{aligned} \quad (2.19)$$

We shall find upper bounds on the compositors of (2.19). Using (2.2)–(2.4), and (2.17), we have in turn:

$$\begin{aligned} &\| -F'(x)^{-1} (F(x^*) - (F(x) + F'(x)(x^* - x))) + (x^* - x) \| \\ &\leq \|F'(x)^{-1} F'(x^*)\| \|F'(x^*)^{-1} (F(x^*) - F(x) - F'(x)(x^* - x))\| + \|x^* - x\| \\ &\leq \|F'(x)^{-1} F'(x^*)\| \int_0^1 \|F'(x^*)^{-1} (F'(x + \tau(x^* - x)) - F'(x))\| d\tau \|x^* - x\| + \\ &\quad \|x^* - x\| \\ &\leq \frac{\int_0^1 (f'(\|x - x^*\|) - f'(\tau \|x - x^*\|)) d\tau \|x - x^*\|}{|f'_0(\|x - x^*\|)|} + \|x^* - x\| \\ &\leq \frac{f(0) - f(\|x - x^*\|) + f'(\|x - x^*\|) \|x - x^*\|}{-f'_0(\|x - x^*\|)} + \|x - x^*\| \\ &= g(\|x - x^*\|) + \|x - x^*\|, \end{aligned} \quad (2.20)$$

where,

$$g(t) = \frac{-f(t) + f'(t) t}{-f'_0(t)}. \quad (2.21)$$

We also have:

$$\| F'(x)^{-1} \mathcal{P}^{-1} \| \| \mathcal{P} F(x) \| \leq \| (\mathcal{P}' F(x))^{-1} \| \| \mathcal{P} F'(x) \| (g(\| x - x^* \|) + \| x - x^* \|). \quad (2.22)$$

Using (2.10), (2.11), (2.15) (2.19), and (2.20), we have in turn:

$$\begin{aligned} \| x_+ - x^* \| &\leq \| \mathcal{B}(x)^{-1} F'(x) \| \| F'(x)^{-1} F'(x^*) \| \times \\ &\quad \| F'(x^*)^{-1} (F(x^*) - (F(x) + F'(x)(x^* - x))) \| + \\ &\quad \| \mathcal{B}(x)^{-1} F'(x) - I \| \| x - x^* \| + \\ &\quad \| \mathcal{B}(x)^{-1} F'(x) \| \| F'(x)^{-1} \mathcal{P}^{-1} \| \| \mathcal{P} r \| \\ &\leq w_1 g(\| x - x^* \|) + w_2 \| x - x^* \| + \\ &\quad w_1 \theta \| F'(x)^{-1} \mathcal{P}^{-1} \| \| \mathcal{P} F(x) \| \\ &\leq w_1 g(\| x - x^* \|) + w_2 \| x - x^* \| + \\ &\quad w_1 \theta (g(\| x - x^* \|) + \| x - x^* \|). \end{aligned} \quad (2.23)$$

By letting $x_+ = x_{n+1}$, $x = x_n$ in (2.24), we get:

$$\begin{aligned} \| x_{n+1} - x^* \| &\leq w_1 g(\| x_n - x^* \|) + w_2 \| x_n - x^* \| + \\ &\quad w_1 \theta (g(\| x_n - x^* \|) + \| x_n - x^* \|) \\ &\leq \alpha_0 \| x_n - x^* \| < \| x_n - x^* \|, \end{aligned} \quad (2.24)$$

which imply $x_{n+1} \in U(x^*, \sigma_0)$, and $\lim_{n \rightarrow \infty} x_n = x^*$.

That completes the proof of Theorem 2.1. \square

Remark 2.2. If $f(t) = f_0(t)$ on $[0, R)$, then, Theorem 2.1 reduces to Theorem 4 in [9]. Note that in view of (2.3), there exists function f_0 satisfying (2.2), (2.4), and (2.5).

Therefore, we have not introduced an additional hypothesis. In practice, the computation of function f requires that of f_0 .

Moreover,

$$f'_0(t) \leq f'(t) \quad t \in [0, R), \quad (2.25)$$

holds in general, and $\frac{f'(t)}{f'_0(t)}$ can be arbitrarily large [3]–[6].

Denote by ρ , σ , α the constants (used in [9]) corresponding to ρ_0 , σ_0 , α_0 respectively by simply replacing f_0 by f in definitions (2.8), (2.9), and (2.15).

In view of (2.25), we have:

$$\rho \leq \rho_0, \quad (2.26)$$

$$\sigma \leq \sigma_0, \quad (2.27)$$

$$\alpha_0 \leq \alpha. \quad (2.28)$$

If strict inequality holds in (2.25), then so does in (2.28). Hence, the estimates on the distances have been improved.

Let us examine some special choices of functions f_0 , and f .

Case 1: (Lipschitz case)

$$f_0(t) = \frac{K_0 t^2}{2} - t, \quad f(t) = \frac{K t^2}{2} - t,$$

$$\mathcal{B}(x_n) = F'(x_n), \quad \mathcal{P}_n = I, \quad w_1 = 1, \quad \text{and} \quad w_2 = 0 \quad (n \geq 0).$$

Then, we get:

$$\alpha_0 = (1 + \theta) \frac{K \|x - x^*\|}{2(1 - K_0 \|x_0 - x^*\|)} + \theta, \quad (2.29)$$

$$\alpha = (1 + \theta) \frac{K \|x - x^*\|}{2(1 - K \|x_0 - x^*\|)} + \theta, \quad (2.30)$$

which imply

$$\sigma_0 = \min \left\{ K, \frac{2(1 - \theta)}{2(1 - \theta)K_0 + (1 + \theta)K} \right\}, \quad (2.31)$$

and

$$\sigma = \min \left\{ K, \frac{2(1 - \theta)}{(3 - \theta)K} \right\}. \quad (2.32)$$

If $\theta = 0$, we obtain:

$$r_0 = \min \left\{ K, \frac{2}{2K_0 + K} \right\}, \quad (2.33)$$

and

$$r = \min \left\{ K, \frac{2}{3K} \right\}, \quad (2.34)$$

which imply

$$r \leq r_0. \quad (2.35)$$

Estimate (2.35) shows that we obtain a larger radius of convergence than in [9]. Below, we provide a numerical example where $r < r_0$ (see Example 2.3).

Case 2: (Smale–Wang case)

$$f_0(t) = \frac{t}{1 - \gamma_0 t} - 2t, \quad f(t) = \frac{t}{1 - \gamma t} - 2t \quad (\gamma \leq \gamma_0),$$

and choose:

$$R = \frac{1}{\gamma_0}.$$

The rest is left to the motivated reader.

More choices of functions f_0 , and f can be found in [9].

Example 2.3. Let $\mathcal{X} = \mathcal{Y} = (-\infty, +\infty)$, $\mathcal{D} = U(x^*, R) = U(0, 1)$, and define function F on \mathcal{D} by:

$$F(x) = e^x - 1. \quad (2.36)$$

Then we obtain:

$$K_0 = e - 1, \quad K = e.$$

In view (2.33), and (2.34), we get:

$$r = \frac{2}{3K} = .245252961 < r_0 = \frac{2}{2K_0 + K} = .324947231. \quad (2.37)$$

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