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Some inequality and Berezin number type inequalities

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Abstract

In this paper, we establish upper bounds that both extend and enhance known inequalities for the Berezin number of bounded linear operators and product of operators. In particular, this paper aims to establish new upper bounds on the Berezin number for operators on functional Hilbert spaces by introducing important improvements to the Buzano inequality. Lastly, a few associated upper bounds are also given.

Keywords: Berezin norm, Berezin number, inequalities.

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1. Introduction

In the literature pertaining to operator theory, the Berezin norm and Berezin number of an operator have been studied for their numerous applications in numerical analysis, quantum physics, engineering, and other fields. To describe the Berezin number and norm, we start by discussing certain concepts and characteristics of bounded linear operators on a Hilbert space.

Let $\mathbb{L}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$ with the an inner product $\langle .,. \rangle$ and corresponding norm $\|.\|$. Recall that the functional Hilbert space (shortly FHS) $\mathcal{H} = \mathcal{H}(\Delta)$ is a Hilbert space of complex-valued functions on a (nonempty) set Δ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in \Delta$, are continuous on $\mathcal{H}$ and for every $\tau \in \Delta$ there exists a function $f_\tau \in \mathcal{H}$ such that $f_\tau(\tau) \neq 0$ or, equivalently, there is no $\tau_0 \in \Delta$ such that $f(\tau_0) = 0$ for all $f \in \mathcal{H}$. The Riesz representation theorem ensures that for each $\tau \in \Delta$ there is a unique element $k_\tau \in \mathcal{H}$ such that $f(\tau) = \langle f, k_\tau \rangle$ for all $f \in \mathcal{H}$. The collection $\{k_\tau : \tau \in \Delta\}$ is called the reproducing kernel of $\mathcal{H}$. For $\tau \in \Delta$, let $\widehat{k}_\tau := \frac{k_\tau}{||k_\tau||}$ be the normalized reproducing kernel of $\mathcal{H}$. The absolute value of positive operator is denoted by $|Z| = (Z^*Z)^{1/2}$.

For a bounded linear operator Z on $\mathcal{H}$, the function $\widetilde{Z}$ defined on Δ by $\widetilde{Z}(\tau) := \langle Z\widehat{k}_\tau(z), \widehat{k}_\tau(z) \rangle$ is the Berezin symbol of Z, which firstly have been introduced by Berezin [7]. In other words, Berezin symbol $\widetilde{Z}$ is the function on Δ defined by restriction of the quadratic form $\langle Zx_1, x_1 \rangle$ with $x_1 \in \mathcal{H}$ to the subset of all

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normalized reproducing kernels of the unit sphere in $\mathcal{H}$. It is clear from the Cauchy-Schwarz inequality that $\widetilde{Z}$ is the bounded function on Δ whose values lie in the numerical range of the operator $\widetilde{Z}$. The Berezin set (or range) and the Berezin number (or radius) of the operator Z are defined by

$$\operatorname{Ber}(Z) := \{\widetilde{Z}(\tau) : \tau \in \Delta\} \text{ and } \operatorname{ber}(Z) := \sup_{\tau\in\Delta} \left|\widetilde{Z}(\tau)\right|,$$

respectively (see [23]). It is obvious that $\operatorname{ber}(Z) \leqslant w(Z) \leqslant \|Z\|$ and $\operatorname{Ber}(Z) \subset W(Z)$, where $w(Z)$ denotes the numerical radius and $W(Z)$ is the numerical range of operator Z. It is well known that

$$\frac{\|Z\|}{2} \leqslant w(Z) \leqslant \|Z\|$$

and

$$\operatorname{ber}(Z) \leqslant w(Z) \leqslant \|Z\|, \tag{1.1}$$

for any $Z \in \mathbb{L}(\mathcal{H})$. In 2022, Huban et al. [21, 22] proved the following results:

$$\operatorname{ber}(Z) \leqslant \frac{1}{2} \||Z| + |Z^*|\|_{\text{ber}} \leqslant \frac{1}{2}\left(\|Z\|_{\text{ber}} + \left\|Z^2\right\|_{\text{ber}}^{\frac{1}{2}}\right) \tag{1.2}$$

and

$$\operatorname{ber}^{2r}(Z) \leqslant \frac{1}{2}\left\||Z|^{2r} + |Z^*|^{2r}\right\|_{\text{ber}}, \text{ where } r \geqslant 1.$$

In [21], Huban et al. significantly improved the upper bound in (1.1) by demonstrating that if $Z \in \mathbb{L}(\mathcal{H})$, then

$$\operatorname{ber}(Z) \leqslant \frac{1}{2} \||Z| + |Z^*|\|_{\text{ber}}.$$

Another improvement for the inequality (1.1) was provided by Huban et al. [20] as

$$\operatorname{ber}^2(Z) \leqslant \frac{1}{2}\left\||Z|^2 + |Z^*|^2\right\|_{\text{ber}},$$

which was further improved in [4] by Başaran and Gürdal as

$$\operatorname{ber}^2(Z) \leqslant \frac{1}{6}\left\||Z|^2 + |Z^*|^2\right\|_{\text{ber}} + \frac{1}{3}\operatorname{ber}(Z) \||Z| + |Z^*|\|_{\text{ber}}.$$

It was shown in [20, 22], respectively, that if $Z \in \mathbb{L}(\mathcal{H}(\Delta))$, then

$$\frac{1}{4}\left\||Z|^2 + |Z^*|^2\right\|_{\text{ber}} \leqslant \operatorname{ber}^2(Z) \leqslant \frac{1}{2}\left\||Z|^2 + |Z^*|^2\right\|_{\text{ber}}, \tag{1.3}$$

and

$$\operatorname{ber}^{2r}(Z) \leqslant \frac{1}{2}\left\||Z|^{2r} + |Z^*|^{2r}\right\|_{\text{ber}},$$

where $r \geqslant 1$. Furthermore, Huban et al. [21] established refinements of (1.2) and (1.3), respectively that can be presented as

$$\operatorname{ber}^j(Z) \leqslant \frac{1}{2}\left\||Z|^{2j\xi} + |Z^*|^{2j(1-\xi)}\right\|_{\text{ber}} \quad \text{and} \quad \operatorname{ber}^{2j}(Z) \leqslant \left\|\xi|Z|^{2j} + (1-\xi)|Z^*|^{2j}\right\|_{\text{ber}},$$

where $Z \in \mathbb{L}(\mathcal{H})$, $0 \leqslant \xi \leqslant 1$, and $j \geqslant 1$. Recently, Başaran and Gürdal [4] obtained some inequalities, showing the following for $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$,

$$\operatorname{ber}^2(Z_1) \leqslant \frac{1}{12}\||Z_1| + |Z_1^*|\|_{\text{ber}}^2 + \frac{1}{3}\operatorname{ber}(Z_1)\||Z_1| + |Z_1^*|\|_{\text{ber}} \leqslant \frac{1}{6}\||Z_1| + |Z_1^*|\|_{\text{ber}}^2 + \frac{1}{3}\operatorname{ber}(Z_1)\||Z_1| + |Z_1^*|\|_{\text{ber}}$$

and
$$\mathrm{ber}^2(Z_2^*Z_1) \leqslant \frac{1}{6}\left\| |Z_1|^4 + |Z_1^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(Z_2^*Z_1)\left\| |Z_1|^2 + |Z_1^*|^2 \right\|_{\mathrm{ber}}.$$
Başaran and Gürdal [4] also obtained that
$$\begin{aligned}\mathrm{ber}^j(Z) &\leqslant \frac{1}{2}\eta \left\| |Z|^{2j\gamma} + |Z^*|^{2j(1-\gamma)} \right\|_{\mathrm{ber}} + \frac{1}{\sqrt{2}}(1-\eta)\,\mathrm{ber}^{\frac{j}{2}}(Z)\left\| |Z|^{2j\gamma} + |Z^*|^{2j(1-\gamma)} \right\|_{\mathrm{ber}}^{1/2} \\ &\leqslant \frac{1}{2}\left\| |Z|^{2\gamma j} + |Z^*|^{2(1-\gamma)j} \right\|_{\mathrm{ber}},\end{aligned}$$
for all $j \geqslant 1$ and $0 \leqslant \gamma, \eta \leqslant 1$. Another important fact about the Berezin number upper bounds that are of our interest are due to Huban et al. in [20]: if $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$ and $j \geqslant 1$, then
$$\mathrm{ber}^j(Z_2^*Z_1) \leqslant \frac{1}{2}\left\| |Z_1|^{2j} + |Z_2|^{2j} \right\|_{\mathrm{ber}}. \tag{1.4}$$
For an in-depth exploration of the intricacies surrounding the Berezin symbol, interested readers are strongly encouraged to refer to [1, 3, 5, 6, 8–10, 12, 13, 15–19, 24–27, 30, 31, 36–38], and the comprehensive references provided therein.

In this paper, motivated by previously reported results, this work aims to develop new Berezin number upper bounds for reproducing kernel Hilbert space operators by introducing new improvements to the well-known Cauchy-Schwarz inequality.

2. Auxiliaries theorems

To prove our generalized Berezin number inequalities, we need several well-known lemmas. The classical Schwarz inequality for positive operators reads that if $Z \in \mathbb{L}(\mathcal{H})$ is a positive operators, then
$$|\langle Zx_1, x_2\rangle|^2 \leqslant \langle Zx_1, x_1\rangle \langle Zx_2, x_2\rangle \tag{2.1}$$
for any $x_1, x_2 \in \mathcal{H}$. A companion of Schwarz inequality (2.1) known as the Kato's inequality or the so called mixed Cauchy Schwarz inequality was first proposed by Kato [28] in 1952. It can be expressed as
$$|\langle Zx_1, x_2\rangle|^2 \leqslant \langle |Z|^{2\gamma}x_1, x_1\rangle\langle |Z^*|^{2(1-\gamma)}x_2, x_2\rangle, \ \gamma \in [0,1],$$
for any operator $Z \in \mathbb{L}(H)$ and any $x_1, x_2 \in \mathcal{H}$. In order to generalize this result, in 1994 Furuta [14] obtained the following result:
$$|\langle Z|Z|^{\gamma+\eta-1}x_1, x_2\rangle|^2 \leqslant \langle |Z|^{2\gamma}x_1, x_1\rangle\langle |Z^*|^{2\eta}x_2, x_2\rangle,$$
for any $x_1, x_2 \in \mathcal{H}$ and $\gamma, \eta \in [0,1]$ with $\gamma + \eta \geqslant 1$. The first lemma follows from the spectral theorem for positive operators and Jensens inequality (see [29]).

Lemma 2.1. *Let $Z \in \mathbb{L}(\mathcal{H})$, $Z \geqslant 0$, and $x_1 \in \mathcal{H}$ be any unit vector. Then, we have*
$$\langle Zx_1, x_1\rangle^j \leqslant (\geqslant) \left\langle Z^j x_1, x_1\right\rangle, \ \ j \geqslant 1 \ \ (j \in [0,1]). \tag{2.2}$$

The next result is the Buzano's generalization of the Cauchy–Schwarz inequality (see [11]).

Lemma 2.2. *Let $x_1, x_2, x_3 \in \mathbb{L}(\mathcal{H})$ with $\|x_3\| = 1$. Then we have*
$$|\langle x_1, x_3\rangle \langle x_3, x_2\rangle| \leqslant \frac{1}{2}\left(\|x_1\|\,\|x_2\| + |\langle x_1, x_2\rangle|\right).$$

Recently, Sababheh et al. [32] proved a new improvement of the Cauchy-Schwarz inequality as follows.

Lemma 2.3. *If $x_1, x_2 \in \mathcal{H}$ and every $\omega \geqslant 0$, then*

$$|\langle x_1, x_2\rangle|^2 \leqslant \frac{1}{\omega+1}\|x_1\|\|x_2\||\langle x_1, x_2\rangle| + \frac{\omega}{\omega+1}\|x_1\|^2\|x_2\|^2 \leqslant \|x_1\|^2\|x_2\|^2.$$

The next result concerns with non-negative convex functions and can be found in [2].

Lemma 2.4. *Let f be a non-negative convex function on $[0,+\infty)$ and $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$ be positive operators. Then*

$$\left\|f\left(\frac{Z_1+Z_2}{2}\right)\right\| \leqslant \left\|\frac{f(Z_1)+f(Z_2)}{2}\right\|.$$

In particular if $j \geqslant 1$, then we have

$$\left\|\left(\frac{Z_1+Z_2}{2}\right)^j\right\| \leqslant \left\|\frac{Z_1^j+Z_2^j}{2}\right\|.$$

Let us assume throughout the paper that $\wp : J \to \mathbb{R}^+$ is a mapping such that $\wp(\Xi)+\wp(1-\Xi) = 1$ and $(0,1) \subset J \subset \mathbb{R}$, unless otherwise stated.

Lemma 2.5 ([33])**.** *Let $\theta_1, \theta_2 \in \mathcal{H}$ and $j \geqslant 1$. Then the following inequality holds*

$$|\langle\theta_1, \theta_2\rangle|^{2j} \leqslant \wp(\Xi)\|\theta_1\|^{2j}\|\theta_2\|^{2j} + \wp(1-\Xi)|\langle\theta_1, \theta_2\rangle|^j\|\theta_1\|^j\|\theta_2\|^j \leqslant \|\theta_1\|^{2j}\|\theta_2\|^{2j}. \tag{2.3}$$

Recently, Gürdal and Stojiljković obtained the following refinement with respect to the mapping $\wp$, which is as follows.

Lemma 2.6 ([34])**.** *Let $\theta_1, \theta_2, \theta_3 \in \mathcal{H}$ such that $\|\theta_3\| = 1$ and let $j \geqslant 1$. Then the following inequality holds*

$$\begin{aligned} &|\langle\theta_1, \theta_3\rangle\langle\theta_3, \theta_2\rangle|^j \\ &\leqslant \min\left\{\frac{\wp(\Xi)+1}{2}\|\theta_1\|^j\|\theta_2\|^j + \frac{\wp(1-\Xi)}{2}|\langle\theta_1, \theta_2\rangle|^j, \frac{\wp(1-\Xi)+1}{2}\|\theta_1\|^j\|\theta_2\|^j + \frac{\wp(\Xi)}{2}|\langle\theta_1, \theta_2\rangle|^j\right\}. \end{aligned} \tag{2.4}$$

Lemma 2.7 ([34])**.** *Let $\theta_1, \theta_2 \in \mathcal{H}$ and let $j \geqslant 1$. Then the following inequality holds*

$$|\langle\theta_1, \theta_2\rangle|^j \leqslant \frac{\wp(\Xi)+1}{2}\|\theta_1\|^j\|\theta_2\|^j + \frac{\wp(1-\Xi)}{2}|\langle\theta_1, \theta_2\rangle|^j \leqslant \|\theta_1\|^j\|\theta_2\|^j. \tag{2.5}$$

Lemma 2.8 ([35])**.** *If $Z_1, Z_2, Z_3, Z_4 \in \mathbb{L}(\mathcal{H})$, then we have*

$$|\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^2 \leqslant \langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle. \tag{2.6}$$

3. Main results

In this part, we now present the first outcome. The following theorem is a refinement of the inequality (2.6).

Theorem 3.1. *If $Z_1, Z_2, Z_3, Z_4 \in \mathbb{L}(\mathcal{H})$ and $\wp$ such that (2.5), then for $j \geqslant 1$ we have*

$$\begin{aligned} |\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j &\leqslant \frac{\wp(\Xi)+1}{2}\sqrt[\frac{j}{2}]{\langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle}\sqrt[\frac{j}{2}]{\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle} \\ &\quad + \frac{\wp(1-\Xi)}{2}|\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j \\ &\leqslant \sqrt[\frac{j}{2}]{\langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle}\sqrt[\frac{j}{2}]{\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle}. \end{aligned} \tag{3.1}$$

Proof. Let $\tau, \sigma \in \Delta$ be an arbitrary. An elementary calculus shows that

$$\left\|Z_2Z_1\widehat{k}_\tau\right\|^2 = \langle Z_2Z_1\widehat{k}_\tau, Z_2Z_1\widehat{k}_\tau\rangle = \langle Z_1^*Z_2^*Z_2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle = \langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle$$

and $\left\|Z_3^*Z_4^*\widehat{k}_\sigma\right\|^2 = \langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle$. If we take $\theta_1 = Z_2Z_1\widehat{k}_\tau$, $\theta_2 = Z_3^*Z_4^*\widehat{k}_\sigma$ in the inequality (2.5), we obtain

$$\begin{aligned}|\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j &\leqslant \frac{\wp(\Xi)+1}{2}\sqrt[\frac{j}{2}]{\langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle}\sqrt[\frac{j}{2}]{\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle} + \frac{\wp(1-\Xi)}{2}|\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j \\ &\leqslant \sqrt[\frac{j}{2}]{\langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle}\sqrt[\frac{j}{2}]{\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle}\end{aligned}$$

which is as desired. □

Note that we get the inequality given by Stojiljković and Gürdal [34, Lemma 4] by putting $j = 2$ in (3.1).

The following corollaries are consequences of (3.1).

Corollary 3.2. *If $Z \in \mathbb{L}(\mathcal{H})$, then for $\gamma, \eta \geqslant 0$ with $\gamma + \eta \geqslant 1$ we have*

$$\begin{aligned}|\langle Z|Z|^{\gamma+\eta-1}\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j &\leqslant \frac{\wp(\Xi)+1}{2}\langle |Z|^{2\gamma}\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2}\langle |Z^*|^{2\eta}\widehat{k}_\sigma, \widehat{k}_\sigma\rangle^{j/2} + \frac{\wp(1-\Xi)}{2}|\langle Z|Z|^{\gamma+\eta-1}\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j \\ &\leqslant \langle |Z|^{2\gamma}\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2}\langle |Z^*|^{2\eta}\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2}.\end{aligned}$$

Proof. Let $Z = U|Z|$ be the polar decomposition of the operator Z, where U is partial isometry and the kernel $N(U) = N(|Z|)$. If we take $Z_4 = U, Z_3 = |Z|^\eta, Z_2 = I$, and $Z_1 = |Z|^\gamma$, simplifying we obtain the result. □

The following corollary can be obtained in a similar way to the one previously given.

Corollary 3.3. *If $Z \in \mathbb{L}(\mathcal{H})$ and $\gamma, \eta \geqslant 1$, then we have*

$$\begin{aligned}|\langle Z|Z|^{\eta-1}Z|Z|^{\gamma-1}\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j &\leqslant \frac{\wp(\Xi)+1}{2}\langle |Z|^{2\gamma}\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2}\langle |Z^*|^{2\eta}\widehat{k}_\sigma, \widehat{k}_\sigma\rangle^{j/2} \\ &\quad + \frac{\wp(1-\Xi)}{2}\left|\langle Z|Z|^{\eta-1}Z|Z|^{\gamma-1}\widehat{k}_\tau, \widehat{k}_\sigma\rangle\right|^j \leqslant \langle |Z|^{2\gamma}\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2}\langle |Z^*|^{2\eta}\widehat{k}_\sigma, \widehat{k}_\sigma\rangle^{j/2}.\end{aligned} \tag{3.2}$$

By taking $\gamma = \eta = 1$ and $j = 1$ in (3.2), the outcome is as follows:

$$\begin{aligned}|\langle Z^2\widehat{k}_\tau, \widehat{k}_\sigma\rangle| &\leqslant \frac{\wp(\Xi)+1}{2}\langle |Z|^2\widehat{k}_\tau, \widehat{k}_\tau\rangle^{1/2}\langle |Z^*|^2\widehat{k}_\sigma, \widehat{k}_\sigma\rangle^{1/2} + \frac{\wp(1-\Xi)}{2}|\langle Z^2\widehat{k}_\tau, \widehat{k}_\sigma\rangle \\ &\leqslant \langle |Z|^2\widehat{k}_\tau, \widehat{k}_\tau\rangle^{1/2}\langle |Z^*|^2\widehat{k}_\sigma, \widehat{k}_\sigma\rangle^{1/2}.\end{aligned}$$

Now we can give some numerical radius inequalities.

Theorem 3.4. *If $Z_1, Z_2, Z_3, Z_4 \in \mathbb{L}(\mathcal{H})$ and $j \geqslant 1$, then we have*

$$\begin{aligned}\mathrm{ber}\,(Z_4Z_3Z_2Z_1)^j &\leqslant \frac{\wp(\Xi)+1}{2}\left\|Z_1^*|Z_2|^2Z_1\right\|_{\mathrm{ber}}^{j/2}\left\|Z_4|Z_3^*|^2Z_4^*\right\|_{\mathrm{ber}}^{j/2} + \frac{\wp(1-\Xi)}{2}\mathrm{ber}\,(Z_4Z_3Z_2Z_1)_{\mathrm{ber}}^j \\ &\leqslant \left\|Z_1^*|Z_2|^2Z_1\right\|_{\mathrm{ber}}^{j/2}\left\|Z_4|Z_3^*|^2Z_4^*\right\|_{\mathrm{ber}}^{j/2}.\end{aligned}$$

Proof. Let $\tau, \sigma \in \Delta$ be an arbitrary. Taking the supremum of the inequality (3.1), we get

$$\begin{aligned}
&\sup_{\tau,\sigma\in\Delta} |\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j \\
&\leqslant \sup_{\tau,\sigma\in\Delta} \left(\frac{\wp(\Xi)+1}{2} \left(\langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle\right)^{j/2} \left(\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle\right)^{j/2} + \frac{\wp(1-\Xi)}{2}|\langle Z_4Z_3Z_2Z_1\widehat{k}_\tau, \widehat{k}_\sigma\rangle|^j \right) \\
&\leqslant \sup_{\tau,\sigma\in\Delta} \left(\langle Z_1^*|Z_2|^2Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle\right)^{j/2} \left(\langle Z_4|Z_3^*|^2Z_4^*\widehat{k}_\sigma, \widehat{k}_\sigma\rangle\right)^{j/2},
\end{aligned}$$

which simplified yields the left hand side, in order to obtain the right hand side, start from the right hand side of the just obtained inequality and apply Lemma 5 given in [34], then using the properties of the mapping $\wp$ and of the supremum, we deduce that

$$\begin{aligned}
\mathrm{ber}\,(Z_4Z_3Z_2Z_1)^j &\leqslant \frac{\wp(\Xi)+1}{2} \left\|Z_1^*|Z_2|^2Z_1\right\|_{\mathrm{ber}}^{j/2} \left\|Z_4|Z_3^*|^2Z_4^*\right\|_{\mathrm{ber}}^{j/2} + \frac{\wp(1-\Xi)}{2}\mathrm{ber}\,(Z_4Z_3Z_2Z_1)^j \\
&\leqslant \left\|Z_1^*|Z_2|^2Z_1\right\|_{\mathrm{ber}}^{j/2} \left\|Z_4|Z_3^*|^2Z_4^*\right\|_{\mathrm{ber}}^{j/2}.
\end{aligned}$$

Thus, we obtain the desired inequality. □

We get the following result by putting $j = 2$ in the above inequality.

Corollary 3.5. *If $Z_1, Z_2, Z_3, Z_4 \in \mathbb{L}(\mathcal{H})$, then we have*

$$\mathrm{ber}\,(Z_4Z_3Z_2Z_1)^2 \leqslant \left\|Z_1^*|Z_2|^2Z_1\right\|_{\mathrm{ber}} \left\|Z_4|Z_3^*|^2Z_4^*\right\|_{\mathrm{ber}}.$$

Theorem 3.6. *If $Z_1 \in \mathbb{L}(\mathcal{H})$ and $j \geqslant 2$, then we have*

$$\begin{aligned}
|\langle Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle\langle \widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle|^j \leqslant{}& \frac{\wp(\Xi)+1}{8} \left\langle \left(|Z_1|^{2j} + |Z_2^*|^{2j}\right)\widehat{k}_\tau, \widehat{k}_\tau \right\rangle \\
&+ \frac{\wp(\Xi)+1}{4} \left|\left\langle |Z_1|^2\,\widehat{k}_\tau, |Z_2^*|^2\,\widehat{k}_\tau \right\rangle\right|^{j/2} + \frac{\wp(1-\Xi)}{2} \left|\left\langle Z_1\widehat{k}_\tau, Z_2^*\widehat{k}_\tau\right\rangle\right|^j.
\end{aligned}$$

In particular, we obtain

$$\begin{aligned}
\mathrm{ber}^{2j}(Z_1) &\leqslant \frac{\wp(\Xi)+1}{8} \left\||Z_1|^{2j} + |Z_1^*|^{2j}\right\|_{\mathrm{ber}} + \frac{\wp(\Xi)+1}{4}\mathrm{ber}^{j/2}\left(|Z_1^*|^2\,|Z_1|^2\right) + \frac{\wp(1-\Xi)}{2}\mathrm{ber}^j(Z_1^2) \\
&\leqslant \frac{\wp(\Xi)+1}{4} \left\||Z_1|^{2j} + |Z_1^*|^{2j}\right\|_{\mathrm{ber}} + \frac{\wp(1-\Xi)}{2}\mathrm{ber}^j(Z_1^2).
\end{aligned}$$

Proof. Let $\tau \in \Delta$ be an arbitrary. Without the loss of generality, let us assume that one of the inequalities hold in (2.4). Setting $\theta_1 = Z_1\widehat{k}_\tau, \theta_2 = Z_2^*\widehat{k}_\tau, \theta_3 = \widehat{k}_\tau$ in (2.4), we obtain

$$\begin{aligned}
&|\langle Z_1\widehat{k}_\tau, \widehat{k}_\tau\rangle\langle \widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle|^j \\
&\leqslant \frac{\wp(\Xi)+1}{2} \left\|Z_1\widehat{k}_\tau\right\|^j \left\|Z_2^*\widehat{k}_\tau\right\|^j + \frac{\wp(1-\Xi)}{2}|\langle Z_1\widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle|^j \\
&\leqslant \frac{\wp(\Xi)+1}{4} \left(\left\||Z_1|^2\,\widehat{k}_\tau\right\|^{j/2} \left\||Z_2^*|^2\,\widehat{k}_\tau\right\|^{j/2} + |\langle |Z_1|^2\widehat{k}_\tau, |Z_2^*|^2\widehat{k}_\tau\rangle|^{j/2} \right) + \frac{\wp(1-\Xi)}{2}|\langle Z_1\widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle|^j \\
&\leqslant \frac{\wp(\Xi)+1}{8} \left(\left\||Z_1|^2\,\widehat{k}_\tau\right\|^j + \left\||Z_2^*|^2\,\widehat{k}_\tau\right\|^j \right) + \frac{\wp(\Xi)+1}{4}|\langle |Z_1|^2\widehat{k}_\tau, |Z_2^*|^2\widehat{k}_\tau\rangle|^{j/2} + \frac{\wp(1-\Xi)}{2}|\langle Z_1\widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle|^j \\
&= \frac{\wp(\Xi)+1}{8} \left(\langle |Z_1|^4\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2} + \langle |Z_2^*|^4\widehat{k}_\tau, \widehat{k}_\tau\rangle^{j/2} \right) + \frac{\wp(\Xi)+1}{4}|\langle |Z_1|^2\widehat{k}_\tau, |Z_2^*|^2\widehat{k}_\tau\rangle|^{j/2} + \frac{\wp(1-\Xi)}{2}|\langle Z_1\widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle|^j \\
&\leqslant \frac{\wp(\Xi)+1}{8}\langle (|Z_1|^{2j} + |Z_2^*|^{2j})\widehat{k}_\tau, \widehat{k}_\tau\rangle + \frac{\wp(\Xi)+1}{4} \left|\langle |Z_1|^2\widehat{k}_\tau, |Z_2^*|^2\widehat{k}_\tau\rangle\right|^{j/2} + \frac{\wp(1-\Xi)}{2} \left|\langle Z_1\widehat{k}_\tau, Z_2^*\widehat{k}_\tau\rangle\right|^j,
\end{aligned}$$

and, so

$$
\begin{aligned}
|\langle Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle\langle\widehat{k}_\tau,Z_2^*\widehat{k}_\tau\rangle|^j \leqslant{}& \frac{\wp(\Xi)+1}{8}\langle(|Z_1|^{2j}+|Z_2^*|^{2j})\widehat{k}_\tau,\widehat{k}_\tau\rangle\\
&+\frac{\wp(\Xi)+1}{4}\left|\langle|Z_1|^2\widehat{k}_\tau,|Z_2^*|^2\widehat{k}_\tau\rangle\right|^{j/2}+\frac{\wp(1-\Xi)}{2}\left|\langle Z_1\widehat{k}_\tau,Z_2^*\widehat{k}_\tau\rangle\right|^j.
\end{aligned}
$$

The second inequality is obtained by taking supremum over all $\tau\in F$, setting $Z_2^*=Z_1^*$, and using Lemma 2.1 and the inequality (1.4) provided by Huban et al. [20]. □

Theorem 3.7. *If $Z_1,Z_2\in\mathbb{L}(\mathcal{H})$, then for any $j\geqslant 1$ the following inequality holds*

$$
\begin{aligned}
\operatorname{ber}^{2j}(Z_2^*Z_1)\leqslant{}&\frac{\wp(\Xi)}{4}\left\||Z_1|^{4j}+|Z_2|^{4j}\right\|_{\mathrm{ber}}+\frac{\wp(1-\Xi)}{2}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}}\operatorname{ber}^j(Z_2^*Z_1)\\
&+\frac{\wp(\Xi)}{2}\operatorname{ber}^j\left(|Z_2|^{2j}|Z_1|^{2j}\right).
\end{aligned}
\tag{3.3}
$$

Proof. Let $\tau\in\Delta$ be an arbitrary. We begin with setting $\theta_1=Z_1\widehat{k}_\tau,\theta_2=Z_2\widehat{k}_\tau$ in (2.3),

$$
\begin{aligned}
|\langle Z_1\widehat{k}_\tau,Z_2\widehat{k}_\tau\rangle|^{2j}\leqslant{}&\wp(\theta)\left\|Z_1\widehat{k}_\tau\right\|^{2j}\left\|Z_2\widehat{k}_\tau\right\|^{2j}+\wp(1-\theta)|\langle Z_1\widehat{k}_\tau,Z_2\widehat{k}_\tau\rangle|^j\left\|Z_1\widehat{k}_\tau\right\|^j\left\|Z_2\widehat{k}_\tau\right\|^j\\
\leqslant{}&\frac{\wp(1-\Xi)}{2}\langle(|Z_1|^{2j}+|Z_2|^{2j})\widehat{k}_\tau,\widehat{k}_\tau\rangle|\langle Z_1\widehat{k}_\tau,Z_2\widehat{k}_\tau\rangle|^j\\
&+\frac{\wp(\Xi)}{2}\left(\left\||Z_1|^{2j}\widehat{k}_\tau\right\|\left\||Z_2|^{2j}\widehat{k}_\tau\right\|+|\langle|Z_1|^{2j}\widehat{k}_\tau,|Z_2|^{2j}\widehat{k}_\tau\rangle|\right)\\
\leqslant{}&\frac{\wp(1-\Xi)}{2}\langle(|Z_1|^{2j}+|Z_2|^{2j})\widehat{k}_\tau,\widehat{k}_\tau\rangle|\langle Z_1\widehat{k}_\tau,Z_2\widehat{k}_\tau\rangle|^j\\
&+\frac{\wp(\Xi)}{4}\langle(|Z_1|^{4j}+|Z_2|^{4j})\widehat{k}_\tau,\widehat{k}_\tau\rangle+\frac{\wp(\Xi)}{2}|\langle|Z_2|^{2j}|Z_1|^{2j}\widehat{k}_\tau,\widehat{k}_\tau\rangle|.
\end{aligned}
$$

Taking the supremum over $\tau\in\Delta$ in the above inequality, we get

$$
\begin{aligned}
\sup_{\tau\in\Delta}|\langle Z_1\widehat{k}_\tau,Z_2\widehat{k}_\tau\rangle|^{2j}\leqslant{}&\frac{\wp(\Xi)}{4}\sup_{\tau\in\Delta}\langle(|Z_1|^{4j}+|Z_2|^{4j})\widehat{k}_\tau,\widehat{k}_\tau\rangle\\
&+\frac{\wp(1-\Xi)}{2}\sup_{\tau\in\Delta}\langle(|Z_1|^{2j}+|Z_2|^{2j})\widehat{k}_\tau,\widehat{k}_\tau\rangle|\langle Z_1\widehat{k}_\tau,Z_2\widehat{k}_\tau\rangle|^j+\frac{\wp(\Xi)}{2}\sup_{\tau\in\Delta}|\langle|Z_2|^{2j}|Z_1|^{2j}\widehat{k}_\tau,\widehat{k}_\tau\rangle|,
\end{aligned}
$$

which is equivalent to

$$
\operatorname{ber}^{2j}(Z_2^*Z_1)\leqslant\frac{\wp(\Xi)}{4}\left\||Z_1|^{4j}+|Z_2|^{4j}\right\|_{\mathrm{ber}}+\frac{\wp(1-\Xi)}{2}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}}\operatorname{ber}^j(Z_2^*Z_1)+\frac{\wp(\Xi)}{2}\operatorname{ber}^j\left(|Z_2|^{2j}|Z_1|^{2j}\right).
$$

This completes the proof. □

Utilizing (1.4) on the third term in the previously obtained inequality, we obtain the following inequality.

Corollary 3.8. *Let $Z_1,Z_2\in\mathbb{L}(H)$ and $\wp$ be such that the conditions from (3.1) hold, then*

$$
\begin{aligned}
\operatorname{ber}^{2j}(Z_2^*Z_1)&\leqslant\frac{\wp(\Xi)}{4}\left\||Z_1|^{4j}+|Z_2|^{4j}\right\|_{\mathrm{ber}}+\frac{\wp(1-\Xi)}{2}\left\||Z_2|^{2j}+|Z_1|^{2j}\right\|_{\mathrm{ber}}\operatorname{ber}^j(Z_2^*Z_1)+\frac{\wp(\Xi)}{2}\operatorname{ber}(|Z_2|^{2j}|Z_1|^{2j})\\
&\leqslant\frac{\wp(\Xi)}{2}\left\||Z_1|^{4j}+|Z_2|^{4j}\right\|_{\mathrm{ber}}+\frac{\wp(1-\Xi)}{2}\left\||Z_2|^{2j}+|Z_1|^{2j}\right\|_{\mathrm{ber}}\operatorname{ber}^j(Z_2^*Z_1)\\
&\leqslant\frac{1}{2}\left\||Z_1|^{4j}+|Z_2|^{4j}\right\|_{\mathrm{ber}}.
\end{aligned}
$$

We get the following result by putting $\wp(\Xi) = \Xi$ and $\Xi = \frac{\omega}{1+\omega}, \omega \geqslant 0$ in (3.3).

Corollary 3.9. *If* $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$, *then we have*

$$\begin{aligned} \mathrm{ber}^{2j}(Z_2^* Z_1) \leqslant{}& \frac{\omega}{2(1+\omega)} \left\| |Z_1|^{2j} + |Z_2|^{2j} \right\|_{\mathrm{ber}} \mathrm{ber}^j(Z_2^* Z_1) \\ &+ \frac{\omega}{4(1+\omega)} \left\| |Z_1|^{4j} + |Z_2|^{4j} \right\|_{\mathrm{ber}} + \frac{\omega}{2(1+\omega)} \mathrm{ber}^j\left(|Z_2|^{2j}|Z_1|^{2j}\right). \end{aligned}$$

Proof. Let $\tau \in \Delta$ be an arbitrary. Utilizing the convexity of the function $f(t) = t^j$ and Lemmas 2.3 and 2.2, we can write

$$\begin{aligned} |\langle Z_2^* Z_1 \widehat{k}_\tau, \widehat{k}_\tau \rangle|^{2j} &= |\langle Z_1 \widehat{k}_\tau, Z_2 \widehat{k}_\tau \rangle|^{2j} \\ &\leqslant \left(\frac{\omega}{1+\omega} \left\| Z_2 \widehat{k}_\tau \right\| \left\| Z_1 \widehat{k}_\tau \right\| |\langle Z_1 \widehat{k}_\tau, Z_2 \widehat{k}_\tau \rangle| + \frac{\omega}{\omega+1} \left\| Z_2 \widehat{k}_\tau \right\|^2 \left\| Z_1 \widehat{k}_\tau \right\|^2 \right)^j \\ &\leqslant \frac{\omega}{1+\omega} \left\| Z_2 \widehat{k}_\tau \right\|^j \left\| Z_1 \widehat{k}_\tau \right\|^j |\langle Z_1 \widehat{k}_\tau, Z_2 \widehat{k}_\tau \rangle|^j + \frac{\omega}{\omega+1} \left\| Z_2 \widehat{k}_\tau \right\|^{2j} \left\| Z_1 \widehat{k}_\tau \right\|^{2j} \\ &\leqslant \frac{\omega}{2(1+\omega)} \left\langle (|Z_2|^{2j} + |Z_1|^{2j}) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle |\langle Z_2^* Z_1 \widehat{k}_\tau, \widehat{k}_\tau \rangle|^j + \frac{\omega}{\omega+1} \left\langle |Z_2|^{2j} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle \widehat{k}_\tau, |Z_1|^{2j} \widehat{k}_\tau \right\rangle \\ &\leqslant \frac{\omega}{2(1+\omega)} \left\langle (|Z_2|^{2j} + |Z_1|^{2j}) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle |\langle Z_2^* Z_1 \widehat{k}_\tau, \widehat{k}_\tau \rangle|^j \\ &\quad + \frac{\omega}{2(\omega+1)} \left(\left\| |Z_2|^{2j} \widehat{k}_\tau \right\| \left\| |Z_1|^{2j} \widehat{k}_\tau \right\| + \left\langle |Z_2|^{2j} \widehat{k}_\tau, |Z_1|^{2j} \widehat{k}_\tau \right\rangle \right) \\ &\leqslant \frac{\omega}{2(1+\omega)} \left\langle (|Z_2|^{2j} + |Z_1|^{2j}) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle |\langle Z_2^* Z_1 \widehat{k}_\tau, \widehat{k}_\tau \rangle|^j \\ &\quad + \frac{\omega}{4(\omega+1)} \left\langle (|Z_2|^{4j} + |Z_1|^{4j}) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \frac{\omega}{2(\omega+1)} \left| \left\langle \left(|Z_1|^{2j} |Z_2|^{2j} \right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right|. \end{aligned}$$

We deduce that

$$\begin{aligned} \sup_{\tau \in F} |\langle Z_2^* Z_1 \widehat{k}_\tau, \widehat{k}_\tau \rangle|^{2j} \leqslant{}& \sup_{\tau \in F} \frac{\omega}{2(1+\omega)} \left\langle (|Z_2|^{2j} + |Z_1|^{2j}) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle |\langle Z_2^* Z_1 \widehat{k}_\tau, \widehat{k}_\tau \rangle|^j \\ &+ \sup_{\tau \in F} \frac{\omega}{4(\omega+1)} \left\langle (|Z_2|^{4j} + |Z_1|^{4j}) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \sup_{\tau \in F} \frac{\omega}{2(\omega+1)} \left| \left\langle \left(|Z_1|^{2j} |Z_2|^{2j} \right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right|, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \mathrm{ber}^{2j}(Z_2^* Z_1) \leqslant{}& \frac{\omega}{2(1+\omega)} \left\| |Z_1|^{2j} + |Z_2|^{2j} \right\|_{\mathrm{ber}} \mathrm{ber}^j(Z_2^* Z_1) \\ &+ \frac{\omega}{4(1+\omega)} \left\| |Z_1|^{4j} + |Z_2|^{4j} \right\|_{\mathrm{ber}} + \frac{\omega}{2(1+\omega)} \mathrm{ber}^j\left(|Z_2|^{2j}|Z_1|^{2j}\right), \end{aligned}$$

and this completes the proof of the theorem. □

We get the following result by putting $\wp(\Xi) = \Xi$ and $\Xi = \frac{\omega}{1+\omega}, \omega \geqslant 0$, in Corollary 3.8.

Corollary 3.10. *If* $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$, *then we have*

$$\begin{aligned} \mathrm{ber}^{2j}(Z_2^* Z_1) \leqslant{}& \frac{\omega}{4(\omega+1)} \left\| |Z_1|^{4j} + |Z_2|^{4j} \right\|_{\mathrm{ber}} + \frac{1}{2(\omega+1)} \left\| |Z_2|^{2j} + |Z_1|^{2j} \right\|_{\mathrm{ber}} \mathrm{ber}^j(Z_2^* Z_1) \\ &+ \frac{\omega}{2(\omega+1)} \mathrm{ber}(|Z_2|^{2j}|Z_1|^{2j}) \leqslant \frac{1}{2} \left\| |Z_1|^{4j} + |Z_2|^{4j} \right\|_{\mathrm{ber}}. \end{aligned}$$

Proof. Let $\tau \in \Delta$ be an arbitrary. Using the inequality (1.4) and Lemma 2.4, we get

$$\begin{aligned}
\mathrm{ber}^{2j}(Z_2^*Z_1) &\leqslant \frac{1}{2(\omega+1)}\left\||Z_2|^{2j}+|Z_1|^{2j}\right\|_{\mathrm{ber}}\mathrm{ber}^j(Z_2^*Z_1)\\
&\quad+\frac{\omega}{4(\omega+1)}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}+\frac{\omega}{2(\omega+1)}\mathrm{ber}(|Z_2|^{2j}|Z_1|^{2j})\\
&\leqslant \frac{1}{4(\omega+1)}\left\||Z_2|^{2j}+|Z_1|^{2j}\right\|_{\mathrm{ber}}^2+\frac{\omega}{4(\omega+1)}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}+\frac{\omega}{4(\omega+1)}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}\\
&\leqslant \frac{1}{2(\omega+1)}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}+\frac{\omega}{4(\omega+1)}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}+\frac{\omega}{4(\omega+1)}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}\\
&= \frac{1}{2}\left\||Z_2|^{4j}+|Z_1|^{4j}\right\|_{\mathrm{ber}}.
\end{aligned}$$

□

We get the following result by putting $\wp(\Xi)=\Xi$, $\Xi=\frac{\omega}{1+\omega}$, $\omega \geqslant 0$, and $j=1$ in Corollary 3.8.

Corollary 3.11. *If $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$, then we have*

$$\begin{aligned}
\mathrm{ber}^2(Z_2^*Z_1) &\leqslant \frac{\omega}{4\omega+4}\left\||Z_1|^4+|Z_2|^4\right\|_{\mathrm{ber}}+\frac{1}{2\omega+2}\left\||Z_2|^2+|Z_1|^2\right\|_{\mathrm{ber}}\mathrm{ber}(Z_2^*Z_1)\\
&\quad+\frac{\omega}{2\omega+2}\mathrm{ber}(|Z_2|^2|Z_1|^2) \leqslant \frac{1}{2}\left\||Z_1|^4+|Z_2|^4\right\|_{\mathrm{ber}},
\end{aligned} \tag{3.4}$$

which was given by Gürdal and Tapdigoglu [19, Theorem 3.1 and Corollary 3.1].

By taking $\omega=\frac{1}{2}$ in (3.4), the outcome is as follows.

Corollary 3.12. *If $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$, then we have*

$$\mathrm{ber}^2(Z_2^*Z_1) \leqslant \frac{1}{12}\left\||Z_1|^4+|Z_2|^4\right\|_{\mathrm{ber}}+\frac{1}{6}\mathrm{ber}(|Z_2|^2|Z_1|^2)+\frac{1}{3}\mathrm{ber}(Z_2^*Z_1)\left\||Z_1|^2+|Z_2|^2\right\|_{\mathrm{ber}} \leqslant \frac{1}{2}\left\||Z_1|^4+|Z_2|^4\right\|_{\mathrm{ber}},$$

which was given by Başaran and Gürdal [4].

Theorem 3.13. *Let $Z_1, Z_2 \in \mathbb{L}(\mathcal{H})$, then for any $j \geqslant 1$ the following inequality holds*

$$\mathrm{ber}^j(Z_2^*Z_1) \leqslant \frac{\wp(\Xi)+1}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|+\frac{\wp(1-\Xi)}{2}\mathrm{ber}^j(Z_2^*Z_1) \leqslant \frac{1}{2}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}}.$$

Proof. Let $\tau \in \Delta$ be an arbitrary. Setting $Z_2 = I$, $Z_3 = I$, $Z_4^* = Z_2$, $\widehat{k}_\tau = \widehat{k}_\sigma$ in (3.1), we obtain the following:

$$|\langle Z_2^*Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle|^j \leqslant \frac{\wp(\Xi)+1}{2}\langle|Z_1|^2\widehat{k}_\tau,\widehat{k}_\tau\rangle^{j/2}\langle|Z_2|^2\widehat{k}_\tau,\widehat{k}_\tau\rangle^{j/2}+\frac{\wp(1-\Xi)}{2}|\langle Z_2^*Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle|^j.$$

Using AG inequality, we obtain

$$\leqslant \frac{\wp(\Xi)+1}{4}\left(\langle|Z_1|^2\widehat{k}_\tau,\widehat{k}_\tau\rangle^j+\langle|Z_2|^2\widehat{k}_\tau,\widehat{k}_\tau\rangle^j\right)+\frac{\wp(1-\Xi)}{2}|\langle Z_2^*Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle|^j.$$

Now using (2.2), we have

$$\leqslant \frac{\wp(\Xi)+1}{4}\left(\langle|Z_1|^{2j}\widehat{k}_\tau,\widehat{k}_\tau\rangle+\langle|Z_2|^{2j}\widehat{k}_\tau,\widehat{k}_\tau\rangle\right)+\frac{\wp(1-\Xi)}{2}|\langle Z_2^*Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle|^j$$

and

$$\sup_{\tau\in\Delta}|\langle Z_2^*Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle|^j \leqslant \frac{\wp(\Xi)+1}{4}\sup_{\tau\in\Delta}\langle\left(|Z_1|^{2j}+|Z_2|^{2j}\right)\widehat{k}_\tau,\widehat{k}_\tau\rangle+\frac{\wp(1-\Xi)}{2}\sup_{\tau\in\Delta}|\langle Z_2^*Z_1\widehat{k}_\tau,\widehat{k}_\tau\rangle|^j,$$

which is equivalent to

$$\mathrm{ber}^j(Z_2^*Z_1) \leqslant \frac{\wp(\Xi)+1}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\| + \frac{\wp(1-\Xi)}{2}\mathrm{ber}^j(Z_2^*Z_1).$$

To obtain the right hand side, notice the following

$$\begin{aligned}\mathrm{ber}^j(Z_2^*Z_1) &\leqslant \frac{\wp(\Xi)+1}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}} + \frac{\wp(1-\Xi)}{2}\mathrm{ber}^j(|Z_2|^*|Z_1|)\\ &\leqslant \frac{\wp(\Xi)+1}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}} + \frac{\wp(1-\Xi)}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}}\\ &= \frac{\wp(\Xi)+1}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}} + \frac{\wp(1-\Xi)}{4}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}}\\ &= \frac{1}{2}\left\||Z_1|^{2j}+|Z_2|^{2j}\right\|_{\mathrm{ber}},\end{aligned}$$

and where the second inequality follows from the inequality (1.4). □

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