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RELATED FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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Abstract. We prove a related fixed point Theorem for four mappings which are not continuous in four fuzzy metric spaces, one of them is a sequentially compact fuzzy metric space. Our Theorem in the metric version generalizes Theorem 4 of [1]. Finally, We give a fuzzy version of Theorem 3 of [1].

1. Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [11] in 1965. George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6]. Recently, Fisher [3], Telci [10] and Aliouche and Fisher [1] proved some related fixed point Theorems in compact metric spaces. Motivated by a work due to Popa [7], we have observed that proving fixed point theorems using an implicit relation is a good idea since it covers several contractive conditions rather than one contractive condition. In this paper, we mainly prove a related fixed point Theorem for four mappings which are not necessarily continuous in four fuzzy metric spaces, using an implicit relation, one of them is a sequentially compact fuzzy metric space. One of our Theorems in the metric version generalizes a theorem of Aliouche and Fisher [1]. We give also a fuzzy version of Theorem 3 of [1]..

Definition 1.1 ([9]). A binary operation $* : [0,1]\times[0,1] \longrightarrow [0,1]$ is a continuous $t-$norm if it satisfies the following conditions:

1) $*$ is associative and commutative,
2) $*$ is continuous,

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3) $a * 1 = a$ for all $a \in [0, 1]$,
4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous $t-$norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.2 ([4])**.** A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous $t-$norm a and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,
1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ if and only if $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Definition 1.3 ([3])**.** Let $(X, M, *)$ be a fuzzy metric space.
1) For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

2) Let $(X, M, *)$ be a fuzzy metric space and τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then, τ is a topology on X induced by the fuzzy metric M.
3) A sequence $\{x_n\}$ in X converges to x if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M(x_n, x, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \to 1$ as $n \to \infty$ for all $t > 0$.
4) A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $M(x_n, x_m, t) > 1 - \epsilon$; i.e., $M(x_n, x_m, t) \to 1$ as $n, m \to \infty$ for all $t > 0$.
5) A fuzzy metric space (X, M, t) in which every Cauchy sequence is convergent is said to be complete.

Definition 1.4. A subset A of X is said to be $F-$bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Lemma 1.5 ([5])**.** *Let $(X, M, *)$ be a fuzzy metric space. Then, $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Lemma 1.6 ([5])**.** *Let $(X, M, *)$ be a fuzzy metric space. Then, M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.7. $(X, M, *)$ is said to be sequentially compact fuzzy metric space if every sequence in X has a convergent sub-sequence in it.

Let Φ be the set of all functions $\phi : [0, 1]^6 \longrightarrow [0, 1]$ such that if either $\phi(u, 1, u, v, v, 1) > 0$ or $\phi(u, u, 1, v, 1, v) > 0$ for all $u, v \in [0, 1)$, then $u > v$.

Example 1.8. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$. Then $\phi \in \Phi$.

2. Main results

Theorem 2.1. *Let (X, M_1, θ_1), (Y, M_2, θ_2), (Z, M_3, θ_3) and (W, M_4, θ_4) be fuzzy metric spaces and $B : X \longrightarrow Y$, $T : Y \longrightarrow Z$, $A : Z \longrightarrow W$, $S : W \longrightarrow X$ be mappings satisfying*

$$(1)\ \ \phi_1\begin{pmatrix} M_1(SATy, SATBx, t), M_1(x, SATy, t), \\ M_1(x, SATBx, t), M_2(y, Bx, t), \\ M_2(y, BSATy, t), M_2(Bx, BSATy, t) \end{pmatrix} > 0$$

for all $x \in X$, $y \in Y$ with $y \neq Bx$ and for all $t > 0$, where $\phi_1 \in \Phi$,

$$(2)\ \ \phi_2\begin{pmatrix} M_2(BSAz, BSATy, t), M_2(y, BSAz, t), \\ M_2(y, BSATy, t), M_3(z, Ty, t), \\ M_3(z, TBSAz, t), M_3(Ty, TBSAz, t) \end{pmatrix} > 0$$

for all $z \in Z$, $y \in Y$ with $z \neq Ty$ and for all $t > 0$, where $\phi_2 \in \Phi$,

$$(3)\ \ \phi_3\begin{pmatrix} M_3(TBSw, TBSAz, t), M_3(z, TBSw, t), \\ M_3(z, TBSAz, t), M_4(w, Az, t), \\ M_4(w, ATBSw, t), M_4(Az, ATBSw, t) \end{pmatrix} > 0$$

for all $z \in Z$, $w \in W$ with $w \neq Az$ and for all $t > 0$, where $\phi_3 \in \Phi$,

$$(4)\ \ \phi_4\begin{pmatrix} M_4(ATBx, ATBSw, t), M_4(w, ATBx, t), \\ M_4(w, ATBSw, t), M_1(x, Sw, t), \\ M_1(x, SATBx, t), M_1(Sw, SATBx, t) \end{pmatrix} > 0$$

for all $x \in X$, $w \in W$ with $x \neq Sw$ and for all $t > 0$, where $\phi_4 \in \Phi$.
Further, suppose that one of the following is true:
(a) (X, M_1, θ_1) is sequentially compact and $SATB$ is continuous on X.
(b) (Y, M_2, θ_2) is sequentially compact and $BSAT$ is continuous on Y.
(c) (Z, M_3, θ_3) is sequentially compact and $TBSA$ is continuous on Z.
(d) (W, M_4, θ_4) is sequentially compact and $ATBS$ is continuous on W.
Then, $SATB$ has a unique fixed point $u \in X$, $BSAT$ has a unique fixed point $v \in Y$, $TBSA$ has a unique fixed point $w \in Z$ and $ATBS$ has a unique fixed point $q \in W$. Further, $Bu = v$, $Tv = w$, $Aw = q$ and $Sq = u$.

Proof. Suppose that (a) holds. For every $t > 0$, define $\phi(x) = M_1(x, SATBx, t)$ for all $x \in X$. Then, there exists $p \in X$ such that $\phi(p) = M_1(p, SATBp, t) = \max\{\phi(x) : x \in X\}$.
Suppose that $BSATBSATBp \neq BSATBSATBSATBp$. Then, $TBSATBp \neq TBSATBSATBp$, $ATBp \neq ATBSATBp$ and $p \neq SATBp$.

Putting $y = BSATBSATBp$ and $x = SATBSATBSATBp$ in (1) we have

$$\phi_1\begin{pmatrix} M_1(SATBSATBSATBp, SATBSATBSATBSATBp, t), \\ M_1(SATBSATBSATBp, SATBSATBSATBp, t), \\ M_1(SATBSATBSATBp, SATBSATBSATBSATBp, t), \\ M_2(BSATBSATBp, BSATBSATBSATBp, t), \\ M_2(BSATBSATBp, BSATBSATBSATBp, t), \\ M_2(BSATBSATBSATBp, BSATBSATBSATBp, t) \end{pmatrix} > 0$$

and so

$$(5)\quad \phi(SATBSATBSATBp) > M_2(BSATBSATBp, BSATBSATBSATBp, t).$$

Putting $y = BSATBSATBp$ and $z = TBSATBp$ in (2) we get

$$\phi_2\begin{pmatrix} M_2(BSATBSATBp, BSATBSATBSATBp, t), \\ M_2(BSATBSATBp, BSATBSATBp, t), \\ M_2(BSATBSATBp, BSATBSATBSATBp, t), \\ M_3(TBSATBp, TBSATBSATBp, t), \\ M_3(TBSATBp, TBSATBSATBp, t), \\ M_3(TBSATBSATBp, TBSATBSATBp, t) \end{pmatrix} > 0.$$

Therefore

$$(6)\quad M_2(BSATBSATBp, BSATBSATBSATBp, t) > M_3(TBSATBp, TBSATBSATBp, t).$$

Putting $z = TBSATBp$ and $w = ATBp$ in (3) we obtain

$$\phi_3\begin{pmatrix} M_3(TBSATBp, TBSATBSATBp, t), \\ M_3(TBSATBp, TBSATBp, t), \\ M_3(TBSATBp, TBSATBSATBp, t), \\ M_4(ATBp, ATBSATBp, t), \\ M_4(ATBp, ATBSATBp, t), \\ M_4(ATBSATBp, ATBSATBp, t) \end{pmatrix} > 0$$

and so

$$(7)\quad M_3(TBSATBp, TBSATBSATBp, t) > M_4(ATBp, ATBSATBp, t).$$

Putting $w = ATBp$ and $x = p$ in (4) we have

$$\phi_4\begin{pmatrix} M_4(ATBp, ATBSATBp, t), M_4(ATBp, ATBp, t), \\ M_4(ATBP, ATBSATBp, t), M_1(p, SATBp, t), \\ M_1(p, SATBp, t), M_1(SATBp, SATBp, t) \end{pmatrix} > 0.$$

Hence

$$(8)\quad M_4(ATBp, ATBSATBp, t) > M_1(p, SATBp, t) = \phi(p).$$

From (5), (6), (7) and (8) we get $\phi(SATBSATBSATBp) > \phi(p)$ which is a contradiction. Therefore

$$(9)\quad BSATBSATBp = BSATBSATBSATBp.$$

Denote $BSATBSATBp = v \in Y$. Then from (9), $v = BSATv$.
Let $Tv = w \in Z$, $Aw = q \in W$, $Sq = u \in X$. Then $v = BSATv = BSAw = BSq = Bu$.
Also, $SATBu = SATv = SAw = Sq = u$, $TBSAw = TBSq = TBu = Tv = w$ and $ATBSq = ATBu = ATv = Aw = q$.

For the uniqueness of u, suppose that $SATBu' = u'$ with $u \neq u'$. Then, $SATBu \neq SATBu'$, $ATBu \neq ATBu'$, $TBu \neq TBu'$ and $Bu \neq Bu'$.

Putting $x = u$ and $y = Bu'$ in (1) we have

$$\phi_1\begin{pmatrix} M_1(SATBu', SATBu, t), M_1(u, SATBu', t), \\ M_1(u, SATBu, t), M_2(Bu', Bu, t), \\ M_2(Bu', BSATBu', t), M_2(Bu, BSATBu', t) \end{pmatrix} > 0$$

and so

$$M_1(u, u', t) > M_2(Bu, Bu', t)....(10).$$

Putting $z = TBu$, $y = Bu'$ in (2) we get

$$\phi_2\begin{pmatrix} M_2(BSATBu, BSATBu', t), M_2(Bu', BSATBu, t), \\ M_2(Bu', BSATBu', t), M_3(TBu, TBu', t), \\ M_3(TBu, TBSATBu, t), M_3(TBu', TBSATBu, t) \end{pmatrix} > 0.$$

Therefore

$$M_2(Bu, Bu', t) > M_3(TBu, TBu', t).....(11).$$

Putting $z = TBu$, $w = ATBu'$ in (3) we obtain

$$\phi_3\begin{pmatrix} M_3(TBSATBu', TBSATBu, t), M_3(TBu, TBSATBu', t), \\ M_3(TBu, TBSATBu, t), M_4(ATBu', ATBu, t), \\ M_4(ATBu', ATBSATBu', t), M_4(ATBu, ATBSATBu', t) \end{pmatrix} > 0.$$

Hence

$$M_3(TBu, TBu', t) > M_4(ATBu, ATBu', t)....(12)$$

Putting $x = SATBu$, $w = ATBu'$ in (4) we have

$$\phi_4\begin{pmatrix} M_4(ATBSATBu, ATBSATBu', t), M_4(ATBu', ATBSATBu, t), \\ M_4(ATBu', ATBSATBu', t), M_1(SATBu, SATBu', t), \\ M_1(SATBu, SATBSATBu, t), M_1(SATBu', SATBSATBu, t)) \end{pmatrix} > 0$$

and so

$$M_4(ATBu, ATBu', t) > M_1(u, u', t)....(13)$$

Using (10), (11), (12) and (13) we get

$$M_1(u, u', t) > M_1(u, u', t)$$

which is a contradiction. Hence, u is the unique fixed point of $SATB$. Similarly, we can prove the uniqueness of fixed points of $BSAT$, $TBSA$ and $ATBS$. In a similar manner, the Theorem holds if either (b) or (c) or (d) is true. □

The following Example illustrates Theorem 2.1.

Example 2.2. Let $X = [0, 1]$, $Y = [1, 2)$, $Z = (2, 3]$ and $W = [3, 4)$ and $M_1(x, y, t) = \dfrac{t}{t + |x - y|}$, $M_2(y, z, t) = \dfrac{t}{t + |y - z|}$, $M_3(z, w, t) = \dfrac{t}{t + |z - w|}$ and $M_4(w, x, t) = \dfrac{t}{t + |w - x|}$.

Define $B : X \longrightarrow Y$ by:

$$Bx = \begin{cases} 1 & \text{if } x \in [0, 3/4], \\ 3/2 & \text{if } x \in (3/4, 1]. \end{cases}$$

$T : Y \longrightarrow Z$ by $Ty = 3$ for all $y \in Y$, $A : Z \longrightarrow W$ by

$$Az = \begin{cases} 7/2 & \text{if } x \in (2, 5/2], \\ 3 & \text{if } x \in (5/2, 3]. \end{cases}$$

and $S : W \longrightarrow X$ by $Sw = 1$ for all $w \in W$. Let

$$\begin{aligned} \phi_1(t_1, t_2, t_3, t_4, t_5, t_6) &= t_1 - \min\{t_2, t_3, t_4, t_5, t_6\} \text{ and} \\ \phi_1 &= \phi_2 = \phi_3 = \phi_4. \end{aligned}$$

In this Example, the inequalities (1), (2), (3) and (4) are satisfied since the value of the left hand side of each inequality is 1.
Clearly, $SATB(1) = 1$, $BSAT(3/2) = 3/2$, $TBSA(3) = 3$, $ATBS(3) = 3$ and $B1 = 3/2$, $T(3/2) = 3$, $A3 = 3$, $S3 = 1$.

If $B = T$, $T = S$, $A = R$, $S = I$ (Identity map) and $W = X$ in Theorem 2.1 we get the following Theorem.

Theorem 2.3. *Let (X, M_1, θ_1), (Y, M_2, θ_2) and (Z, M_3, θ_3) be fuzzy metric spaces and $T : X \longrightarrow Y$, $S : Y \longrightarrow Z$, $R : Z \longrightarrow X$ be mappings satisfying*

$$(1)\ \ \phi_1 \begin{pmatrix} M_1(RSy, RSTx, t), M_1(x, RSy, t), M_1(x, RSTx, t), \\ M_2(y, Tx, t), M_2(y, TRSy, t), M_2(Tx, TRSy, t) \end{pmatrix} > 0$$

for all $x \in X$, $y \in Y$ with $y \neq Tx$ and for all $t > 0$, where $\phi_1 \in \Phi$,

$$(2)\ \ \phi_2 \begin{pmatrix} M_2(TRz, TRSy, t), M_2(y, TRz, t), M_2(y, TRSy, t), \\ M_3(z, Sy, t), M_3(z, STRz, t), M_3(Sy, STRz, t) \end{pmatrix} > 0$$

for all $z \in Z$, $y \in Y$ with $z \neq Sy$ and for all $t > 0$, where $\phi_2 \in \Phi$,

$$(3)\ \ \phi_3 \begin{pmatrix} M_3(STx, STRz, t), M_3(z, STx, t), M_3(z, STRz, t), \\ M_1(x, Rz, t), M_1(x, RSTx, t), M_1(Rz, RSTx, t) \end{pmatrix} > 0$$

*for all $z \in Z$, $x \in X$ with $x \neq Rz$ and for all $t > 0$, where $\phi_3 \in \Phi$.
Further, suppose that one of the following is true:*
(a) (X, M_1, θ_1) is sequentially compact and RST is continuous on X.
(b) (Y, M_2, θ_2) is sequentially compact and TRS is continuous on Y.
(c) (Z, M_3, θ_3) is sequentially compact and STR is continuous on Z.
Then, RST has a unique fixed point $u \in X$, TRS has a unique fixed point $v \in Y$ and STR has a unique fixed point $w \in Z$. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

If $R = I$ (Identity map) and $Z = X$ in Theorem 2.3 we obtain

Theorem 2.4. *Let (X, M_1, θ_1) and (Y, M_2, θ_2) be fuzzy metric spaces and $T : X \longrightarrow Y$, $S : Y \longrightarrow X$ be mappings satisfying*

$$(1)\ \ \phi_1 \begin{pmatrix} M_1(Sy, STx, t), M_1(x, Sy, t), M_1(x, STx, t), \\ M_2(y, Tx, t), M_2(y, TSy, t), M_2(Tx, TSy, t) \end{pmatrix} > 0$$

for all $x \in X$, $y \in Y$ with $y \neq Tx$ and for all $t > 0$, where $\phi_1 \in \Phi$,

$$(2)\ \ \phi_2 \begin{pmatrix} M_2(Tx, TSy, t), M_2(y, Tx, t), M_2(y, TSy, t), \\ M_1(x, Sy, t), M_1(x, STx, t), M_1(Sy, STx, t) \end{pmatrix} > 0$$

*for all $x \in X$, $y \in Y$ with $x \neq Sy$ and for all $t > 0$, where $\phi_2 \in \Phi$.
Further, suppose that one of the following is true:*

(a) (X, M_1, θ_1) *is sequentially compact and* ST *is continuous on* X*.*
(b) (Y, M_2, θ_2) *is sequentially compact and* TS *is continuous on* Y*.*
Then, ST *has a unique fixed point* $u \in X$ *and* TS *has a unique fixed point* $v \in Y$*. Further,* $Tu = v$ *and* $Sv = u$*.*

1) The metric version of Theorem 2.4 in compact metric spaces generalizes and improves Theorem 4 of Aliouche and Fisher [1] under the implicit relation $\phi : \mathbb{R}_+^6 \to \mathbb{R}$ such that $\phi(u, u, 0, v, 0, v) < 0$ or $\phi(u, 0, u, v, v, 0) < 0$ implies $u < v$.

2) If $\phi_1(t_1, t_2, t_3, t_4, t_5, t_6) = \phi_2(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5, t_6\}$ in Theorem 2.4, we get a fuzzy version of a Theorem of Fisher [3].

Finally, we give a fuzzy version of Theorem 3 of Aliouche and Fisher [1] using the following implicit relations.
We denote by Ψ the set of all functions $\psi : [0, 1]^4 \longrightarrow [0, 1]$ such that
:(i) ψ is upper semi continuous in each coordinate variable,
(ii) ψ is decreasing in 3rd and 4th variable,
(iii) if either $\psi(u, v, 1, u) \geq 0$ or $\psi(u, 1, v, 1) \geq 0$ or $\psi(u, v, u, 1) \geq 0$ for all $u, v \in [0, 1]$, then $u \geq v$.

Example 2.5. $\psi(t_1, t_2, t_3, t_4) = t_1 - \min\{t_2, t_3, t_4\}$,

Example 2.6. $\psi(t_1, t_2, t_3, t_4) = t_1 - \phi_1(\min\{t_2, t_3, t_4\})$, where $\phi_1 : (0, 1] \longrightarrow (0, 1]$ is an increasing and continuous function with $\phi(t) > t$ for $0 < t < 1$. For example $\phi_1(t) = \sqrt{t}$ or $\phi_1(t) = t^h$ for $0 < h < 1$.

We need the following Lemma of [2].

Lemma 2.7 ([2])**.** *Let* $\{x_n\}$ *be a sequence in a fuzzy metric space* $(X, M, *)$ *with* $M(x, y, t) \longrightarrow 1$ *as* $t \longrightarrow \infty$ *for all* $x, y \in X$*. If there exists a number* $k \in (0, 1)$ *such that*

$$M(x_{n+1}, x_n, kt) \geq M(x_n, x_{n-1}, t)$$

for all $t > 0$ *and* $n = 1, 2, 3, \ldots$ *, then* $\{x_n\}$ *is a Cauchy sequence in* X*.*

Theorem 2.8. *Let* (X, M_1, θ_1) *and* (Y, M_2, θ_2) *be complete fuzzy metric spaces with* $M_1(x, x', t) \longrightarrow 1$ *as* $t \longrightarrow \infty$ *for all* $x, x' \in X$ *and* $M_2(y, y', t) \longrightarrow 1$ *as* $t \longrightarrow \infty$ *for all* $y, y' \in Y$ *. Let* $T : X \longrightarrow Y$*,* $S : Y \longrightarrow X$ *be mappings satisfying:*

$$(1)\ \ \psi_1\Big(\ M_1(Sy, STx, kt), M_2(y, Tx, t), M_1(x, Sy, t), M_1(x, STx, t)\ \Big) \geq 0,$$

$$(2)\ \ \psi_2\Big(\ M_2(Tx, TSy, kt), M_1(x, Sy, t), M_2(y, Tx, t), M_2(y, TSy, t)\ \Big) \geq 0$$

for all $x \in X$*,* $y \in Y$ *and for all* $t > 0$*, where* $\psi_1, \psi_2 \in \Psi$ *and* $0 < k < 1$*.*
Then, ST *has a unique fixed point* $u \in X$ *and* TS *has a unique fixed point* $v \in Y$*. Further,* $Tu = v$ *and* $Sv = u$*.*

Proof. Let x_0 be an arbitrary point in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by: $y_n = Tx_{n-1}$, $x_n = Sy_n$ for $n = 1, 2, \ldots.$
Putting $x = x_n$ and $y = y_n$ in (1), we have

$$\psi_1(M_1(x_n, x_{n+1}, kt), M_2(y_n, y_{n+1}, t), 1, M_1(x_n, x_{n+1}, t)) \geq 0.$$

Since ψ_1 is decreasing in 4th variable, we get

$$\psi_1(M_1(x_n, x_{n+1}, kt), M_2(y_n, y_{n+1}, t), 1, M_1(x_n, x_{n+1}, kt)) \geq 0.$$

From (iii), we obtain

$$M_1(x_n, x_{n+1}, kt) \geq M_2(y_n, y_{n+1}, t)...(3)$$

Putting $x = x_{n-1}$and $y = y_n$ in (2), we have

$$\psi_2(M_2(y_n, y_{n+1}, kt), M_1(x_{n-1}, x_n, t), 1, M_2(y_n, y_{n+1}, t)) \geq 0.$$

As ψ_2 is decreasing in 4th variable, we get

$$\psi_2(M_2(y_n, y_{n+1}, kt), M_1(x_{n-1}, x_n, t), 1, M_2(y_n, y_{n+1}, kt)) \geq 0.$$

From (iii), we obtain

$$M_2(y_n, y_{n+1}, kt) \geq M_1(x_{n-1}, x_n, t)....(4).$$

Using (3) and (4) we have for $n = 1, 2, ...$

$$\begin{aligned} M_1(x_n, x_{n+1}, t) &\geq M_1(x_{n-1}, x_n, t/k^2) \text{ and} \\ M_2(y_n, y_{n+1}, t) &\geq M_2(y_{n-1}, y_n, t/k^2). \end{aligned}$$

From Lemma 2.7, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Hence, $\{x_n\}$ converges to $u \in X$ and $\{y_n\}$ converges to $v \in Y$.
Putting $x = x_{n-1}$ and $y = v$ in (1), we get

$$\psi_1(M_1(Sv, STx_{n-1}, kt), M_2(v, Tx_{n-1}, t), M_1(x_{n-1}, Sv, t), M_1(x_{n-1}, STx_{n-1}, t)) \geq 0$$

Letting $n \longrightarrow \infty$, we have

$$\psi_1(M_1(Sv, u, kt), 1, M_1(u, Sv, t), 1) \geq 0.$$

Using (iii), we obtain

$$M_1(Sv, u, kt) \geq M_1(u, Sv, t)$$

and so $Sv = u$. Similarly, we can show that $Tu = v$. Now, $STu = Sv = u$ and $TSv = Tu = v$.
To prove the uniqueness of u, suppose that ST has a second fixed point u' in X.
Putting $x = u'$, $y = v$ in (1), we get

$$\psi_1(M_1(u, u', kt), M_2(Tu, Tu', t), M_1(u', u, t), 1)) \geq 0.$$

Since ψ_1 is decreasing in 3rd variable, we have

$$\psi_1(M_1(u, u', kt), M_2(Tu, Tu', t), M_1(u', u, kt), 1)) \geq 0.$$

From (iii), we obtain

$$M_1(u, u', kt) \geq M_2(Tu, Tu', t).$$

Similarly, we have

$$M_2(Tu, Tu', kt) \geq M_1(u, u', t).$$

Hence

$$M_1(u, u', t) \geq M_1(u, u', t/k^2)$$

and so $u = u'$. The uniqueness of v follows in a similar manner. □

1) If $\psi_1(t_1,t_2,t_3,t_4) = \psi_2(t_1,t_2,t_3,t_4) = t_1 - \min\{t_2,t_3,t_4\}$ in Theorem 2.7, we get a fuzzy version of a Theorem of Fisher [3].

2) As in Theorems 2.4 and 2.7, we can obtain fuzzy versions of Theorems of [9].

The following Example support our Theorem 2.7.

Example 2.9. Let $X = [0,1] = Y$ and $M_1(x,y,t) = M_2(y,x,t) = \dfrac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Define $T : X \longrightarrow Y$ and $S : Y \longrightarrow X$ by:

$$Tx = \begin{cases} x/2 & \text{if } x \in (0,1], \\ 1/2 & \text{if } x = 0. \end{cases},$$

$Sy = 1/2$ for all $y \in Y$. Let

$$\begin{aligned} \psi_1(t_1,t_2,t_3,t_4) &= \psi_2(t_1,t_2,t_3,t_4) \\ &= t_1 - \min\{t_2,t_3,t_4\}. \end{aligned}$$

In this Example, the inequality (1) is satisfied since the value of the left hand side of inequality is 1 and the inequality (2) is satisfied with $k = 1/2$.
Clearly, $ST(1/2) = 1/2$, $TS(1/4) = 1/4$, $S(1/4) = 1/2$ and $T(1/2) = 1/4$.

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