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On the oscillatory behavior of solutions of canonical and noncanonical even-order neutral differential equations with distributed deviating arguments

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Abstract

The oscillatory behavior of solutions of an even-order neutral differential equation with distributed deviating arguments is considered using Riccati, generalized Riccati transformations, integral averaging technique of Philos type and the theory of comparison. New sufficient conditions are established in both canonical and noncanonical cases. Two examples are given to support our results.

Keywords: Oscillation, even-order, neutral differential equations.

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1. Introduction

In this article, we discuss the oscillation property of solutions of the even-order neutral differential equation

$$\left(a(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma}\right)' + \int_a^b q(\tau,s)\, f(x(g(\tau,s)))\, ds = 0, \qquad \tau \geqslant \tau_0, \tag{1.1}$$

where $v(\tau) = x(\tau) + \int_c^d p(\tau,\mu)\, x(\sigma(\tau,\mu))\, d\mu$, γ is a quotient of odd positive integers and $n \geqslant 4$ is an even integer under the condition

$$\delta(\tau_0) = \int_{\tau_0}^{\infty} \frac{1}{a^{\frac{1}{\gamma}}(\tau)} d\tau = \infty, \qquad \text{or} \qquad \delta(\tau_0) = \int_{\tau_0}^{\infty} \frac{1}{a^{\frac{1}{\gamma}}(\tau)} d\tau < \infty.$$

Throughout the paper, we assume that

(H_1) $a(\tau) \in C^1([\tau_0,\infty),(0,\infty))$, $a'(\tau) \geqslant 0$;

(H_2) $p(\tau,\mu) \in C([\tau_0,\infty)\times[c,d],[0,\infty))$, $0 \leqslant \int_c^d p(\tau,\mu)\, d\mu \leqslant p < 1$;

(H_3) $\sigma(\tau,\mu) \in C([\tau_0,\infty)\times[c,d],\mathbb{R})$, $\frac{\partial\sigma(\tau,\mu)}{\partial\mu} \geqslant 0$, $\sigma(\tau,\mu) \leqslant \tau$, and $\liminf\limits_{\tau\to\infty} \sigma(\tau,\mu) = \infty$;

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(H_4) $q(\tau, s) \in C([\tau_0, \infty) \times [a, b], [0, \infty))$, $f \in C(R, R)$, with $f(z)/z^{\beta} \geqslant K$, for all $z \neq 0$, and for some positive constant K and β is a quotient of odd positive integers;

(H_5) $g(\tau, s) \in C([\tau_0, \infty) \times [a, b], R)$, $\frac{\partial g(\tau,s)}{\partial s} \geqslant 0$, $g(\tau, s) \leqslant \tau, g_*'(\tau) > 0$, where $g_*(\tau) = g(\tau, a)$, and $\liminf_{\tau\to\infty} g(\tau, s) = \infty$.

By a solution of (1.1), we mean a nontrivialfunction $x(\tau) \in C^{(n-1)}([\tau_x, \infty)), \tau_x \geqslant \tau_0$, which has the property $a(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma} \in C^1([\tau_1, \infty))$ and satisfies (1.1). We consider only those solutions $x(\tau)$ of (1.1), which satisfy $\sup\{|x(\tau)| : \tau \geqslant \tau^*\} > 0$ for $\tau^* \geqslant \tau_x$. A solution $x(\tau)$ of (1.1) is termed oscillatory if it has arbitrarily large zeros on $[\tau_x, \infty)$; otherwise, it is said to be nonoscillatory. Equation (1.1) is termed oscillatory if all its solutions oscillate.

In dynamical models, delay and oscillation effects are often formulated by means of external sources and/or nonlinear diffusion, perturbing the natural evolution of related systems; see, e.g., [10, 11, 18]. It is notable that in the last few decades, there has been considerable interest in studying the oscillation property of solutions of differential equations and applications; see [1, 2, 4, 8, 9, 12, 15–17] and the references therein. Moreover, in the modeling of various problems arising in engineering and natural sciences, the authors often use differential equations with distributed deviating arguments. Therefore, the analysis of qualitative properties of solutions of such equations is crucial for applications, see [23, 24]. Meanwhile, we mention here the works of Zhang et al. [25, 26] and Li and Rogovchenko [14], where they studied the higher-order half-linear delay differential equation

$$\left(r(\tau)\left(x^{(n-1)}(\tau)\right)^{\alpha}\right)' + q(\tau)x^{\beta}(g(\tau)) = 0,$$

α, β are the ratios of odd positive integers, $\beta \leqslant \alpha$. More recently Li and Rogovchenko [13] were concerned with the oscillation of solutions of equation

$$v^{(n)}(\tau) + h(\tau)x(\sigma(\tau)) = 0,$$

with $v(\tau) = x(\tau) + p(\tau)x(g(\tau))$. It is notable that the asymptotic behavior of higher-order neutral differential equations with distributed deviating arguments has received few interest in the literature. Recently, Moaaz et al. [19] studied the oscillation of the differential equation

$$\left(a(\tau)\left(v^{(n-1)}(\tau)\right)\right)' + \int_a^b h(\tau, u) f(x(g(\tau, u)))\, du = 0, \qquad \tau \geqslant \tau_0,$$

with $v(\tau) = x^{\alpha}(\tau) + p(\tau)x(\sigma(\tau))$ in the canonical case.

In this article, we are concerning with the oscillatory behavior of solutions of Eq. (1.1) by applying the Riccati, generalized Riccati transformations, integral averaging technique of Philos type and the theory of comparison. In this work we shall consider both canonical and noncanonical cases.

2. Auxiliary lemmas

In this section, we outline some lemmas needed for our results.

Lemma 2.1 ([20]). *Let $y(\tau)$ be a positive and m-times differentiable function on an interval $[\tau_0, \infty)$, with non-positive mth derivative $y^{(m)}(\tau)$, which is not identically zero on any interval $[\tau_1, \infty)$, $\tau_1 \geqslant \tau_0$, and such that $y^{(m-1)}(\tau)y^{(m)}(\tau) \leqslant 0$. Then there exist constants $0 < \xi < 1$ and $P > 0$ such that $y'(\xi\tau) \geqslant P\tau^{m-2}y^{(m-1)}(\tau)$ for all sufficient large τ.*

Lemma 2.2 ([2]). *Let $y^{(m)}(\tau)$ be of fixed sign and $y^{(m-1)}(\tau)y^{(m)}(\tau) \leqslant 0$, for all $\tau \geqslant \tau_0$. If $\lim_{\tau\to\infty} y(\tau) \neq 0$, then for every $\xi \in (0, 1)$ there may exist $\tau_{\xi} \geqslant \tau$ such that $y(\tau) \geqslant \frac{\xi}{(m-1)!}\tau^{m-1}\left|y^{(m-1)}(\tau)\right|$ for $\tau \geqslant \tau_{\xi}$.*

Lemma 2.3 (Philos, [21]). *Let $y \in C^{I}([\tau_0,\infty), R^{+})$. If $y^{(I)}(\tau)$ is eventually of one sign for all large τ, then there exist a $\tau_1 \geqslant \tau_0$ and an integer $i, 0 \leqslant i \leqslant I$ with $I+i$ even for $y^{(I)}(\tau) \geqslant 0$, or $I+i$ odd for $y^{(I)}(\tau) \leqslant 0$ such that*

$$i > 0 \text{ yields } y^{(j)}(\tau) > 0 \text{ for } \tau \geqslant \tau_1, j = 0, 1, \ldots, i-1, \text{ and}$$
$$i \leqslant I-1 \text{ yields } (-1)^{i+j} y^{(j)}(\tau) > 0 \text{ for } \tau \geqslant \tau_1, j = i, i+1, \ldots, I-1.$$

Lemma 2.4. *Suppose that $x(\tau)$ is a solution of (1.1), which is eventually positive. Assume that $v'(\tau) > 0$. Then*

$$\left(a(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma}\right)' \leqslant -q_1(\tau)\, v^{\beta}(g_*(\tau)), \tag{2.1}$$

where $q_1(\tau) = K(1-p)^{\beta} \int_a^b q(\tau,s)\, ds.$

Proof. The proof is very similar to the proof of Lemma 2.4 of [6]. □

3. Oscillation results

In this section, we begin with the canonical case $\delta(\tau_0) = \infty$.

Theorem 3.1. *If there exist $\vartheta(\tau) \in C^1([\tau_0,\infty),(0,\infty))$, $b(\tau) \in C^1([\tau_0.\infty),[0,\infty))$, $\kappa \in (0,1)$, and $\epsilon, C > 0, \beta \geqslant \gamma \geqslant 1$ such that*

$$\int_{\tau_0}^{\infty} \left(\Omega(\xi) - a(\xi)\vartheta(\xi) \frac{\left[\frac{\vartheta'(\xi)}{\vartheta(\xi)} + (\gamma+1)\kappa\epsilon g_*'(\xi) g_*^{n-2}(\xi) b^{\frac{1}{\gamma}}(\xi)\right]^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left[\kappa\epsilon g_*'(\xi) g_*^{n-2}(\xi)\right]^{\gamma}} \right) d\xi = \infty, \tag{3.1}$$

where $\Omega(\tau) = \vartheta(\tau) \left(C^{\beta-\gamma} q_1(\tau) - [a(\tau) b(\tau)]'\right) + \kappa\epsilon g_*'(\tau) g_*^{n-2}(\tau) a(\tau)\vartheta(\tau) b^{1+\frac{1}{\gamma}}(\tau)$, *then Eq.* (1.1) *is oscillatory.*

Proof. For the sake of contradiction, suppose that $x(\tau)$ is an eventually positive solution of (1.1) on $[\tau_0,\infty)$. Then there exists $\tau_1 \geqslant \tau_0$ such that $x(\sigma(\tau,\mu)) > 0$ and $x(g(\tau,s)) > 0$ for $\tau \geqslant \tau_1$. Hence we deduce that $v(\tau) > 0$ for $\tau \geqslant \tau_1$, and

$$\left(a(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma}\right)' = -\int_a^b q(\tau,s) f(x(g(\tau,s)))\, ds \leqslant 0.$$

Using Lemma 2.3, we obtain

$$v(\tau) > 0, v'(\tau) > 0, v^{(n-1)}(\tau) \geqslant 0 \text{ and } v^{(n)}(\tau) \leqslant 0, \tag{3.2}$$

for $\tau \geqslant \tau_1$, and by using Lemma 2.4, we obtain (2.1). Define

$$w(\tau) = \vartheta(\tau)\left[\frac{a(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma}}{v^{\gamma}(\kappa g_*(\tau))} + a(\tau) b(\tau)\right]. \tag{3.3}$$

It is clear by (3.3) that $w(\tau) > 0$ for $\tau \geqslant \tau_1$, and

$$\begin{aligned} w'(\tau) = {} & \frac{\vartheta'(\tau)}{\vartheta(\tau)} w(\tau) + \vartheta(\tau)[a(\tau) b(\tau)]' + \vartheta(\tau) \frac{\left(a(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma}\right)'}{v^{\gamma}(\kappa g_*(\tau))} \\ & - \vartheta(\tau) \frac{\gamma\kappa a(\tau) g_*'(\tau)\left(v^{(n-1)}(\tau)\right)^{\gamma} v'(\kappa g_*(\tau))}{v^{\gamma+1}(\kappa g_*(\tau))}, \end{aligned}$$

i.e.,

$$w'(\tau) \leqslant \frac{\vartheta'(\tau)}{\vartheta(\tau)} w(\tau) + \vartheta(\tau)[a(\tau)b(\tau)]' - \vartheta(\tau)\frac{q_1(\tau)\nu^{\beta}(g_*(\tau))}{\nu^{\gamma}(\kappa g_*(\tau))} - \vartheta(\tau)\frac{\gamma\kappa a(\tau)g_*'(\tau)\left(\nu^{(n-1)}(\tau)\right)^{\gamma}\nu'(\kappa g_*(\tau))}{\nu^{\gamma+1}(\kappa g_*(\tau))}.$$

By Lemma 2.1, and since $\nu^{(n-1)}(\tau)$ is nonincreasing, we have

$$\nu'(\kappa g_*(\tau)) \geqslant \epsilon g_*^{n-2}(\tau)\nu^{(n-1)}(g_*(\tau)) \geqslant \epsilon g_*^{n-2}(\tau)\nu^{(n-1)}(\tau). \tag{3.4}$$

Since $\nu(\tau)$ is positive and increasing, we have

$$\nu(g_*(\tau)) \geqslant \nu(\kappa g_*(\tau)).$$

Moreover, since there may exist a positive constant C such that $\nu(g_*(\tau)) \geqslant C$, then

$$w'(\tau) \leqslant \frac{\vartheta'(\tau)}{\vartheta(\tau)} w(\tau) + \vartheta(\tau)[a(\tau)b(\tau)]' - C^{\beta-\gamma}\vartheta(\tau)q_1(\tau) - \vartheta(\tau)\frac{\gamma\kappa\epsilon a(\tau)g_*'(\tau)\left(\nu^{(n-1)}(\tau)\right)^{\gamma+1}g_*^{n-2}(\tau)}{\nu^{\gamma+1}(\kappa g_*(\tau))},$$

and from the definition of w, we have

$$\frac{\nu^{(n-1)}(\tau)}{\nu(\kappa g_*(\tau))} = \frac{1}{a^{\frac{1}{\gamma}}(\tau)}\left[\frac{w(\tau)}{\vartheta(\tau)} - [a(\tau)b(\tau)]\right]^{\frac{1}{\gamma}},$$

then

$$w'(\tau) \leqslant \frac{\vartheta'(\tau)}{\vartheta(\tau)} w(\tau) + \vartheta(\tau)[a(\tau)b(\tau)]' - C^{\beta-\gamma}\vartheta(\tau)q_1(\tau) - \gamma\kappa\epsilon g_*'(\tau)g_*^{n-2}(\tau)\frac{\vartheta(\tau)}{a^{\frac{1}{\gamma}}(\tau)}\left(\frac{w(\tau)}{\vartheta(\tau)} - [a(\tau)b(\tau)]\right)^{\frac{\gamma+1}{\gamma}}. \tag{3.5}$$

Now we define

$$\Phi = \frac{w(\tau)}{\vartheta(\tau)} \quad \text{and} \quad \Psi = a(\tau)b(\tau).$$

And by using the inequality (see [22])

$$\Phi^{1+\frac{1}{\gamma}} - (\Phi-\Psi)^{1+\frac{1}{\gamma}} \leqslant \Psi^{\frac{1}{\gamma}}\left[\left(1+\frac{1}{\gamma}\right)\Phi - \frac{1}{\gamma}\Psi\right], \qquad \Phi\Psi \geqslant 0, \quad \gamma \geqslant 1,$$

we have

$$\left(\frac{w(\tau)}{\vartheta(\tau)} - [a(\tau)b(\tau)]\right)^{\frac{\gamma+1}{\gamma}} \geqslant \left[\frac{w(\tau)}{\vartheta(\tau)}\right]^{1+\frac{1}{\gamma}} + \frac{1}{\gamma}[a(\tau)b(\tau)]^{1+\frac{1}{\gamma}} - \left(1+\frac{1}{\gamma}\right)\frac{[a(\tau)b(\tau)]^{\frac{1}{\gamma}}}{\vartheta(\tau)} w(\tau). \tag{3.6}$$

Using (3.5) and (3.6), for $\tau \geqslant \tau_1$, we have

$$w'(\tau) \leqslant \frac{\vartheta'(\tau)}{\vartheta(\tau)} w(\tau) + \vartheta(\tau)[a(\tau)b(\tau)]' - C^{\beta-\gamma}\vartheta(\tau)q_1(\tau) + \gamma\kappa\epsilon g_*'(\tau)g_*^{n-2}(\tau)\frac{\vartheta(\tau)}{a^{\frac{1}{\gamma}}(\tau)}\left[\left(1+\frac{1}{\gamma}\right)\frac{[a(\tau)b(\tau)]^{\frac{1}{\gamma}}}{\vartheta(\tau)} w(\tau) - \frac{1}{\gamma}[a(\tau)b(\tau)]^{1+\frac{1}{\gamma}} - \frac{w^{1+\frac{1}{\gamma}}(\tau)}{\vartheta^{1+\frac{1}{\gamma}}(\tau)}\right].$$

Then

$$w'(\tau) \leqslant \vartheta(\tau) \left([a(\tau) b(\tau)]' - C^{\beta-\gamma} q_1(\tau) \right) + \left[\frac{\vartheta'(\tau)}{\vartheta(\tau)} + (\gamma+1) \kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau) b^{\frac{1}{\gamma}}(\tau) \right] w(\tau)$$
$$- \frac{\gamma \kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau)}{a^{\frac{1}{\gamma}}(\tau) \vartheta^{\frac{1}{\gamma}}(\tau)} w^{1+\frac{1}{\gamma}}(\tau) - \kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau) a(\tau) \vartheta(\tau) b^{1+\frac{1}{\gamma}}(\tau).$$

Now letting

$$T = \frac{\vartheta'(\tau)}{\vartheta(\tau)} + (\gamma+1) \kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau) b^{\frac{1}{\gamma}}(\tau), \quad R = \frac{\gamma \kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau)}{a^{\frac{1}{\gamma}}(\tau) \vartheta^{\frac{1}{\gamma}}(\tau)}, \text{ and } Y = w(\tau),$$

and using the inequality (see [3])

$$TY - RY^{\frac{\kappa+1}{\kappa}} \leqslant \frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}} \frac{T^{\kappa+1}}{R^\kappa}, R > 0 \text{ and } T \text{ are constants,} \tag{3.7}$$

then we have

$$w'(\tau) \leqslant -\Omega(\tau) + a(\tau) \vartheta(\tau) \frac{\left[\frac{\vartheta'(\tau)}{\vartheta(\tau)} + (\gamma+1) \kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau) b^{\frac{1}{\gamma}}(\tau) \right]^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left[\kappa \epsilon g_*'(\tau) g_*^{n-2}(\tau) \right]^\gamma}. \tag{3.8}$$

Integrating (3.8) from τ_1 to τ, we obtain

$$\int_{\tau_1}^{\tau} \left(\Omega(\xi) - a(\xi) \vartheta(\xi) \frac{\left[\frac{\vartheta'(\xi)}{\vartheta(\xi)} + (\gamma+1) \kappa \epsilon g_*'(\xi) g_*^{n-2}(\xi) b^{\frac{1}{\gamma}}(\xi) \right]^{\gamma+1}}{(\gamma+1)^{\gamma+1} \left[\kappa \epsilon g_*'(\xi) g_*^{n-2}(\xi) \right]^\gamma} \right) d\xi \leqslant w(\tau_1) - w(\tau),$$

which contradicts (3.1) and so the proof is completed. □

Theorem 3.2. *If there exist* $\xi(\tau) \in C^1([\tau_0, \infty), (0, \infty))$ *and a positive constant* L *such that* $\beta \leqslant \gamma$. *If*

$$\int_{\tau_0}^{\infty} \left[\xi(u) q_1(u) - \frac{[\xi'(u)]^2 a(u)}{4L \xi(u) g_*'(u) g_*^{(n-1)\beta-1}(u)} \right] du = \infty, \tag{3.9}$$

then equation (1.1) *oscillates.*

Proof. For the sake of contradiction, suppose that $x(\tau)$ is an eventually positive solution of (1.1) on $[\tau_0, \infty)$. Then there exists $\tau_1 \geqslant \tau_0$ such that $x(\sigma(\tau, \mu)) > 0$ and $x(g(\tau, s)) > 0$ for $\tau \geqslant \tau_1$. As in the proof of Theorem 3.1, we obtain (3.2), and also (2.1) is satisfied. Now letting

$$\phi(\tau) = \frac{\xi(\tau) a(\tau) \left[v^{(n-1)}(\tau) \right]^\gamma}{[v(\kappa g_*(\tau))]^\beta},$$

then $\phi(\tau) > 0$, and

$$\phi'(\tau) = \frac{\xi'(\tau)}{\xi(\tau)} \phi(\tau) + \frac{\xi(\tau) \left[a(\tau) \left[v^{(n-1)}(\tau) \right]^\gamma \right]'}{[v(\kappa g_*(\tau))]^\beta}$$
$$- \xi(\tau) \frac{\beta \kappa a(\tau) g_*'(\tau) \left(v^{(n-1)}(\tau) \right)^\gamma v^{\beta-1}(\kappa g_*(\tau)) v'(\kappa g_*(\tau))}{v^{2\beta}(\kappa g_*(\tau))} \tag{3.10}$$
$$\leqslant \frac{\xi'(\tau)}{\xi(\tau)} \phi(\tau) - \frac{\xi(\tau) q_1(\tau) [v(g_*(\tau))]^\beta}{[v(\kappa g_*(\tau))]^\beta} - \xi(\tau) \frac{\beta \kappa a(\tau) g_*'(\tau) \left(v^{(n-1)}(\tau) \right)^\gamma v^{\beta-1}(\kappa g_*(\tau)) v'(\kappa g_*(\tau))}{v^{2\beta}(\kappa g_*(\tau))}.$$

Now from Lemma 2.2, for every $\lambda \in (0,1)$, we have

$$\nu(\kappa g_*(\tau)) \geqslant \frac{\lambda \kappa^{n-1} g_*^{n-1}(\tau)\, \nu^{(n-1)}(\kappa g_*(\tau))}{n-1!}, \quad \tau \geqslant \tau_\lambda \geqslant \tau_1. \tag{3.11}$$

Now since ν is increasing and as in Theorem 3.1, we have (3.4). Now using (3.4), (3.10), and (3.11), we obtain

$$\begin{aligned}\phi'(\tau) \leqslant{} & \frac{\xi'(\tau)}{\xi(\tau)}\phi(\tau) - \xi(\tau) q_1(\tau) \\ & - \frac{\beta \epsilon \kappa \left[\frac{\lambda\kappa^{n-1}}{n-1!}\right]^{\beta-1} a(\tau)\,\xi(\tau)\, g_*'(\tau)\, g_*^{n-2}(\tau) \left[g_*^{n-1}(\tau)\right]^{\beta-1} \left(\nu^{(n-1)}(\tau)\right)^{2\gamma}}{\nu^{2\beta}(\kappa g_*(\tau)) \left(\nu^{(n-1)}(\tau)\right)^{\gamma-\beta}}.\end{aligned}$$

Moreover, since $\nu^{(n-1)}(\tau)$ is a positive non-increasing function then there exists a positive integer L^* such that $\nu^{(n-1)}(\tau) \leqslant L^*$. Then

$$\phi'(\tau) \leqslant \frac{\xi'(\tau)}{\xi(\tau)}\phi(\tau) - \xi(\tau) q_1(\tau) - \frac{L g_*'(\tau)\, g_*^{(n-1)\beta-1}(\tau)}{\xi(\tau)\, a(\tau)}\phi^2(\tau),$$

where $L = (L^*)^{\beta-\gamma} \epsilon\beta\kappa \left[\frac{\lambda\kappa^{n-1}}{n-1!}\right]^{\beta-1}$, by completing square, we obtain

$$\phi'(\tau) \leqslant -\xi(\tau) q_1(\tau) + \frac{[\xi'(\tau)]^2 a(\tau)}{4L\xi(\tau)\, g_*'(\tau)\, g_*^{(n-1)\beta-1}}.$$

By integrating from τ_2 to τ, we get

$$0 < \phi(\tau) \leqslant \phi(\tau_2) - \int_{\tau_2}^{\tau} \left[\xi(s) q_1(s) - \frac{[\xi'(s)]^2 a(s)}{4L\xi(s)\, g_*'(s)\, g_*^{(n-1)\beta-1}(s)}\right] ds.$$

This is a contradiction with (3.9), and so the proof is completed. □

Theorem 3.3. *If the differential equation*

$$z'(\tau) + q_1(\tau)\left[\frac{\varepsilon g_*(\tau)}{n-1!\,(a(g_*(\tau)))^{\frac{1}{\gamma}}}\right]^{\beta} z^{\frac{\beta}{\gamma}}(g_*(\tau)) = 0 \tag{3.12}$$

is oscillatory for some constant $\varepsilon \in (0,1)$, then equation (1.1) is oscillatory.

Proof. For the sake of contradiction, suppose that $x(\tau)$ is an eventually positive solution of (1.1) on $[\tau_0,\infty)$. Then there exists such that $x(\sigma(\tau,\mu)) > 0$ and $x(g(\tau,s)) > 0$ for $\tau \geqslant \tau_1 \geqslant \tau_0$. As in the proof of Theorem 3.1, we obtain (3.2). By using Lemma 2.2, we find

$$\nu(\tau) \geqslant \frac{\varepsilon \tau^{n-1} \nu^{(n-1)}(\tau)}{n-1!}.$$

Thus, from (2.1), we have

$$\left(a(\tau)\left(\nu^{(n-1)}(\tau)\right)^{\gamma}\right)' + q_1(\tau)\left[\frac{\varepsilon (g_*(\tau))^{n-1}}{[a(g_*(\tau))]^{\frac{1}{\gamma}}\, n-1!}\right]^{\beta} \left([a(g_*(\tau))]^{\frac{1}{\gamma}} \nu^{(n-1)}(g_*(\tau))\right)^{\beta} \leqslant 0,$$

we see that $z(\tau) = a(\tau)\left(\nu^{(n-1)}(\tau)\right)^{\gamma}$ is a positive solution of the differential inequality

$$z'(\tau) + q_1(\tau)\left[\frac{\varepsilon (g_*(\tau))^{n-1}}{[a(g_*(\tau))]^{\frac{1}{\gamma}}\, n-1!}\right]^{\beta} (z(g_*(\tau)))^{\frac{\beta}{\gamma}} \leqslant 0.$$

Using [20, Corollary 1], we see that (3.12) has a positive solution, this is a contradiction and so the proof is completed. □

4. The noncanonical case $\delta(\tau_0) < \infty$

Definition 4.1. Let $D = \{(\tau, s) \in R^2 : \tau \geqslant s \geqslant \tau_0\}$, $\varpi, \varpi^* \in C(D, R)$, $\theta(\tau) \in C^1([\tau_0, \infty), (0, \infty))$ such that $\varpi(\tau, \tau) = 0, \tau \geqslant \tau_0, \varpi(\tau, s) > 0, \tau > s \geqslant \tau_0$ and ϖ has nonpositive continuous partial derivative $\frac{\partial \varpi}{\partial s}$ satisfying for all sufficient large $\tau_1 \geqslant \tau_0$,

$$\frac{\partial \varpi(\tau, s)}{\partial s} + \frac{\theta'(\tau)}{\theta(\tau)} \varpi(\tau, s) = -\frac{\varpi^*(\tau, s)}{\theta(\tau)} [\varpi(\tau, s)]^{\frac{\gamma}{\gamma+1}}.$$

Theorem 4.2. *Suppose that (3.9) holds. If there exist positive constants $M, \lambda \in (0,1), \beta \leqslant \gamma$ such that*

$$\limsup_{\tau \to \infty} \int_{\tau_0}^{\tau} \left[\varpi(\tau, h) M^{\beta-\gamma} \theta(h) q_1(h) \left[\frac{\lambda}{(n-2)!} g_*^{n-2}(h) \right]^{\beta} - \frac{a(\xi) [\varpi^*(\tau, h)]^{(\gamma+1)}}{(\gamma+1)^{\gamma+1} \theta^{\gamma}(h)} \right] dh > 0, \tag{4.1}$$

and either

$$\int_{\tau_0}^{\infty} \delta(s) ds = \infty, \text{ or} \tag{4.2}$$

$$\int_{\tau_0}^{\infty} \int_{u}^{\infty} \delta(s) ds du = \infty, \tag{4.3}$$

then Eq. (1.1) is oscillatory.

Proof. For the sake of contradiction, suppose that x is an eventually positive solution of (1.1). Now from the definition of $\nu(\tau)$ we deduce that $\nu(\tau) > 0$ for $\tau \geqslant \tau_1$ and

$$\left(a(\tau) \left(\nu^{(n-1)}(\tau) \right)^{\gamma} \right)' = -\int_a^b q(\tau, s) f(x(g(\tau, s))) ds \leqslant 0.$$

Using Lemma 2.3 there exist three possible cases for $\tau \geqslant \tau_1$ large enough.

(S_1) $\nu(\tau) > 0, \nu'(\tau) > 0, \nu^{(n-1)}(\tau) > 0, \nu^{(n)}(\tau) \leqslant 0, \left(a(\tau) \left(\nu^{(n-1)}(\tau) \right)^{\gamma} \right)' \leqslant 0$;

(S_2) $\nu(\tau) > 0, \nu'(\tau) > 0, \nu^{(n-2)}(\tau) > 0, \nu^{(n-1)}(\tau) < 0, \left(a(\tau) \left(\nu^{(n-1)}(\tau) \right)^{\gamma} \right)' \leqslant 0$;

(S_3) $\nu(\tau) > 0, \nu^{(i)}(\tau) < 0, \nu^{(i+1)}(\tau) > 0$, for every odd integer $i \in \{1, 2, \ldots, n-3\}$ and $\nu^{(n-1)}(\tau) < 0$, $\left(a(\tau) \left(\nu^{(n-1)}(\tau) \right)^{\gamma} \right)' \leqslant 0$.

Letting (S_1) hold, then it follows by Theorem 3.2 that every solution of (1.1) oscillates when the condition (3.9) holds. Assume that (S_2) holds. Define the function

$$\Phi(\tau) = \theta(\tau) \frac{a(\tau) \left[\nu^{(n-1)}(\tau) \right]^{\gamma}}{\left[\nu^{(n-2)}(\tau) \right]^{\gamma}}, \quad \tau \geqslant \tau_0. \tag{4.4}$$

Then $\Phi(\tau) < 0$, and

$$\Phi'(\tau) = \frac{\theta'(\tau)}{\theta(\tau)} \Phi(\tau) + \theta(\tau) \frac{\left[a(\tau) \left[\nu^{(n-1)}(\tau) \right]^{\gamma} \right]'}{\left[\nu^{(n-2)}(\tau) \right]^{\gamma}} - \frac{\gamma \theta(\tau) a(\tau) \left[\nu^{(n-1)}(\tau) \right]^{\gamma+1}}{\left[\nu^{(n-2)}(\tau) \right]^{\gamma+1}}.$$

This with (2.1) and (4.4) leads to

$$\Phi'(\tau) \leqslant \frac{\theta'(\tau)}{\theta(\tau)} \Phi(\tau) - \theta(\tau) \frac{q_1(\tau) \nu^{\beta}(g_*(\tau))}{\left[\nu^{(n-2)}(\tau) \right]^{\gamma}} - \frac{\gamma \Phi^{\frac{\alpha+1}{\alpha}}(\tau)}{a^{\frac{1}{\gamma}}(\tau) \theta^{\frac{1}{\gamma}}(\tau)},$$

i.e.,

$$\Phi'(\tau) \leqslant \frac{\theta'(\tau)}{\theta(\tau)}\Phi(\tau) - \theta(\tau)\frac{q_1(\tau)\nu^{\beta}(g_*(\tau))\left[\nu^{(n-2)}(g_*(\tau))\right]^{\gamma}}{\left[\nu^{(n-2)}(g_*(\tau))\right]^{\beta}\left[\nu^{(n-2)}(\tau)\right]^{\gamma}}\left[\nu^{(n-2)}(g_*(\tau))\right]^{\beta-\gamma} - \frac{\alpha\Phi^{\frac{\gamma+1}{\gamma}}(\tau)}{a^{\frac{1}{\gamma}}(\tau)\,\theta^{\frac{1}{\gamma}}(\tau)}.$$

Since $\nu^{(n-2)}(\tau)$ is positive and decreasing, then $\nu^{(n-2)}(g_*(\tau)) > \nu^{(n-2)}(\tau)$, and there may exist a positive constant M such that $\nu^{(n-2)}(g_*(\tau)) \leqslant M$,

$$\Phi'(\tau) \leqslant \frac{\theta'(\tau)}{\theta(\tau)}\Phi(\tau) - M^{\beta-\gamma}\theta(\tau)\,q_1(\tau)\frac{\nu^{\beta}(g_*(\tau))}{\left[\nu^{(n-2)}(g_*(\tau))\right]^{\beta}} - \frac{\gamma\Phi^{\frac{\gamma+1}{\gamma}}(\tau)}{a^{\frac{1}{\gamma}}(\tau)\,\theta^{\frac{1}{\gamma}}(\tau)}.$$

Using Lemma 2.2, we have $\nu(g_*(\tau)) \geqslant \frac{\lambda}{n-2!}g_*^{n-2}(\tau)\,\nu^{(n-2)}(g_*(\tau))$. Then

$$\Phi'(\tau) \leqslant \frac{\theta'(\tau)}{\theta(\tau)}\Phi(\tau) - M^{\beta-\gamma}\theta(\tau)\,q_1(\tau)\left[\frac{\lambda}{(n-2)!}g_*^{n-2}(\tau)\right]^{\beta} - \frac{\gamma\Phi^{\frac{\gamma+1}{\gamma}}(\tau)}{a^{\frac{1}{\gamma}}(\tau)\,\theta^{\frac{1}{\gamma}}(\tau)}.$$

Multiplying by $\varpi(\tau,s)$ and integrating from τ_1 to τ, we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau}\varpi(\tau,\xi)\,M^{\beta-\gamma}\theta(\xi)\,q_1(\xi)\left[\frac{\lambda}{(n-2)!}g_*^{n-2}(\xi)\right]^{\beta}d\xi\\
&\quad\leqslant \varpi(\tau,\tau_1)\,\Phi(\tau_1) + \int_{\tau_1}^{\tau}\left[\frac{\partial\varpi(\tau,\xi)}{\partial\xi} + \frac{\theta'(\xi)}{\theta(\xi)}\varpi(\tau,\xi)\right]\Phi(\xi)\,d\xi - \gamma\int_{\tau_1}^{\tau}\varpi(\tau,\xi)\frac{\Phi^{\frac{\gamma+1}{\gamma}}(\xi)}{a^{\frac{1}{\gamma}}(\xi)\,\theta^{\frac{1}{\gamma}}(\xi)}d\xi,\\
&\quad= \varpi(\tau,\tau_1)\,\Phi(\tau_1) - \int_{\tau_1}^{\tau}\frac{\varpi^*(\tau,\xi)}{\theta(\xi)}\left[\varpi(\tau,\xi)\right]^{\frac{\gamma}{\gamma+1}}\Phi(\xi)\,d\xi - \gamma\int_{\tau_1}^{\tau}\varpi(\tau,\xi)\frac{\Phi^{\frac{\gamma+1}{\gamma}}(\xi)}{a^{\frac{1}{\gamma}}(\xi)\,\theta^{\frac{1}{\gamma}}(\xi)}d\xi.
\end{aligned}$$

Set

$$R = \frac{\gamma\varpi(\tau,\xi)}{a^{\frac{1}{\gamma}}(\xi)\,\theta^{\frac{1}{\gamma}}(\xi)},\quad T = \frac{\varpi^*(\tau,\xi)}{\theta(\xi)}\left[\varpi(\tau,\xi)\right]^{\frac{\gamma}{\gamma+1}},\ \text{and } Y = -\Phi(\xi).$$

Then by using the inequality (3.7), we have

$$\frac{\varpi^*(\tau,\xi)}{\theta(\eta)}\left[\varpi(\tau,\xi)\right]^{\frac{\gamma}{\gamma+1}}(-\Phi(\xi)) - \frac{\gamma\varpi(\tau,\xi)\,(-\Phi(\xi))^{\frac{\gamma+1}{\gamma}}}{a^{\frac{1}{\gamma}}(\xi)\,\theta^{\frac{1}{\gamma}}(\xi)} \leqslant \frac{1}{(\gamma+1)^{\gamma+1}}\left[\varpi^*(\tau,\xi)\right]^{(\gamma+1)}\frac{a(\xi)}{\theta^{\gamma}(\xi)}.$$

Hence

$$\begin{aligned}
&\int_{\tau_1}^{\tau}\left[\varpi(\tau,\xi)\,M^{\beta-\gamma}\theta(\xi)\,q_1(\xi)\left[\frac{\lambda}{(n-2)!}g_*^{n-2}(\xi)\right]^{\beta} - \frac{1}{(\gamma+1)^{\gamma+1}}\left[\varpi^*(\tau,\xi)\right]^{(\gamma+1)}\frac{a(\xi)}{\theta^{\gamma}(\xi)}\right]d\xi\\
&\quad\leqslant \varpi(\tau,\tau_1)\,\Phi(\tau_1) < 0,
\end{aligned}$$

which contradicts (4.1). For case (S_3) assume that $\nu(\tau)$ satisfies (S_3). Since $\left(a\left(\nu^{(n-1)}\right)^{\gamma}\right)' \leqslant 0$, we have for $s \geqslant \tau \geqslant \tau_1$,

$$a^{\frac{1}{\gamma}}(s)\,\nu^{(n-1)}(s) \leqslant a^{\frac{1}{\gamma}}(\tau)\,\nu^{(n-1)}(\tau).$$

Multiplying by $a^{-\frac{1}{\gamma}}(s)$ and integrating the resulting inequality from τ to χ we get

$$\nu^{(n-2)}(\chi) \leqslant \nu^{(n-2)}(\tau) + a^{\frac{1}{\gamma}}(\tau)\,\nu^{(n-1)}(\tau)\int_{\tau}^{\chi}a^{-\frac{1}{\gamma}}(s)\,ds.$$

Letting $\chi \to \infty$, we get

$$0 \leqslant v^{(n-2)}(\tau) + a^{\frac{1}{\gamma}}(\tau)\, v^{(n-1)}(\tau)\, \delta(\tau),$$

which yields

$$v^{(n-2)}(\tau) \geqslant -a^{\frac{1}{\gamma}}(\tau)\, v^{(n-1)}(\tau)\, \delta(\tau).$$

Hence there may exist a constant $m > 0$ such that

$$v^{(n-2)}(\tau) \geqslant m\delta(\tau). \tag{4.5}$$

Integrating (4.5) from τ_1 to τ, we get

$$v^{(n-3)}(\tau) - v^{(n-3)}(\tau_1) \geqslant m \int_{\tau_1}^{\tau} \delta(s)\, ds.$$

This yields

$$-v^{(n-3)}(\tau_1) \geqslant m \int_{\tau_1}^{\tau} \delta(s)\, ds,$$

which contradicts (4.2). Now we consider the case when (4.2) is not satisfied. Then integrating (4.5) from τ to ∞, we obtain

$$-v^{(n-3)}(\tau) \geqslant m \int_{\tau}^{\infty} \delta(s)\, ds.$$

Integrating again from τ_1 to τ, we have

$$-v^{(n-4)}(\tau) + v^{(n-4)}(\tau_1) \geqslant m \int_{\tau_1}^{\tau} \int_{u}^{\infty} \delta(s)\, ds du.$$

This implies that

$$v^{(n-4)}(\tau_1) \geqslant m \int_{\tau_1}^{\tau} \int_{u}^{\infty} \delta(s)\, ds du,$$

which contradicts (4.3), and this completes the proof . □

Remark 4.3. Theorem 4.2 remains true if we used the condition (3.12) of Theorem 3.3 instead of (3.9).

5. Examples and conclusion

Example 5.1. Consider the differential equation

$$\left[\tau \left(x(\tau) + \int_{\frac{1}{2}}^{1} \frac{\mu}{\tau} x\left(\frac{\tau+\mu}{3}\right) d\mu\right)'''\right]' + \int_0^1 \frac{8192 q_0 u}{125\tau^3} x^3 \left(\frac{\tau+u}{2}\right) du = 0, \quad \tau \geqslant 1, \tag{5.1}$$

where $n = 4, q_0 > 0, \gamma = 1, \beta = 3, c = \frac{1}{2}, d = 1, a = 0, b = 1, K = 1, a(\tau) = \tau, 0 \leqslant \int_c^d p(\tau,\mu)\, d\mu = \int_{\frac{1}{2}}^1 \frac{\mu}{\tau} d\mu = \frac{3}{8\tau} < \frac{3}{8} < 1$. Choosing $p = \frac{3}{8}$, $\vartheta(\tau) = \tau^2$, $b(\tau) = \frac{1}{\tau^3}$, then $q_1(\tau) = \left(\frac{5}{8}\right)^3 \int_0^1 \frac{8192 q_0 u}{125\tau^3} du = \frac{8q_0}{\tau^3}$,

$$\Omega(\tau) = \vartheta(\tau)\left(-[a(\tau)\, b(\tau)]' + C^{\beta-\gamma} q_1(\tau)\right) + \kappa\epsilon g_*'(\tau)\, g_*^{n-2}(\tau)\, a(\tau)\, \vartheta(\tau)\, b^{1+\frac{1}{\gamma}}(\tau),$$
$$\Omega(\tau) = \left[2 + \frac{k\epsilon}{8} + 8q_0 C^2\right] \frac{1}{\tau},$$

and

$$\int_{\tau_0}^{\infty}\left(\Omega(u)-a(u)\vartheta(u)\frac{\left[\frac{\vartheta'(u)}{\vartheta(u)}+(\gamma+1)\kappa\epsilon g_*'(u)g_*^{n-2}(u)b^{\frac{1}{\gamma}}(u)\right]^{(\gamma+1)}}{(\gamma+1)^{(\gamma+1)}\left[\kappa\epsilon g_*'(u)g_*^{n-2}(u)\right]^{\gamma}}\right)du$$
$$=\int_{\tau_0}^{\infty}\left(\left[2+\frac{\epsilon\kappa}{8}+8q_0C^2\right]\frac{1}{u}-\frac{(8+\epsilon\kappa)^2}{8\epsilon\kappa}\frac{1}{u}\right)du=\infty.$$

If $C^2q_0>\frac{1}{\epsilon\kappa}, \kappa\in(0,1), \epsilon, C>0$. From Theorem 3.1, it follows that Eq. (5.1) oscillates.

Example 5.2. Consider the differential equation

$$\left[\tau\left(x(\tau)+\int_1^2\frac{\mu}{3\tau^2}x\left(\frac{\tau+\mu}{3}\right)d\mu\right)'''\right]'+\int_0^1\frac{q_0u}{\tau^{\frac{4}{3}}}x\left(\frac{\tau+u}{4}\right)du=0,\quad \tau\geqslant 1, \tag{5.2}$$

where $n=4, q_0>0, \gamma=1, \beta=1, c=1, d=2, a=0, b=1, K=1, a(\tau)=\tau$, $0\leqslant\int_c^d p(\tau,\mu)\,d\mu=\int_1^2\frac{\mu}{3\tau^2}d\mu=\frac{1}{2\tau^2}<\frac{1}{2}<1$. Choosing $p=\frac{1}{2}, \xi(\tau)=\tau^{\frac{1}{3}}$, then

$$\int_{\tau_0}^{\infty}\left[\xi(u)q_1(u)-\frac{[\xi'(u)]^2a(u)}{4L\xi(u)g_*'(u)g_*^{(n-1)\beta-1}(u)}\right]du=\int_1^{\infty}\left[\frac{q_0}{4u}-\frac{16}{9Lu^{\frac{8}{3}}}\right]du=\infty.$$

From Theorem 3.2, we deduce that every solution of (5.2) oscillates.

Conclusion 5.3. In this paper, we consider a class of even-order neutral differential equations with distributed deviating arguments of the type (1.1) in both cases of canonical case $\delta(\tau_0)=\infty$ and noncanonical case $\delta(\tau_0)<\infty$. We discuss new oscillation criteria using Riccati, generalized Riccati transformations, integral averaging technique of Philos type, and the method of comparison. For interested researchers, we suggest studying (1.1) in the case $\int_{\tau_0}^{\infty}\int_u^{\infty}\delta(s)\,ds\,du<\infty$.

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