

MULTIPLE POSITIVE SOLUTIONS FOR NONLINEAR
SINGULAR THIRD-ORDER BOUNDARY VALUE PROBLEM IN
ABSTRACT SPACES

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ABSTRACT. In this paper, we study the nonlinear singular boundary value problem in abstract spaces:

$$\begin{cases} u''' + f(t, u) = \theta, & t \in (0, 1), \\ u(0) = u'(0) = \theta, & u'(1) = \xi u'(\eta), \end{cases}$$

where $0 < \eta < 1$ and $1 < \xi < \frac{1}{\eta}$, θ denotes the zero element of E , E is a real Banach space, and $f(t, u)$ is allowed to be singular at both end point $t = 0$ and $t = 1$. We show the existence of at least two positive solutions of this problem.

1. INTRODUCTION

In this paper, we consider the following singular boundary value problem (BVP) for third-order differential equations in a Banach space E :

$$u'''(t) + f(t, u(t)) = \theta, \quad 0 < t < 1 \quad (1.1)$$

subject to the boundary conditions

$$u(0) = u'(0) = \theta, \quad u'(1) = \xi u'(\eta), \quad (1.2)$$

where $0 < \eta < 1$ and $1 < \xi < \frac{1}{\eta}$, $f \in C[(0, 1/2) \times P, P]$ which may be singular at $t = 0, t = 1$; P is a cone of Banach space E , which will be stated in detail in section 2; θ is the zero element of E .

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Boundary value problems arise from applied mathematics and physics, and they have received a great deal of attention in the literature. Problems of the form (1.1) subject to (1.2), for example, are used to model such phenomena as the deflection of a curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1]. Third-order boundary value problems have been studied widely in the literature (see [1]–[12] and references therein). However, all of the above-mentioned references consider (1.1) only in scalar space. On the other hand, the theory of ordinary differential equations (ODE) in abstract spaces is becoming an important branch of mathematics in last thirty years because of its application in partial differential equations and ODE's in appropriately infinite dimensional spaces (see, for example [13], [14]). As a result the goal of this paper is to fill up the gap in this area, that is, to investigate the existence of multiple positive solutions of (1.1) with (1.2) in a Banach space E .

This paper is organized as follows. Section 2 gives some preliminaries and some lemmas. Section 3 is devoted to the main results and the proof of the results; an example is worked out to indicate the application of our main results.

2. PRELIMINARIES AND LEMMAS

In this paper, we suppose throughout that E is a real Banach space. A nonempty closed convex subset P in E is said to be a cone which defined a partial ordering in E by $x \leq y$ if and only if $y - x \in P$, P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E , and the smallest N is called the normal constant of P (it is clear, $N \geq 1$). For details on cone theory, see [15]. Let S be a bounded subset of a Banach space. $\alpha(S)$ denotes the Kuratowski's measure of noncompactness of S . In this paper, $\alpha(\cdot)$ denotes the Kuratowski's measure of noncompactness of a bounded subset of both E and $C[[0, 1], E]$. Let

$$C[[0, 1], E] = \{u : [0, 1] \rightarrow E \mid u(t) \text{ is continuous on } [0, 1]\},$$

$$C^2[[0, 1], E] = \{u : [0, 1] \rightarrow E \mid u(t) \text{ is second order continuously differentiable in } [0, 1]\},$$

$$C^3((0, 1), E) = \{u : (0, 1) \rightarrow E \mid u(t) \text{ is third order continuously differentiable in } (0, 1)\}.$$

For $u = u(t) \in C[[0, 1], E]$, let $\|u\|_C = \max_{0 \leq t \leq 1} \|u(t)\|$, then $C[[0, 1], E]$ becomes a Banach space. Let $Q = \{u \in C[[0, 1], E] \mid u(t) \geq \theta, t \in [0, 1]\}$, then Q is a cone in $C[[0, 1], E]$. An operator $u(t) \in C[[0, 1], E] \cap C^3((0, 1), E)$ is called a positive solution of the BVP (1.1)–(1.2) if $u(t)$ satisfies (1.1)–(1.2) and $u \in Q$, $u(t) \neq \theta$, $t \in [0, 1]$.

Lemma 2.1. [12] *Let $\xi\eta \neq 1$. Then for $y \in C[[0, 1], E]$, the BVP:*

$$\begin{aligned} u''' + y(t) &= \theta, \quad t \in (0, 1), \\ u(0) = u'(0) &= \theta, \quad u'(1) = \xi u'(\eta), \end{aligned}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

where

$$G(t, s) = \frac{1}{2(1 - \xi\eta)} \begin{cases} (2ts - s^2)(1 - \xi\eta) + t^2s(\xi - 1), & s \leq \min\{\eta, t\}, \\ t^2(1 - \xi\eta) + t^2s(\xi - 1), & t \leq s \leq \eta, \\ (2ts - s^2)(1 - \xi\eta) + t^2(\xi\eta - s), & \eta \leq s \leq t, \\ t^2(1 - s), & \max\{\eta, t\} \leq s \end{cases} \quad (2.1)$$

is called the Green's function.

Lemma 2.2. [12] *Let $0 < \eta < 1$ and $1 < \xi < \frac{1}{\eta}$. Then for $G(t, s)$ defined in (2.1), we have estimates*

(i) *for any $(t, s) \in [0, 1] \times [0, 1]$, $0 \leq G(t, s) \leq \Phi(s)$, where*

$$\Phi(s) = \frac{1 + \xi}{1 - \xi\eta} s(1 - s).$$

(ii) *for any $(t, s) \in [\frac{\eta}{\xi}, \eta] \times [0, 1]$, $G(t, s) \geq \gamma\Phi(s)$, where*

$$0 < \gamma = \frac{\eta^2}{2\xi^2(1 + \xi)} \min\{\xi - 1, 1\} < 1.$$

Let us list some conditions.

(H1) $f(t, u) \leq g(t)h(u)$, $t \in (0, 1)$, $u \in P$, where $g : (0, 1) \rightarrow [0, \infty)$ is continuous, and $h : P \rightarrow P$ is continuous and maps bounded subsets to bounded subsets.

(H2) $0 < \int_0^1 \Phi(s)g(s)ds < \infty$.

(H3) There exists a $\varphi \in P^*$ such that $u > \theta$ implies that $\varphi(u) > 0$ and

$$m_0 < \lim_{\substack{\|u\| \rightarrow 0 \\ u \in P}} \inf_{t \in [\frac{\eta}{\xi}, \eta]} \frac{\varphi(f(t, u))}{\varphi(u)} \leq \infty,$$

where $P^* = \{\varphi \in E^* | \varphi(u) \geq 0, u \in P\}$ denotes the dual cone of P , and $m_0 = (\min_{t \in [\frac{\eta}{\xi}, \eta]} \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)ds)^{-1}$.

(H4) There exists a $\varphi_1 \in P^*$ such that $u > \theta$ implies that $\varphi_1(u) > 0$ and

$$m_0 < \lim_{\substack{\|u\| \rightarrow \infty \\ u \in P}} \inf_{t \in [\frac{\eta}{\xi}, \eta]} \frac{\varphi_1(f(t, u))}{\varphi_1(u)} \leq \infty,$$

where P^* and m_0 are the same as in (H3).

(H5) There exists a $\eta_0 > 0$ such that $u \in P$, $\|u\| \leq \eta_0$ implies that

$$\|h(u)\| < M_0\eta_0,$$

where $M_0 = (N \int_0^1 \Phi(s)g(s)ds)^{-1}$, and N denotes the normal constant of P .

(H6) $0 \leq \overline{\lim}_{\substack{\|u\| \rightarrow 0 \\ u \in P}} \frac{\|h(u)\|}{\|u\|} < M_0$, where M_0 is as in (H5).

(H7) $0 \leq \overline{\lim}_{\substack{\|u\| \rightarrow \infty \\ u \in P}} \frac{\|h(u)\|}{\|u\|} < M_0$, where M_0 is as in (H5).

(H8) There exists a $\varphi_2 \in P^*$ such that $u > \theta$ implies that $\varphi_2(u) > 0$. Then exists a $\eta_1 > 0$ such that $t \in [\frac{\eta}{\xi}, \eta]$, $u \in P$ and $\frac{\gamma}{N}\eta_1 \leq \|u\| \leq \eta_1$ imply

$$\varphi_2(f(t, u)) > m_0\varphi_2(u),$$

where P^* and m_0 are the same as in (H3).

(H9) For $l > 0$, $[a, b] \subset (0, 1)$, f is uniformly continuous on $[a, b] \times (P \cap T_l)$, where $T_l = \{u \in E \mid \|u\| \leq l\}$, and there exists a constant $L : 0 \leq L < \frac{1}{2M_1}$ such that for $t \in (0, 1)$ and bounded subsets $D \subset P$,

$$\alpha(f(t, D)) \leq L\alpha(D) \quad (2.2)$$

holds, where $M_1 = \max_{s \in [0, 1]} \Phi(s)$.

Now we define

$$(Au)(t) = \int_0^1 G(t, s)f(s, u(s))ds, \quad \forall u \in Q, \quad (2.3)$$

then $(Au)(t) \geq \theta$, $t \in [0, 1]$, and using the Lebesgue dominated convergence theorem we know that $(Au)(t)$ is continuous on $[0, 1]$, hence the integral operator $A : Q \rightarrow Q$. Further, we can easily show that

- (i) If $u \in Q$, then $(Au)'''(t) = -f(t, u)$, $t \in (0, 1)$, hence $Au \in Q \cap C^3[(0, 1), E]$.
- (ii) If $u \in Q$ satisfies $Au = u$, then u is a solution of the BVP (1.1)–(1.2).

Therefore, the BVP (1.1)–(1.2) is equivalent to the operator equation $Au = u$, $u \in Q$.

Lemma 2.3. [15], [16] *Let D be a bounded set of E and the map $f : [a, b] \times D \rightarrow E$ be bounded. Assume that $f(t, u)$ is uniformly continuous with respect to t . Then, we have*

$$\alpha(f([a, b] \times S)) = \max_{t \in [a, b]} \alpha(f(t, S)), \quad S \subset D.$$

Lemma 2.4. [15], [16] *If $H \subset C[[a, b], E]$ is bounded and equicontinuous, then*

$$\alpha(H([a, b])) = \max_{t \in [a, b]} \alpha(H(t)),$$

where $H([a, b]) = \{u(t) \mid u \in H, t \in [a, b]\}$.

Lemma 2.5. *Assume that (H1), (H2), (H9) hold. Then, operator $A : Q \cap T_l \rightarrow Q$ is a strict set contraction operator.*

Proof. It is clear that A is a bounded and continuous operator from $Q \cap T_l$ into Q . Since $f(t, u)$ is bounded and uniformly continuous in t on $[a, b] \times T_l$ for any $l > 0$, it follows from Lemma 2.3 and (2.2) that

$$\alpha(f([a, b] \times D)) = \max_{t \in [a, b]} \alpha(f(t, D)) \leq L\alpha(D), \quad D \subset P \cap T_l. \quad (2.4)$$

Let $S \subset C[[a, b], E]$ be bounded. We know that $A(S) \subset C[[a, b], E]$ is bounded and equicontinuous, so, by Lemma 2.4,

$$\alpha(A(S)) = \max_{t \in [a, b]} \alpha(A(S(t))), \quad (2.5)$$

where $A(S(t)) = \{(Au)(t)|u \in S\} \subset E$ (t is fixed). Using the formula

$$\int_0^1 y(t)dt \in \overline{co}\{y(t)|t \in [0, 1]\} \text{ for } y \in C[[0, 1], E],$$

and observing Lemma 2.2 and (2.4), we find

$$\begin{aligned} \alpha(A(S(t))) &= \alpha(\{\int_0^1 G(t, s)f(s, u(s))ds|u \in S\}) \\ &\leq \alpha(\overline{co}\{G(t, s)f(s, u(s))|s \in [0, 1], u \in S\}) \\ &\leq M_1\alpha(\{f(s, u(s))|s \in [0, 1], u \in S\}) \\ &\leq M_1\alpha(f([0, 1] \times B) \leq M_1L\alpha(B), t \in [a, b], \end{aligned} \quad (2.6)$$

where $B = \{u(s)|s \in [0, 1], u \in S\}$. For any given $\varepsilon > 0$, there is a partition $S = \bigcup_{j=1}^n S_j$ such that

$$\text{diam}(S_j) < \alpha(S) + \frac{\varepsilon}{3}, \quad j = 1, 2, \dots, n. \quad (2.7)$$

Choosing $u_j \in S_j (j = 1, s, \dots, n)$ and a partition of $J : a = t_0 < t_1 < \dots < t_m = b$ such that

$$\|u_j(t) - u_j(s)\| < \frac{\varepsilon}{3}, \quad \forall j = 1, s, \dots, n; t, s \in [t_{i-1}, t_i], i = 1, 2, \dots, m. \quad (2.8)$$

Obviously, $B = \bigcup_{i=1}^m \bigcup_{j=1}^n B_{ij}$, where $B_{ij} = \{u(s)|s \in [t_{i-1}, t_i], u \in S_j\}$. For any $u(t), \bar{u}(\bar{t}) \in B_{ij}(t, \bar{t} \in [t_{i-1}, t_i], u, \bar{u} \in S_j)$. It follows from (2.7), (2.8) that

$$\begin{aligned} \|u(t) - \bar{u}(\bar{t})\| &\leq \|u(t) - u_j(t)\| + \|u_j(t) - u_j(\bar{t})\| + \|u_j(\bar{t}) - \bar{u}(\bar{t})\| \\ &\leq \|u - u_j\|_C + \frac{\varepsilon}{3} + \|u_j - \bar{u}\|_C \leq 2\text{diam}(S_j) + \frac{\varepsilon}{3} < 2\alpha(S) + \varepsilon. \end{aligned}$$

Consequently,

$$\text{diam}(B_{ij}) \leq 2\alpha(S) + \varepsilon, \quad \forall i = 1, 2, \dots, m, j = 1, s, \dots, n,$$

and so $\alpha(B) \leq 2\alpha(S) + \varepsilon$, which implies, since ε is arbitrary,

$$\alpha(B) \leq 2\alpha(S). \quad (2.9)$$

It follows then from (2.5), (2.6),(2.9) that $\alpha(A(S)) \leq 2M_1L\alpha(S)$, $S \subset Q \cap T_l$ with $2M_1L < 1$, and the Lemma 2.5 is proved. \square

We will apply the following fixed point theorem to obtain solutions of the BVP (1.1)–(1.2).

Lemma 2.6. [16] *Let E be a Banach space, $K \subset E$ be a cone in E , $R > r > 0$, $K(r, R) = \{u \in K|r \leq \|u\| \leq R\}$, and let $A : K(r, R) \rightarrow K$ be a strict set contract operator such that either*

(i) $Au \not\leq u, \forall u \in K, \|u\| = r$ and $Au \not\leq u, \forall u \in K, \|u\| = R$; or

(ii) $Au \not\leq u, \forall u \in K, \|u\| = r$ and $Au \not\geq u, \forall u \in K, \|u\| = R$.

Then A has at least a fixed point in $K(r, R)$.

3. MAIN RESULTS

Theorem 3.1. *Let P be a normal cone in E . Suppose that the conditions (H1)–(H5) and (H9) are satisfied. Then the singular boundary value problem (1.1)–(1.2) has at least two positive solutions.*

Proof. Let

$$K = \{u \in Q \mid u(t) \geq \gamma u(s), t \in [\frac{\eta}{\xi}, \eta], s \in [0, 1]\},$$

then K is a cone in $C[[0, 1], E]$ and satisfies $K \subset Q$.

We show that $A(K) \subset K$. In fact, for $u \in Q$, $t \in [\frac{\eta}{\xi}, \eta]$, by Lemma 2.2 we get

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t, s)f(s, u(s))ds \geq \gamma \int_0^1 \Phi(s)f(s, u(s))ds \\ &= \gamma(Au)(x), \forall x \in [0, 1]. \end{aligned}$$

This shows that $Au \in K$, hence $A(Q) \subset K$ and naturally $A(K) \subset K$.

By (H3) there exist a $r_0 \in (0, \eta_0)$ and $\tau_1 > 0$ such that $u \in P$, $\|u\| \leq r_0$ implies

$$\varphi(f(t, u)) \geq (m_0 + \tau_1)\varphi(u), \quad t \in [\frac{\eta}{\xi}, \eta], \quad (3.1)$$

where η_0 is as in (H5). We show that

$$Au \not\leq u, \quad \forall u \in K, \quad \|u\|_C = r_0. \quad (3.2)$$

In fact, if there exists $u_0 \in K$, $\|u_0\|_c = r_0$ such that $Au_0 \leq u_0$, then, by the definition of K and the normality of P we have that

$$N\|u_0(t)\| \geq \gamma\|u_0(s)\|, \quad t \in [\frac{\eta}{\xi}, \eta], \quad s \in [0, 1],$$

hence $\|u_0(t)\| \geq \frac{\gamma}{N}\|u_0\|_C = \frac{\gamma}{N}r_0$, $t \in [\frac{\eta}{\xi}, \eta]$. Taking

$$\lambda = \min\{\varphi(u_0(t)) \mid t \in [\frac{\eta}{\xi}, \eta]\},$$

then $\lambda > 0$. On the other hand, for $t \in [\frac{\eta}{\xi}, \eta]$, we have that

$$u_0(t) \geq (Au_0)(t) \geq \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)f(s, u_0(s))ds,$$

so, by (3.1) we get, for $t \in [\frac{\eta}{\xi}, \eta]$,

$$\begin{aligned} \varphi(u_0(t)) &\geq \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)\varphi(f(s, u_0(s)))ds \geq (m_0 + \tau_1) \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)\varphi(u_0(s))ds \\ &\geq (m_0 + \tau_1)\lambda \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)ds \geq \frac{m_0 + \tau_1}{m_0}\lambda > \lambda. \end{aligned}$$

This is a contradiction to the definition of λ . Therefore, (3.2) holds.

By (H4) there exist $R_0 > 0$ and $\tau_2 > 0$ such that $u \in P$, $\|u\| \geq R_0$ implies

$$\varphi_1(f(t, u)) \geq (m_0 + \tau_2)\varphi_1(u), \quad t \in [\frac{\eta}{\xi}, \eta]. \quad (3.3)$$

Take $R > \max\{\frac{NR_0}{\gamma}, \eta_0\}$, where η_0 is as in (H5), then $R > \eta_0 > r_0$. We show that

$$Au \not\leq u, \quad \forall u \in K, \quad \|u\|_C = R. \quad (3.4)$$

In fact, if there exists $u_0 \in K$, $\|u_0\|_C = R$ such that $Au_0 \leq u_0$, then, by the definition of K and the normality of P we have that

$$N\|u_0(t)\| \geq \gamma\|u_0(s)\|, \quad t \in [\frac{\eta}{\xi}, \eta], \quad s \in [0, 1].$$

Hence $\|u_0(t)\| \geq \frac{\gamma}{N}\|u_0\|_C = \frac{\gamma}{N}R > R_0$, $t \in [\frac{\eta}{\xi}, \eta]$. Taking

$$\lambda_1 = \min\{\varphi_1(u_0(t)) | t \in [\frac{\eta}{\xi}, \eta]\},$$

then $\lambda_1 > 0$. On the other hand, for $t \in [\frac{\eta}{\xi}, \eta]$, we have that

$$u_0(t) \geq (Au_0)(t) \geq \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)f(s, u_0(s))ds,$$

so, by (3.3) we get, for $t \in [\frac{\eta}{\xi}, \eta]$,

$$\begin{aligned} \varphi_1(u_0(t)) &\geq \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)\varphi_1(f(s, u_0(s)))ds \geq (m_0 + \tau_2) \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)\varphi_1(u_0(s))ds \\ &\geq (m_0 + \tau_2)\lambda_1 \int_{\frac{\eta}{\xi}}^{\eta} G(t, s)ds \geq \frac{m_0 + \tau_2}{m_0}\lambda_1 > \lambda_1. \end{aligned}$$

This is a contradiction to the definition of λ_1 . Therefore, (3.4) holds.

Now, we show that

$$Au \not\geq u, \quad \forall u \in Q, \quad \|u\|_C = \eta_0. \quad (3.5)$$

In fact, if there exists $u_0 \in Q$, $\|u_0\|_C = \eta_0$ such that $Au_0 \geq u_0$, then, by Lemma 2.2 and (H1) we know that

$$\begin{aligned} u_0(t) &\leq (Au_0)(t) = \int_0^1 G(t, s)f(s, u_0(s))ds \\ &\leq \int_0^1 \Phi(s)g(s)h(u_0(s))ds, \quad t \in [0, 1]. \end{aligned}$$

Hence, it follows from the normality of P and (H5) that

$$\begin{aligned} \|u_0(t)\| &\leq N \int_0^1 \Phi(s)g(s)\|h(u_0(s))\|ds \\ &\leq NM_0\eta_0 \int_0^1 \Phi(s)g(s)ds, \quad t \in [0, 1]. \end{aligned}$$

Therefore, $\eta_0 = \|u_0\|_C < \eta_0 M_0 N \int_0^1 \Phi(s)g(s)ds = \eta_0$, which is a contradiction. Thus (3.5) holds.

By (3.2), (3.5), Lemmas 2.5 and 2.6 we know that the operator A has at least one fixed point u_1 in $K(r_0, \eta_0) = \{u \in K | r_0 \leq \|u\|_C \leq \eta_0\}$. Similarly, by (3.5), (3.4), Lemmas 2.5 and 2.6 we know that A has at least one fixed point u_2 in $K(\eta_0, R) = \{u \in K | \eta_0 \leq \|u\|_C \leq R\}$. Thus, the BVP (1.1)–(1.2) has at least two

positive solutions $u_1(t)$ and $u_2(t)$ which satisfy $0 < \|u_1\|_C < \|u_2\|_C$. Theorem 3.1 is proved. \square

Theorem 3.2. *Let P be a normal cone in E . Suppose that the conditions (H1), (H2) and (H6) – (H9) are satisfied. Then the singular boundary value problem (1.1)–(1.2) has at least two positive solutions.*

Proof. The proof is similar to Theorem 3.1. \square

Example 3.3. Consider the following one dimensional singular boundary value problem

$$\begin{cases} u'''(t) + \frac{u^2}{2(1-t)^{\frac{3}{2}}} + \frac{\sqrt{u}}{16t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}} = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, \quad u'(1) = 2u'(\frac{1}{4}). \end{cases} \quad (3.6)$$

Conclusion. The BVP (3.6) possesses at least two positive solutions $u_1(t)$ and $u_2(t)$ which satisfy $u_i(t) > 0$, $t \in (0, 1)$, $i = 1, 2$.

Proof. Let $E = \mathbb{R}^1$, $P = [0, \infty)$, then P is a normal cone in E . Hence

$$f(t, u) = \frac{u^2}{2(1-t)^{\frac{3}{2}}} + \frac{\sqrt{u}}{16t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}},$$

$f : (0, 1) \times P \rightarrow P$ is continuous, the Green's function satisfies $G(t, s) \leq \Phi(s) = 6s(1-s)$, $(t, s) \in [0, 1] \times [0, 1]$.

It is clear that $f(t, u) \leq g(t)h(u)$, where $g(t) = \frac{1}{t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}}$, $h(u) = \frac{u^2}{2} + \frac{\sqrt{u}}{16}$, and

$$\int_0^1 \Phi(s)g(s)ds = \int_0^1 \frac{6}{\sqrt{s(1-s)}}ds = 6\pi.$$

Hence, (H1) and (H2) are satisfied. Taking $\eta = \frac{1}{4}$, $\xi = 2$, and $\varphi \in P^*$, $\varphi(u) = u$, we have that

$$\lim_{u \rightarrow 0^+} \inf_{t \in [\frac{1}{8}, \frac{1}{4}]} \frac{\varphi(f(t, u))}{\varphi(u)} = \lim_{u \rightarrow 0^+} \inf_{t \in [\frac{1}{8}, \frac{1}{4}]} \left(\frac{u}{2(1-t)^{\frac{3}{2}}} + \frac{1}{16t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}\sqrt{u}} \right) = \infty,$$

and

$$\lim_{u \rightarrow \infty} \inf_{t \in [\frac{1}{8}, \frac{1}{4}]} \frac{\varphi(f(t, u))}{\varphi(u)} = \lim_{u \rightarrow \infty} \inf_{t \in [\frac{1}{8}, \frac{1}{4}]} \left(\frac{u}{2(1-t)^{\frac{3}{2}}} + \frac{1}{16t^{\frac{3}{2}}(1-t)^{\frac{3}{2}}\sqrt{u}} \right) = \infty.$$

Hence, (H3) and (H4) are satisfied. Moreover, since $N = 1$ (the normal constant of P) and $M_0 = (N \int_0^1 \Phi(s)g(s)ds)^{-1} = \frac{1}{6}(\pi)^{-1}$, for $\eta_0 = \frac{3}{2}$, it can follow easily that (H9) is satisfied. By Theorem 3.1, our conclusion follows. \square

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