J. Nonlinear Sci. Appl. 1 (2008), no. 4, 203–205

THE JOURNAL OF NONLINEAR SCIENCES AND APPLICATIONS
http://www.tjnsa.com

COMMENT ON AND A CHARACTERIZATION OF THE CONCEPT OF COMPLETE RESIDUATED LATTICE

FATHEI M. ZEYADA[1] AND M. A. ABD-ALLAH[2]

ABSTRACT. We prove that some properties of the definition of complete residuated lattice [2,4] can be derived from the other properties. Furthermore we prove that the concept of strictly two-sided commutative quantale [1,3] and the concept of complete residuated lattice are equivalent notions.

1. INTRODUCTION

Definition 1. A structure $(L,\vee,\wedge,*,\rightarrow,\perp,\top)$ is called a complete residuated lattice iff

(1) $(L,\vee,\wedge,\perp,\top)$ is a complete lattice whose greatest and least element are $\top,\perp$ respectively,

(2) $(L,*,\top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L, and

(b) $\forall\ a\in L, a*\top=\top*\ a=a$,

(3)(a) $*$ is isotone,

(b) $\rightarrow$ is a binary operation on L which is antitone in the first and isotone in the second variable,

(c) $\rightarrow$ is couple with $*$ as: $a*b\leq c$ iff $a\leq b\rightarrow c\quad\forall\ a,b,c\in L.$

The following proposition illustrates that the conditions (3)(a) and (3)(b) are consequences from the other conditions. Therefore conditions (3)(a) and (3)(b) should be omit from Definition 1 to be consistent.

Date: Received:March 2008 ; Revised: May 2008

* Corresponding author.

2000 *Mathematics Subject Classification.* Primary 42A20; Secondary 42A32.

Key words and phrases. Complete residuated lattice; Quantale; Complete MV-algebra.

Proposition 1. The conditions (3)(a) and (3)(b) are obtained from the commutativety of $*$ and from (3)(c).

Proof. Let $a_1, a_2, b \in L$ $s.t.$ $a_1 \leq a_2$.

(3)(a) Since $a_2 * b \leq a_2 * b$, then $a_2 \leq b \to (a_2 * b)$ and so $a_1 \leq b \to (a_2 * b)$. So $a_1 * b \leq a_2 * b$. Since $*$ is commutative, then $b * a_1 \leq b * a_2$. Hence $*$ is isotone.

(3)(b) Since $a_2 \to b \leq a_2 \to b$, then $(a_2 \to b) * a_2 \leq b$. So, $(a_2 \to b) * a_1 \leq b$ which implies that $a_2 \to b \leq a_1 \to b$,i.e., $\to$ is antitone in the first variable. Since $b \to a_1 \leq b \to a_1$, then $(b \to a_1) * b \leq a_1 \leq a_2$. So, $b \to a_1 \leq b \to a_2$, *i.e.*, $\to$ is isotone in the second variable.

For the following definition we refer to [1,3].

Definition 2. A structure $(L,\vee,\wedge,*,\to,\perp,\top)$ is called a strictly two-sided commutative quantale iff

(1) $(L,\vee,\wedge,\perp,\top)$ is a complete lattice whose greatest and least element are $\top, \perp$ respectively,

(2) $(L,*,\top)$ is a commutative monoid,

(3)(a) $*$ is distributive over arbitrary joins, i.e., $a * \vee_{j\in J} b_j = \vee_{j\in J}(a * b_j)$ $\forall a \in L$, $\forall \{b_j \,|j \in J\} \subseteq L$,

(b) $\to$ is a binary operation on L defined by : $a \to b = \vee_{\lambda * a \leq b}\lambda \quad \forall a, b \in L$.

Lemma 1. In any strictly two-sided commutative quantale $(L,\vee,\wedge,*,\to,\perp,\top)$, $*$ is isotone.

Proof. Let $a_1, a_2, b \in L$ $s.t.$ $a_1 \leq a_2$. Now, $b * a_2 = b * (a_1 \vee a_2) =$
$(b * a_1) \vee (b * a_2)$. Then $b * a_1 \leq b * a_2$. *nce* $*$ is commutative,
then $a_1 * b \leq a_2 * b$. Hence $*$ is isotone.

Theorem 1. A structure $(L,\vee,\wedge,*,\to,\perp,\top)$ is complete residuated lattice iff it is strictly two-sided commutative quantale.

Proof. $\Longrightarrow$: First, since for every $\lambda \in L$ s.t. $a * \lambda \leq b$ we have $\lambda \leq a \to b$. Then $\vee_{\lambda * a \leq b}\lambda \leq a \to b$. Since $a \to b \leq a \to b$, then $(a \to b) * a \leq b$. So, $a \to b \in \{\lambda \in L \,|\lambda * a \leq b\}$. Hence $\vee_{\lambda * a \leq b}\lambda = a \to b$.

Second, since $*$ is isotone, then $\vee_{j\in J}(a * b_j) \leq a * \vee_{j\in J} b_j$. Now, $\forall j \in J$, $b_j \leq a \to (a * b_j)$ which implies that $\vee_{j\in J} b_j \leq a \to \vee_{j\in J}(a * b_j)$. Thus $a * \vee_{j\in J} b_j \leq \vee_{j\in J}(a * b_j)$. Hence $a * \vee_{j\in J} b_j = \vee_{j\in J}(a * b_j)$.

$\Longleftarrow$: Let $a * b \leq c$. Then $b \to c = \vee_{\lambda * b \leq c}\lambda \geq a$. Conversely, let $a \leq b \to c$. Then, $a \leq \vee_{\lambda * b \leq c}\lambda$. So from Lemma 1 , $a * b \leq (\vee_{\lambda * b \leq c}\lambda) * b = \vee_{\lambda * b \leq c}(\lambda * b) \leq c$.

Definition 3 [1]. A structure $(L,\vee,\wedge,*,\to,\perp,\top)$ is called a complete MV- algebra iff the following conditions are satisfied:

(1) $(L,\vee,\wedge,*,\to,\perp,\top)$ is a strictly two-sided commutative quantale;

(2) $\forall a, b \in L$, $(a \to b) \to b = a \vee b$.

Corollary 1. $(L,\vee,\wedge,*,\rightarrow,\perp,\top)$ is a complete MV- algebra iff $(L,\vee,\wedge,*,\rightarrow,\perp,\top)$ is a complete residuated lattice satisfies the additinal property

(MV) $(a \rightarrow b) \rightarrow b = a \vee b \ \ \forall a, b \in L.$

References

1. U.Höhle, Characterization of L-topologies by L-valued neighborhoods, in: U. Höhle, S.E.Rodabaugh (Eds.), The Handbooks of Fuzzy Sets Series, Vol.3, Kluwer ACademic Publishers, Dordrecht, 1999.pp.389-432.
2. J.Pavelka, On fuzzy logic II, Z. Math. Logic Gvundlagen Math. 25:119-134(1979).
3. K.I. Rosenthal, Quantales and their applications, Pitman Research Notes in Mathematics 234(Longman, Burnt Mill, Harlow 1990).
4. M.S.Ying, Fuzzifying topology based on complete residuated lattice-valued logic (I), Fuzzy Sets and Systems 56(1993)337-373.

[1] Center of Mathematics and Theoretical Computer Sciences
Assiut, Egypt

[2] Department of Mathematics, Faculty of Science, Al Azhar University,
Assiut, Egypt