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SOME PROPERTIES OF C-FRAMES OF SUBSPACES

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Abstract. In [13] frames of subspaces extended to continuous version namely c-frame of subspaces. In this article we consider to the relations between c-frames of subspaces and local c-frames. Also in this article we give some important relation about duality and parseval c-frames of subspaces.

1. Introduction and preliminaries

Throughout this paper H will be a Hilbert space and $\mathbb{H}$ will be the collection of all closed subspace of H, respectively. Also, (X, μ) will be a measure space, and $v : X \to [0, +\infty)$ a measurable mapping such that $v \neq 0$ a.e. We shall denote the unit closed ball of H by H_1.

Frames were introduced in the context of non-harmonic Fourier series [9], and frame of subspaces introduced by Casazza and Kutyniok in [4]. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of [8]. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly, several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission on [6], and to design high-rate constellation with full diversity in multiple-antenna code design [12]. In [1, 2, 3] some applications have been developed.

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The frames of subspaces were considered by Casazza and Kutyniok [12]. In that paper they formulated a general method for piecing together local frames to get global frames. The importance of that approach is that it is both necessary and sufficient for the construction of global frames of local frames.

In this paper we try to obtain relations between c-frames of subspaces and local c-frames. Also we get many useful relation about dual discussions.

Definition 1.1. Let $\{f_i\}_{i\in I}$ be a sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \sum_{i\in I} |\langle f_i, h\rangle|^2 \leq B\|h\|^2. \tag{1.1}$$

The constants A and B are called frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A-tight frame and if $A = B = 1$ it is called a parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded,

$$T : l^2(I) \to H, T(c_i) = \sum_{i\in I} c_i f_i \tag{1.2}$$

$$T^* : H \to l^2(I), T^*(f) = \{\langle f, f_i\rangle\}_{i\in I} \tag{1.3}$$

$$Sf = TT^*f = \sum_{i\in I} \langle f, f_i\rangle f_i. \tag{1.4}$$

This operators are called synthesis operator, analysis operator and frame operator, respectively.

Theorem 1.2. *Let $\{f_i\}_{i\in I}$ be a frame with frame operator S. Then*

$$h = \sum_{i\in I} \langle h, S^{-1}f_i\rangle f_i, \quad \forall h \in H.$$

The series converges unconditionally for all $h \in H$.

Proof. See [7] Theorem 5.1.6. □

In case $\{f_i\}_{i\in I}$ is an overcomplete frame, by Theorem 5.6.1 in [7] there exist frames $\{g_i\}_{i\in I} \neq \{S^{-1}f_i\}_{i\in I}$ for which $h = \sum_{i\in I} \langle h, g_i\rangle f_i$, for each $h \in H$. Such frame is called dual frame of $\{f_i\}_{i\in I}$ and $\{S^{-1}f_i\}_{i\in I}$ is called canonical dual frame of $\{f_i\}_{i\in I}$.

Lemma 1.3. *Assume that $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are Bessel sequences in H. Then the following are equivalent.*
(i) $h = \sum_{i\in I} \langle h, f_i\rangle g_i$, for each $h \in H$.
(ii) $h = \sum_{i\in I} \langle h, g_i\rangle f_i$, for each $h \in H$.
(iii) $\langle h, k\rangle = \sum_{i\in I} \langle h, f_i\rangle\langle g_i, k\rangle$, for each $h, k \in H$.
In case the equivalent condition are satisfied, $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are dual frame for H.

Proof. See [7] lemma 5.6.2. □

Definition 1.4. For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspaces in H, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i\rangle 0$ for all $i \in I$. Then $\{W_i\}_{i\in I}$ is a frame of subspace with respect to $\{v_i\}_{i\in I}$ for H if there exist $0 < C \leq D < \infty$ such that for all $h \in H$

$$C\|h\|^2 \leq \sum_{i\in I} {v_i}^2\|\pi_{W_i}(h)\|^2 \leq D\|h\|^2 \tag{1.5}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i.

We call C and D the frame bounds. The family $\{W_i\}_{i\in I}$ is called a C-tight frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H, if in 1.5 the constants C and D can be chosen so that $C = D$, a parseval frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H provided $C = D = 1$ and an orthonormal basis of subspaces if $H = \bigoplus_{i\in I} W_i$. If $\{W_i\}_{i\in I}$ possesses an upper frame bound, but not necessarily a lower bound, we call it is a Bessel sequence of subspace with respect to $\{v_i\}_{i\in I}$ for H with Bessel bound D. The representation space employed in this setting is

$$(\sum_{i\in I} \oplus W_i)_{l_2} = \{\{f_i\}_{i\in I}| f_i \in W_i \quad and \quad \{||f_i||\}_{i\in I} \in l^2(I)\}.$$

Let $\{W_i\}_{i\in I}$ be a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H. The synthesis operator, analysis operator and frame operator are defined by

$$T_W : (\sum_{i\in I} \oplus W_i)_{l_2} \to H \quad with \quad T_W(f) = \sum_{i\in I} v_i f_i,$$

$$T_W^* : H \to (\sum_{i\in I} \oplus W_i)_{l_2} \quad with \quad T_W^*(h) = \{v_i \pi_{W_i}(h)\}_{i\in I},$$

$$S_W(h) = T_W T_W^*(h) = \sum_{i\in I} v_i^2 \pi_{W_i}(h).$$

By proposition 3.16 in [13], if $\{W_i\}_{i\in I}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H with frame bounds C and D then S_W is a positive and invertible operator on H with $CId \leq S_W \leq DId$.
Continuous frame or frames associated with measurable space was introduced in [10]. The basic definitions and properties are below:

Definition 1.5. Let (X, μ) be a measure space. Let $f : X \to H$ be weakly measurable (i.e., for all $h \in H$, the mapping $x \to \langle f(x), h\rangle$ is measurable). Then f is called a continuous frame or c-frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X |\langle f(x), h\rangle|^2 d\mu \leq B\|h\|^2. \tag{1.6}$$

The representation space employed in this setting is

$$L^2(X, \mu)$$

$$= \{\varphi : X \to H| \quad \varphi \quad is \quad measurable \quad and \int_X ||\varphi(x)||^2 d\mu < \infty\}.$$

The synthesis operator, analysis operator and frame operator are defined by

$$T_f : L^2(X, \mu) \to H$$

$$\langle T_f \varphi, h \rangle = \int_X \varphi(x) \langle f(x), h \rangle d\mu(x). \tag{1.7}$$

$$T_f^* : H \to L^2(X, \mu)$$

$$(T_f^* h)(x) = \langle h, f(x) \rangle. \tag{1.8}$$

$$S_f : H \to H$$

$$S_f = T_f T_f^*, \tag{1.9}$$

$$\langle S_f(h), k \rangle = \langle T_f T_f^*(h), k \rangle = \int_X \langle f(x), h \rangle \langle k, f(x) \rangle d\mu(x).$$

Also by Theorem 2.5. in [15] S_f is positive, self-adjoint and invertible.

Theorem 1.6. *Let f be a continuous frame for H with a frame operator S_f and let $V : H \to K$ be a bounded and invertible operator. Then $V \circ f$ is a continuous frame for K with the frame operator VS_fV^*.*

Proof. See [15]. □

Definition 1.7. Let f, g be c-frames for H. We say that g is a dual frame for f if for each $h, k \in H$

$$\langle h, k \rangle = \int_X \langle h, f(x) \rangle \langle g(x), k \rangle d\mu(x).$$

By Theorem 1.6 $S_f^{-1} f$ is a c-frame for H with frame operator S_f^{-1}. Also we have

$$\begin{aligned}
\int_X \langle h, f(x) \rangle \langle S_f^{-1} f(x), k \rangle d\mu(x) &= \int_X \langle h, f(x) \rangle \langle f(x), S_f^{-1}(k) \rangle d\mu(x) \\
&= \langle S_f(h), S_f^{-1}(k) \rangle \\
&= \langle h, k \rangle,
\end{aligned}$$

thus $S_f^{-1} f$ is a dual frame for f and called canonical dual frame.

Now we will introduce the continuous version of frames of subspaces and we shall obtain some useful properties as out it.

Definition 1.8. Let $F : X \to \mathbb{H}$ be such that for each $h \in H$, the mapping $x \mapsto \pi_{F(x)}(h)$ is measurable (i.e. is weakly measurable). We say that F is a c-frame of subspaces with respect to v for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X v^2(x) \|\pi_{F(x)}(h)\|^2 d\mu \leq B\|h\|^2. \tag{1.10}$$

F is called a tight c-frame of subspaces with respect to v for H if A, B can be chosen so that $A = B$, and parseval if $A = B = 1$. If we only have the upper bound, we call F is a c-Bessel mapping with respect to v for H.

Definition 1.9. Let $F : X \to \mathbb{H}$. Let $L^2(X,H,F)$ be the class of all measurable mapping $f : X \to H$ such that for each $x \in X$, $f(x) \in F(x)$ and

$$\int_X \|f(x)\|^2 d\mu < \infty.$$

It can be verified that $L^2(X,H,F)$ is a Hilbert space with inner product defined by

$$\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle d\mu$$

for $f, g \in L^2(X,H,F)$.

Remark 1.10. For brevity, we shall denote $L^2(X,H,F)$ by $L^2(X,F)$. Let F be a c-Bessel mapping with respect to v for H, $f \in L^2(X,F)$ and $h \in H$. Then:

$$\begin{aligned}
|\int_X v(x)\langle f(x), h\rangle d\mu| &= |\int_X v(x)\langle \pi_{F(x)}(f(x)), h\rangle d\mu| \\
&= |\int_X v(x)\langle f(x), \pi_{F(x)}(h)\rangle d\mu| \\
&\leq \int_X v(x)\|f(x)\|.\|\pi_{F(x)}(h)\|d\mu \\
&\leq (\int_X \|f(x)\|^2 d\mu)^{1/2}(\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu)^{1/2} \\
&\leq B^{1/2}\|h\|(\int_X \|f(x)\|^2 d\mu)^{1/2}.
\end{aligned}$$

So we may define:

Definition 1.11. Let F be a c-Bessel mapping with respect to v for H. We define the pre-frame operator (synthesis operator) $T_F : L^2(X,F) \to H$, by

$$\langle T_F(f), h\rangle = \int_X v(x)\langle f(x), h\rangle d\mu, \tag{1.11}$$

where $f \in L^2(X,F)$ and $h \in H$.

By the remark 1.10, $T_F : L^2(X,F) \to H$ is a bounded linear mapping. Its adjoint

$$T_F^* : H \to L^2(X,F)$$

will be called analysis operator, and $S_F = T_F \circ T_F^*$ will be called c-frame of subspaces operator. The representation space in this setting is $L^2(X,F)$.

Remark 1.12. Let F be a c-Bessel mapping with respect to v for H. Then $T_F : L^2(X,F) \to H$ is indeed a vector-valued integral, which for $f \in L^2(X,F)$ we shall denote by

$$T_F(f) = \int_X vf d\mu \tag{1.12}$$

where
$$\Big\langle \int_X vf d\mu, h\Big\rangle = \Big\langle \int_X v(x)\big\langle f(x), h\big\rangle d\mu, h \in H.$$
For each $h \in H$ and $f \in L^2(X, F)$, we have
$$\begin{aligned} \big\langle T_F^*(h), f\big\rangle &= \big\langle h, T_F(f)\big\rangle \\ &= \int_X v(x)\big\langle h, f(x)\big\rangle d\mu \\ &= \int_X v(x)\big\langle \pi_{F(x)}(h), f(x)\big\rangle d\mu \\ &= \big\langle v\pi_F(h), f\big\rangle. \end{aligned}$$
Hence for each $h \in H$,
$$T_F^*(h) = v\pi_F(h). \tag{1.13}$$
So $T_F^* = v\pi_F$.

Definition 1.13. Let F and G be c-Bessel mapping with respect to vs for H. We say F and G are weakly equal if $T_F^* = T_G^*$, which is equivalent with $v\pi_F(h) = v\pi_G(h)$, *a.e.* for all $h \in H$. Since, $v \neq 0$ *a.e.*, F and G are weakly equal if $\pi_F(h) = \pi_G(h)$, *a.e.* for all $h \in H$.

Remark 1.14. Let $T_F = 0$. Now, let $O : X \to \mathbb{H}$ be defined by $O(x) = \{0\}$, for almost all $x \in X$. Then O is a c-Bessel mapping with respect to v and $T_O = 0$. Let $h \in H$. Since $v\pi_F(h) \in L^2(X, F)$, so
$$\begin{aligned} \int_X v^2(x)\big\langle \pi_{F(x)}(h), \pi_{F(x)}(h)\big\rangle d\mu &= \int_X v(x)\big\langle v(x)\pi_{F(x)}(h), h\big\rangle d\mu \\ &= \big\langle T_F(v\pi_F(h)), h\big\rangle = 0. \end{aligned}$$
Thus, $\pi_{F(x)}(h) = 0$, *a.e.* Therefore, $\pi_F(h) = \pi_O(h)$, *a.e.* Hence F and O are weakly equal.

2. Main Result

Definition 2.1. Let F be a c-Bessel mapping with respect to v for H, we shall denote
$$A_{F,v} = \inf_{h \in H_1} \|v\pi_F(h)\|^2, \tag{2.1}$$
$$B_{F,v} = \sup_{h \in H_1} \|v\pi_F(h)\|^2 = \|v\pi_F\|^2. \tag{2.2}$$

Remark 2.2. Let F be a c-Bessel mapping with respect to v for H. Since, for each $h \in H$
$$\big\langle T_F T_F^*(h), h\big\rangle = \|v\pi_F(h)\|^2,$$
$A_{F,v}$ and $B_{F,v}$ are optimal scalars which satisfy
$$A_{F,v} \leq T_F T_F^* \leq B_{F,v}.$$
So F is a c-frame of subspaces with respect to v for H if and only if $A_{F,v} > 0$.

Proposition 2.3. *The following conditions are equivalent.*
(i) F is a c-frame of subspaces with respect to v for H with bounds C and D.
(ii) $CId \leq S_F \leq DId$.
Moreover the optimal bounds are $\|S_F\|$ and $\|S_F^{-1}\|^{-1}$.

Proof. $(i) \Rightarrow (ii)$ is obvious. For $(ii) \Rightarrow (i)$, let T_F^* denote the analysis operator of F. Since $S_F = T_F T_F^*$ and then $\|T_F\|^2 = \|S_F\|$, for each $h \in H$, we have

$$\int_X v^2 \|\pi_F(h)\|^2 \, d\mu = \|T_F^*(h)\|^2 \leq \|T_F^*\|^2 \|h\|^2 = \|S_F\| \|h\|^2 \leq D\|h\|^2$$

Also for all $h \in H$,

$$\|T_F^*(h)\|^2 = \langle TT^*(h), h \rangle = \langle S_F h, h \rangle = \langle S^{\frac{1}{2}} h, S^{\frac{1}{2}} h \rangle = \|S^{\frac{1}{2}} h\|^2 \geq C\|h\|^2.$$

Also

$$\|S_F\| = \sup_{h \in H_1} \langle S_F(h), h \rangle = \sup_{h \in H_1} \|v \pi_F(h)\|^2 = B_{F,v}.$$

So the optimal upper bound is $\|S_F\|$. For the optimal lower bound, if C be the lower bound we have

$$C\|h\|^2 \leq \langle S_F^{1/2}(h), S_F^{1/2}(h) \rangle \leq D\|h\|^2,$$

now put $h = S_F^{-1/2}(h)$. We have

$$C\|S_F^{-1/2}(h)\|^2 \leq \langle h, h \rangle \leq D\|S_F^{-1/2}(h)\|^2,$$

thus

$$\|S_F^{-1}\| = \sup_{h \in H_1} \|S_F^{-1/2}(h)\|^2 \leq C^{-1}.$$

We conclude that $A_{F,v} \leq \|S_F^{-1}\|^{-1}$. In other implication we have

$$\|h\| \leq \|S_F^{-1/2}\| \|S_F^{1/2}(h)\|.$$

Hence

$$\inf_{h \in H_1} \|S_F^{1/2}(h)\|^2 \geq \inf_{h \in H_1} \|h\|^2 \|S_F^{-1/2}\|^{-2} = \|S_F^{-1}\|^{-1},$$

we conclude that $A_{F,v} \geq \|S_F^{-1}\|^{-1}$. Finally $A_{F,v} = \|S_F^{-1}\|^{-1}$. ☐

We will need the following results on operators from [11].

Lemma 2.4. *Let V be a closed subspace of H and let T be a bounded and invertible operator on H. Then the following assertions are equivalent:*
(i) $\pi_{\overline{TV}} T = T \pi_V$.
(ii) $T^* T V \subset V$.

Lemma 2.5. *Let V be a closed subspace of H and let T be a bounded and invertible operator on H. Then*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

Lemma 2.6. *Let F be a c-Bessel mapping with respect to v for H. Then F is c-frame of subspaces with respect to v for H if and only if T_F is surjective.*

Proof. Let $A_{F,v} > 0$. Since, for each $h \in H$

$$\begin{aligned} \langle T_F T_F^*(h), h \rangle &= \int_X v^2(x) ||\pi_{F(x)}(h)||^2 d\mu \\ &= ||v\pi_F(h)||^2 \\ &\geq A_{F,v}\|h\|^2. \end{aligned}$$

Therefore, $T_F : L^2(X, F) \to H$ is surjective. Now let T_F is surjective. Let

$$T_F^\dagger : L^2(X, F) \to H$$

be its pseudo-inverse. Since for each $h \in H$

$$\begin{aligned} \|h\| &= \|T_F^{\dagger *} T_F^*(h)\| \\ &\leq \|T_F^{\dagger *}\| \|T_F^*(h)\| \\ &= \|T_F^{\dagger *}\| \|v\pi_F(h)\|, \end{aligned}$$

so $A_{F,v} \geq \|T_F^{\dagger *}\|^{-2} > 0$. □

Lemma 2.7. *Let F be a c-Bessel mapping with respect to v for H. Then the frame operator $S_F = T_F T_F^*$ is invertible if and only if F is a c-frame of subspaces with respect to v for H.*

Proof. Let $S_F = T_F T_F^*$ be invertible. We have

$$A_{F,v} \leq \inf_{h \in H} \|T_F^*\| = \inf_{h \in H} \langle T_F T_F^*(h), h \rangle \in \sigma(T_F T_F^*),$$

so $A_{F,v} > 0$. Now let F be a c-frame of subspaces with respect to v for H. So by the Lemma 2.6, T_F is surjective. Thus there exist $A > 0$ such that for all $h \in H$

$$A\|h\| \leq \|T_F^*(h)\|.$$

Hence for all $h \in H$

$$\langle S_F(h), h \rangle = \|T_F^*(h)\|^2 \geq A^2 \|h\|^2.$$

Thus by Proposition 3.2.12 in [14], S_F is invertible. □

Proposition 2.8. *Let F be a c-frame of subspaces with respect to v for H, with c-frame of subspaces operator S_F and let T be a bounded and invertible operator on H. Then TF is a c-frame of subspaces with respect to v for H with c-frame of subspaces operator S_{TF} satisfying*

$$\frac{TS_F T^*}{\|T\|^2} \leq S_{TF} \leq \|T^{-1}\|^2 T S_F T^*.$$

Proof. By employing Lemma 2.5, for each $h \in H$ we have,

$$\begin{aligned}
\Big\langle \frac{TS_FT^*}{\|T\|^2}h, h\Big\rangle &= \frac{1}{\|T\|^2}\big\langle S_FT^*h, T^*h\big\rangle \\
&= \frac{1}{\|T\|^2}\int_X v^2\|\pi_{F(x)}(T^*h)\|^2\,d\mu \\
&= \frac{1}{\|T\|^2}\int_X v^2\|\pi_{F(x)}T^*\pi_{TF(x)}(h)\|^2\,d\mu \\
&\leq \frac{\|T^*\|^2}{\|T\|^2}\int_X v^2\|\pi_{TF(x)}(h)\|^2\,d\mu \\
&= \big\langle S_{TF}h, h\big\rangle.
\end{aligned}$$

Now for the upper bound, by applying Lemma 2.5 to $TF(x)$ and T^{-1} we have, $\pi_{TF(x)} = \pi_{TF(x)}{T^*}^{-1}\pi_{F(x)}T^*$. Thus

$$\begin{aligned}
\big\langle S_{TF}h, h\big\rangle &= \int_X v^2(x)\|\pi_{TF(x)}(h)\|^2 d\mu \\
&= \int_X v^2(x)\|\pi_{TF(x)}{T^*}^{-1}\pi_{F(x)}T^*(h)\|^2 d\mu \\
&\leq \|T^{-1}\|^2\int_X v^2(x)\|\pi_{F(x)}T^*(h)\|^2 d\mu \\
&= \|T^{-1}\|^2\big\langle S_F(T^*h), (T^*h)\big\rangle \\
&= \|T^{-1}\|^2\big\langle TS_FT^*(h), h\big\rangle,
\end{aligned}$$

and this completes the proof, by Proposition 2.3. $\square$

By imposing some extra conditions on Proposition 2.8, we determine S_{TF} precisely as follow.

Proposition 2.9. *Let F be a c-frame of subspaces with respect to v for H, with c-frame of subspaces operator S_F and let T be a bounded, self-adjoint and invertible operator, on H satisfying*

$$T^*T(F(x)) \subseteq F(x), \tag{2.3}$$

for all $x \in X$. Then TF is a c-frame of subspaces with respect to v for H with c-frame of subspaces operator TS_FT^{-1}.

Proof. Since T is invertible and bounded, then TF is a c-frame of subspaces with respect to v for H. Now for determining the related c-frame of subspaces operator precisely, we apply the Lemma 2.4

$$\begin{aligned}\langle S_{TF}h, k\rangle &= \int_X v^2(x)\langle \pi_{TF(x)}(h), k\rangle d\mu \\ &= \int_X v^2(x)\langle T\pi_{F(x)}(T^{-1}h), k\rangle d\mu \\ &= \int_X v^2(x)\langle \pi_{F(x)}(T^{-1}h), Tk\rangle d\mu \\ &= \langle \int_X v^2\pi_F(T^{-1}h)d\mu, Tk\rangle \\ &= \langle T\int_X v^2\pi_F(T^{-1}h)d\mu, k\rangle \\ &= \langle TS_FT^{-1}h, k\rangle,\end{aligned}$$

for all $h, k \in H$. Now the proof follows by proposition 2.3. □

Now we shall have some reconstruction formula for a signal by employing local c- frames properties and then combining them globally with a focus on c- frame of subspace property of a system of local c-frames. Also we will show some other relations between these local c-frames of a system and the system itself, with a frame of subspaces structure.

Definition 2.10. Let (X, μ) and (Y, λ) be two measure spaces. Let $f : X \times Y \to H$ and $F : X \to \mathbb{H}$. Let for each $x \in X$, $f(x, .) : Y \to F(x)$ be a c-frame for $F(x)$. Then (F, f) is called a system of local c-frames. Also, the system of local c-frames (F, f) is called a system of c-frame of subspaces with respect to v for H if F is a c-frame of subspaces with respect to v for H.

Theorem 2.11. *Let (X, μ) and (Y, λ) be two σ-finite measure space. Let $f : X \times Y \to H$, $F : X \to \mathbb{H}$ be weakly measurable mappings. Let (F, f) be the system of local c-frames. Let*

$$0 < A(x) = \inf_{x \in F(x)_1} \int_Y |\langle f(x, y), h\rangle|^2 d\lambda$$

$$\leq \sup_{x \in F(x)_1} \int_Y |\langle f(x, y), h\rangle|^2 d\lambda = B(x) < \infty,$$

and let

$$0 < A = \inf_x A(x) \leq \sup_x B(x) = B < \infty.$$

Then, (F, f) is a system of c-frame of subspaces with respect to v for H if and only if

$$v.f : X \times Y \to H,$$

$$(x, y) \mapsto v(x)f(x, y)$$

is a continuous frame for H.

Proof. For each $h \in H$ we have

$$\begin{aligned}
A\|v\pi_{F(x)}(h)\|^2 &= A\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \\
&\leq \int_X A(x)v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \\
&\leq \int_X\int_Y |\langle v(x)\pi_{F(x)}(h), f(x,y)\rangle|^2 d\lambda d\mu \\
&= \int_X\int_Y |\langle h, v(x)f(x,y)\rangle|^2 d\lambda d\mu \\
&= \int_{X\times Y} |\langle v(x)f(x,y), h\rangle|^2 d(\mu\times\lambda) \\
&= \int_X\int_Y |\langle \pi_{F(x)}(h), v(x)f(x,y)\rangle|^2 d\lambda d\mu \\
&\leq \int_X B(x)v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \\
&\leq B\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu,
\end{aligned}$$

and the Theorem is proved. □

Proposition 2.12. *Let (X,μ) and (Y,λ) be two σ-finite measure spaces, $f : X\times Y \to H$ and $F : X \to \mathbb{H}$ be weakly measurable mappings. Let (F,f) be a system of c-frame of subspaces with c-frame of subspaces operator S_F. Let $\tilde{f}(x,.) : Y \to F(x)$ be a dual frame of $f(x,.) : Y \to F(x)$ for all $x \in X$, such that $\tilde{f} : X\times Y \to H$ be weakly measurable. Then for each $h \in H$, we have:*
(i)Reconstruction formula

$$h = \int_X v^2 S_F^{-1}\pi_F(h)\, d\mu.$$

(ii)For each $h,k \in H$ we have

$$\langle h,k\rangle = \int_{X\times Y} \langle h, vf\rangle\langle S_F^{-1}v\tilde{f}, k\rangle d(\mu\times\lambda),$$

so $S_F^{-1}v\tilde{f} : X\times Y \to H$ is a dual frame for c-frame $v.f : X\times Y \to H$.

Proof. (i) Since S_F is invertible and bounded, for all $h \in H$ by applying remark 1.12 we have

$$\begin{aligned}
\langle h, k\rangle &= \langle S_F^{-1} S_F h, k\rangle \\
&= \Big\langle \int_X v^2 \pi_F(h)\, d\mu, S_F^{-1} k\Big\rangle \\
&= \int_X v^2(x) \langle \pi_{F(x)}(h), S_F^{-1} k\rangle d\mu \\
&= \int_X v^2(x) \langle S_F^{-1} \pi_{F(x)}(h), k\rangle d\mu \\
&= \Big\langle \int_X v^2 S_F^{-1} \pi_F(h)\, d\mu, k\Big\rangle.
\end{aligned}$$

(ii) By (i) we have

$$\begin{aligned}
\langle h, k\rangle &= \int_X v^2(x) \langle S_F^{-1} \pi_{F(x)}(h), k\rangle d\mu \\
&= \int_X v^2(x) \langle \pi_{F(x)}(h), S_F^{-1} k\rangle d\mu \\
&= \int_X v^2(x) \int_Y \langle \pi_F(h), f(x,y)\rangle \langle \tilde{f}(x,y), S_F^{-1} k\rangle d\lambda d\mu \\
&= \int_X \int_Y \langle h, v(x) f(x,y)\rangle \langle S_F^{-1} v(x) \tilde{f}(x,y), k\rangle d\lambda d\mu \\
&= \int_{X\times Y} \langle h, vf\rangle \langle S_F^{-1} v\tilde{f}, k\rangle d(\mu \times \lambda).
\end{aligned}$$

□

By theorem 2.11, $v.f$ is a c-frame for H with associated c-frame operator S_f, then we can use the (global) canonical dual frame $S_f^{-1} v.f$ to perform centralized reconstruction,

$$\langle h, k\rangle = \int_{X\times Y} \langle h, v.f\rangle \langle S_f^{-1} v.\tilde{f}, k\rangle \, d(\mu \times \lambda).$$

The c-frame of subspaces operators can be expressed in terms of local c-frame operators, as follows:

We denote the synthesis and analysis operator for $f(x,.)$ by T_x and T_x^*, respectively.

Proposition 2.13. *Let $T_{\tilde{x}}^*, T_x$ be the associated analysis operators of $\tilde{f}(x,.) : Y \to F(x)$ and synthesis operators of $f(x,.) : Y \to F(x)$. Let $x \mapsto T_x T_{\tilde{x}}^*$ from X to $B(H,H)$ is measurable. With the hypothesis of Proposition 2.12 the c-frame of subspaces operator can be written as*

$$S_F(h) = \int_X v^2 T_x T_{\tilde{x}}^*(h)\, d\mu.$$

Proof. For each $h, k \in H$

$$\begin{aligned}
\langle S_F(h), k\rangle &= \int_X v^2(x)\langle \pi_{F(x)}(h), k\rangle \, d\mu \\
&= \int_X v^2(x)\langle \pi_{F(x)}(h), \pi_{F(x)}k\rangle d\mu \\
&= \int_X v^2(x)\int_Y \langle h, \tilde{f}(x,y)\rangle\langle f(x,y), k\rangle d\lambda d\mu \\
&= \int_X v^2(x)\int_Y T^*_{\tilde{x}}(h)(y)\langle f(x,y), k\rangle d\lambda d\mu \\
&= \int_X v^2(x)\langle T_x T^*_{\tilde{x}}(h), k\rangle d\mu \\
&= \langle \int_X v^2 T_x T^*_{\tilde{x}}(h) d\mu, k\rangle.
\end{aligned}$$

□

Similarly, we have

$$S_F(h) = \int_X v T_{\tilde{x}} T^*_x(h)\, d\mu.$$

Theorem 2.14. *With the hypothesis of proposition 2.12 if for each $x \in X$, $f(x,.) : Y \to F(x)$, is a parseval c-frame for $F(x)$. Then F is a parseval c-frame of subspaces.*

Proof. Since $f(x,.) : Y \to F(x)$ are Pareval c-frames for all $x \in X$, then $S_{f(x,.)} = I$. We have

$$\begin{aligned}
\langle S_F h, k\rangle &= \int_X v^2(x)\langle \pi_F(h), k\rangle d\mu \\
&= \int_X v^2(x)\int_Y \langle \pi_F(h), f(x,y)\rangle\langle f(x,y), k\rangle d\lambda d\mu \\
&= \int_{X\times Y} \langle h, v(x)f(x,y)\rangle\langle v(x)f(x,y), k\rangle d(\mu\times\lambda) \\
&= \int_{X\times Y} |\langle h, v(x)f(x,y)\rangle|^2 d(\mu\times\lambda) \\
&= \langle h, h\rangle.
\end{aligned}$$

Hence $S_F = I$. This complete the proof. □

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