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Dynamic behaviour of a single-species nonlinear fishery model with infection: the role of fishing tax and time-dependent market price

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Abstract

Taxation policy for fishing received global consent to protect fisheries from drastic harvesting. Still, it should be applied sustainably for a greater ecological and economic benefit because over-taxation may impair fishers' earnings and reduce the overall societal revenue. The fish disease may alter the system dynamics and reduce the revenue generation from the fishery. This paper proposes a nonlinear bioeconomic harvesting model of a single-species fishery with infection, variable market price, and nonlinear demand to explore taxation's ecological and economic effects. We provide the stability results of the system's different ecological and economic equilibrium points. The analytical conditions for the existence of transcritical bifurcation are also established. The computational results show that the system exhibits three dynamical regimes depending on the fishing tax. Taxation might control intensive harvesting but augment disease spreading and price hiking. Higher regulatory tax may even cause a regime shift, where the system enters into a non-harvesting regime from the harvesting one, causing an ecological and economic imbalance. Using Pontryagin's maximum principle, we decipher that some optimal fishing tax exists for the maximum societal benefit in a disease-induced fishery.

Keywords: Bioeconomic model, variable market price, quadratic demand function, local stability, bifurcation, optimal tax.

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1. Introduction

World fisheries have increased tremendously in the last fifty years due to the high demand for fishery products, the use of sophisticated fishing gear & vessel technology, and growing trade [72]. Global fish production (inland plus marine) has increased from 89.6 million tonnes in 2016 [27] to the highest ever, 96.4 million tonnes in 2018 [28]. Overexploitation has led many fisheries under stress, or its extinction [41, 75]. Different policies have been implemented regionally, nationally, and globally to protect world fisheries and promote sustainable development. To this effect, FAO (Food and Agriculture Organization) introduced the Code of Conduct for Responsible Fisheries (CCRF) in 1995 [29]. CCRF was further

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intensified in 2015 by implementing Sustainable Development Goal (SDG) 14 to conserve, protect, and sustainably use the oceans, seas, and marine resources [31, 69].

Several governing agencies apply many actions to protect overexploitation and preserve marine resources and habitats for sustainable use. For example, creating a marine protected area (MPA) is a well-accepted conservation policy for the fish, fisheries, and marine environment [50]. However, the success of MPA has been questioned. MPA is more likely to improve the biological goals (like increased fish abundance and improved fish habitat) but, in many cases, fails to revamp social benefit [21]. Fishing has a direct effect on the harvested biomass. Fixing a harvesting quota for a particular fish species may protect the species from being overharvested [18, 30]. A fishing license or vessel buy-back policy is another means to reduce overharvesting [20]. Furthermore, a fishing fee or tax is usually considered one of the crucial measures for controlling overharvesting. These regulatory measures help protect fish and fisheries and achieve the SDG 14 targets at large [51]. Policymakers may use the tax revenue earned through such fiscal policy for the socio-economic upliftment of the fishers and the marine ecosystem. Iceland is one of such countries that successfully implemented fishing fees for pelagic and demersal fishes [34].

Infection in fish is ubiquitous and known for a very long period. Fish production and revenue generation may be severely affected due to disease [65, 73]. However, the reason and distribution of fish infection must be better understood, particularly for marine fish [74]. Water pollution is considered one of the significant causes of fish infection in the coastal areas [4, 46, 56]. Some other reasons behind the increasing infection rate are water temperature variation, changes in coastal dynamics, and lack of proper governance [37]. Recently, new and transboundary diseases have augmented epidemiological studies of aquatic fish in the presence of infection [59]. Infection may cause a low level of fish productivity [22]. The economic loss due to the production loss of fish for the disease may be huge despite complimentary price hikes due to short supply. Thailand reported a financial loss of US$ 7.38 billion during 2010-2017 for decreased shrimp production due to episodes of disease [70]. Peterman and Posadas [62] reported a total of 16.9M$ loss in 2016 due to the catfish disease in the east Mississippi catfish industry. Therefore, a global challenge is protecting fish and fishery from diseases and reducing economic loss by maintaining sustainable production.

Modern bioeconomic fishery received global attention as it can give insights into how to deal with the multi-difficulties of fisheries [12, 58] and prescribe suitable protective measures that could be ecologically and economically viable [67]. However, it is shown that a conflict exists between conservation policy and socio-economic objectives [19]. For example, a higher fishing tax may relieve the fish stock from over-harvesting but may jeopardize the livelihood of local fishing people. It is particularly true in underdeveloped countries where fishermen have limited alternatives for their livelihood. Therefore, imposing a fishing tax scientifically and sensibly is essential.

The price of many commodities, like fish, is determined by instantaneous demand and supply in an open market. Demand is an essential tool that enhances market price fluctuation. Price tends to increase if there is a shortfall in supply and vice-versa. The intricacy of demand, tax, and infection plays a role in the fishery system and revenue generation and needs better understood. Using an ecological model for the harvested species with the market-linked price might be more effective in deciding the control measure. Here, we propose and analyze a dynamic model of fish stock in the presence of infection, where harvesting effort depends on the profitability of the fishery. The model also considers a fishing tax on the landed fish, and the market price of fish depends on the difference between instantaneous demand and supply. Our analysis revealed that taxation might control intensive fish harvesting but augment disease spreading and price hiking. Higher regulatory tax may cause a regime shift, where the system enters a non-harvesting regime from the harvesting one. Using the optimal control theory, we show some trade-offs between revenue generation and regulatory tax. The overall societal revenue, defined here as the sum of fishers' income from selling fish plus the tax revenue earned by the regulatory body, is highest at the optimal tax level. However, the individual earnings in these heads are different at different tax levels.

The rest of the study is organized as follows. Section 2 describes the bioeconomic model formulation for a single-species fishery. Section 3 contains the positivity boundedness, the existence of equilibrium points of the model, and their local stability properties. The impacts of variation in the regulatory tax on the equilibrium values are also presented in this section. The existence of some optimal policies is discussed in Section 4. The study ends with a discussion in Section 5.

2. Model construction

Suppose $F(t)$ be the current stock level of a fish and $h(t)$ be the harvesting rate then the fish growth equation may be represented by

$$\frac{dF}{dt} = jF\left(1 - \frac{F}{L}\right) - h(t), \tag{2.1}$$

where j is the intrinsic growth rate of the fish population and L is the environmental carrying capacity.

Many fish harvesting models [16, 17, 66] consider $h(t)$ as a constant and independent of the stock size. We, however, consider here that the harvesting rate follows the catch per unit of effort (CPUE) hypothesis, where harvesting at any time is proportional to the fish biomass of that time [5, 49, 54, 61]. Thus, $h(t) = q_1H(t)F(t)$, where $H(t)$ is the harvesting effort at time t, measured in terms of the number of boats, fishing gears, individuals involved in the fishing; and q_1 is the catchability coefficient, measured in terms of the mesh size of the net, gear sophistication, etc. Then the rate equation (2.1) reads

$$\frac{dF}{dt} = jF\left(1 - \frac{F}{L}\right) - q_1HF.$$

Presume that the fish stock is infected by some parasites, giving rise to two fish sub-populations: a susceptible class, S, and an infected class, I. So, the net fish stock at any time t is $F(t) = S(t) + I(t)$, and at any time t, $S(t) + I(t) \leqslant L$, meaning that the entire fish population never exceeds the environmental carrying capacity. Then the interactive dynamics of the fish population can be represented as

$$\frac{dS}{dt} = j(S+I)\left(1 - \frac{S+I}{L}\right) - fSI - q_1HS, \qquad \frac{dI}{dt} = fSI - \mu I - q_2HI, \tag{2.2}$$

where the rate parameters f, μ, and q_2 represent, respectively, the disease transmission rate, death of the infected fish, and catchability coefficient of the infected fish. It is assumed here that the mixing of susceptible and infected fishes is homogeneous, the disease spreads through horizontal transmission following the density-dependent rule, infected fishes do not recover, harvesting is non-selective, and all biological processes are instantaneous. Since infection may induce morbidity through hypoxia, reduce swimming ability, and the conspicuousness of the infected fish [25, 33], the catchability may be higher for infected fish compared to healthy fish under the same effort, i.e., $q_2 \geqslant q_1$.

The fishing agency assigns more manpower, boats, etc, to harvesting if there is a profit. However, the case will be the opposite if profitability reduces. Therefore, harvesting effort, which is usually assumed to be time-independent [6, 15, 76], should be time-dependent. Here we assume that the harvesting effort varies with time and is proportional to the profit margin (selling price-cost of fishing) [7]. If c is the cost of per unit harvesting effort and M is the market price per unit fish biomass at time t, then the system (2.2) with variable harvesting effort can be represented as

$$\begin{aligned}
\frac{dS}{dt} &= j(S+I)\left(1 - \frac{S+I}{L}\right) - fSI - q_1HS, \\
\frac{dI}{dt} &= fSI - \mu I - q_2HI, \\
\frac{dH}{dt} &= \phi_1\left\{(q_1S + q_2I)M - c\right\}H,
\end{aligned} \tag{2.3}$$

where ϕ_1 is a proportionality constant.

Many authors have considered taxation policy in harvesting models [26, 32, 39, 45, 57, 68] to control overfishing. However, none of these has considered infection in the fish stock, which may cause a significant change in the system dynamics. If the fisherman pays a tax $\tau(>0)$ to the regulating agency for per unit biomass of the harvested fish, then the model (2.3) takes the form

$$\frac{dS}{dt} = j(S+I)\left(1-\frac{S+I}{L}\right) - fSI - q_1HS, \quad \frac{dI}{dt} = fSI - \mu I - q_2HI, \quad \frac{dH}{dt} = \phi_1\left\{(q_1S+q_2I)(M-\tau)-c\right\}H.$$

The fish price is adjusted daily in the open market, balancing demand and supply. In such a case, price should be regarded as a time variable [13, 53] instead of a constant as usually considered in many models [7, 54]. Then the per capita rate of price change should be proportional to the difference between the market demand (D) and the amount of supplied fish (Q) at that time [5]. Considering a quadratic market demand

$$D(M) = A - A_1M - A_2M^2,$$

where A, A_1, A_2 are positive constants with $A_2 << A_1 << A$ [9], and noting that the supplied fish at any time t is

$$Q(t) = q_1HS + q_2HI,$$

the dynamic bioeconomic fishery model in the presence of infection, harvesting, and taxation can be expressed as

$$\begin{aligned}
\frac{dS}{dt} &= j(S+I)\left(1-\frac{S+I}{L}\right) - fSI - q_1SH = F_1(S,I,H,M),\\
\frac{dI}{dt} &= fSI - \mu I - q_2IH = F_2(S,I,H,M),\\
\frac{dH}{dt} &= \phi_1\Big((q_1S+q_2I)(M-\tau)-c\Big)H = F_3(S,I,H,M),\\
\frac{dM}{dt} &= \phi_2M(D-Q) = F_4(S,I,H,M),
\end{aligned} \tag{2.4}$$

where ϕ_2 is a proportionality constant, and $F_i(S,I,H,M)$ $(i = 1,2,3,4)$ are the functional forms of the rate of change of the respective state variables. Table 1 represents the state variables and parameters considered to formulate the model (2.4) and their default parameter values to be used subsequently. Many authors have studied the harvesting model in the presence and absence of infection. For example, Hu and Cao [38] considered saturated harvesting in a predator-prey model and analyzed its stability and bifurcations. In [42], the authors considered a predator-prey model with constant harvesting and prey refuge to show the existence and uniqueness of the limit cycle. Juneja and Agnihotri [43] studied a predator-prey model with prey infection and predator harvesting. They mainly observed the infection recovery effect on the system dynamics and optimized the net profit taking tax as the controlling parameter. They, however, ignored the dynamic market price of the harvested species. The dynamics of a single-species fishery model, having variable harvesting effort and market price, were explored in [7]. The harvesting tax and its optimality were not considered here, and the per capita demand was considered constant. Ang and Safuan [3] analyzed a harvested predator-prey model with variable carrying capacity and in the presence of environmental toxicants. It is shown that bionomic equilibrium has a strong dependence on resource density. In addition, using the Pontryagin maximum principle, they prescribed the optimal harvesting policy. The effects of fear and refuge on the optimal harvesting in a predator-prey model with cross-diffusion were analyzed by Ma et al. [47]. The harvesting rate was considered a constant, and they did not consider the economic aspect of harvested species. Variable harvesting and the demand-dependent market price of the harvested stocks were considered in [5, 49, 53]. These studies did not consider disease in the harvested fish and ignored the optimal tax policy and the corresponding societal revenue.

Table 1: State variables and parameters with their descriptions and default values.

State variable	Description	Unit	
$S(t)$	Healthy fish biomass at time t	metric tons	
$I(t)$	Infected fish biomass at time t	metric tons	
$H(t)$	Fishing effort at time t	SFU*	
$M(t)$	Market price per unit fish biomass at time t	M$**/metric ton	
Parameter	Description	Default Value	Reference
j	Intrinsic growth rate of healthy fish	0.9 /year	[7]
L	Environmental carrying capacity	5 metric tons	[64]
f	Disease transmission rate	0.04 /metric ton/year	[55]
q_1	Catchability coefficient of susceptible fish	0.8 /SFU/year	[1]
μ	Death rate of infected fish	0.05 /year	[7]
q_2	Catchability coefficient of infected fish	0.9 /SFU/year	[1]
c	Cost per unit of fishing effort	9 M$/SFU/year	[44]
A	Maximum demand	0.9 metric tons/year	Assumed
A_1	Demand sensitivity parameter	0.01 (metric tons)^2/M$/year	Assumed
A_2	Demand sensitivity parameter	0.005 (metric tons)^3/(M$)^2/year	Assumed
ϕ_1	Stiffness parameter	0.1 SFU/M$	[7]
ϕ_2	Proportionality constant	0.15 /metric ton	[7]
τ	Tax per unit biomass of harvested fish	M$/metric ton	Variable

* SFU stands for Standardized Fishing Unit [14, 36] and ** M$ indicates million USD.

3. Model analysis

3.1. Well-posedness of the system

The well-posedness of an ecological model can be justified by its positivity and boundedness results. One can ensure that the system (2.4) is positive and bounded by the subsequent lemma.

Lemma 3.1. *With the initial condition* $\mathcal{I} = (S_0, I_0, H_0, M_0) \in \mathbb{R}_+^{4,0}$, *the positivity and boundedness of the system* (2.4) *is guaranteed in* $\mathcal{G}_{\mathcal{L}}$, *where* $\mathcal{G}_{\mathcal{L}} = (S, I, H, M) : 0 < (S+I) < \iota + \zeta_1,\ 0 < M < \hat{\iota} + \zeta_2,\ 0 < X(S, I, H, M) < \frac{s_4}{s_3} + \zeta$, *for any positive* $\zeta_1,\ \zeta_2,\ \zeta$.

Proof. Consider the Banach space of continuous functions $B = B([0,t], \mathbb{R}_+^{4,0})$, which maps the interval $[0,t]$ into $\mathbb{R}_+^{4,0}$ having norm

$$||\mathcal{N}|| = \sup_{0<\theta<t} (|\mathcal{N}_1(\theta)|, |\mathcal{N}_2(\theta)|, |\mathcal{N}_3(\theta)|, |\mathcal{N}_4(\theta)|),$$

where $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4)$. One can assume the initial state of the system (2.4) as

$$S_\theta = \mathcal{N}_1(\theta) > 0,\ I_\theta = \mathcal{N}_2(\theta) > 0,\ H_\theta = \mathcal{N}_3(\theta) > 0, \text{ and } M_\theta = \mathcal{N}_4(\theta) > 0,\ \theta \in [0,t], \tag{3.1}$$

where $(\mathcal{N}_1(0), \mathcal{N}_2(0), \mathcal{N}_3(0), \mathcal{N}_4(0)) \in B$. Then from the fundamental theory of functional differential equations [35], the system (2.4) has a unique solution with initial point (3.1).

Next we prove that the solutions of the model (2.4) are positive for all positive t. From Eq. (2.4), one can notice

$$S(t) = \left[S_0 + \int_0^t \left\{ j\left(1 - \frac{I(z_2)}{L}\right) e^{-\int_0^{z_2} \left[j\left(1 - \frac{S(z_1)+I(z_1)}{L}\right) - fI(z_1) - q_1 H(z_1) \right] dz_1} \right\} dz_2 \right] \mathcal{P} > 0 \text{ if } S_0 > 0,$$

where

$$\mathcal{P} = e^{\int_0^t \left[j\left(1 - \frac{S(z_3)+I(z_3)}{L}\right) - fI(z_3) - q_1 H(z_3) \right] dz_3},$$

$$
\begin{aligned}
I(t) &= I_0 e^{\int_0^t (fS(z)-\mu-q_2H(z))dz} > 0 \text{ if } I_0 > 0,\\
H(t) &= H_0 e^{\int_0^t \phi_1\left(\left(q_1S(z)+q_2I(z)\right)(M(z)-\tau)-c\right)dz} > 0 \text{ if } H_0 > 0,\\
M(t) &= M_0 e^{\int_0^t \phi_2\left(A-A_1M(z)-A_2M^2(z)-\left(q_1S(z)+q_2I(z)\right)H(z)\right)dz} > 0 \text{ if } M_0 > 0.
\end{aligned}
$$

Therefore, irrespective of the choice of any positive initial point, the system always exhibits positive solution for all $t > 0$. Consequently, the solutions of the system (2.4) is positive in $\mathbb{R}_+^{4,0}$, which is the interior of $\mathbb{R}_+^4$, making the system positively invariant.

Presume that $(S(t), I(t), H(t), M(t))$ is a solution of the system (2.4) with initial state $(S_0, I_0, H_0, M_0) \in \mathbb{R}_+^{4,0}$. Adding the first two equations of system (2.4), one can reach

$$
\frac{d(S+I)}{dt} \leqslant j(S+I)\left(1-\frac{S+I}{L}\right).
$$

Applying standard comparison theorem, one gets

$$
\limsup_{t\to\infty}(S(t)+I(t)) \leqslant \iota, \text{ where } \iota = \max\{S_0+I_0, L\}. \tag{3.2}
$$

Similarly, the last equation of (2.4) gives, $\frac{dM}{dt} \leqslant \hat{j}M\left(1-\frac{M}{\hat{L}}\right)$, where $\hat{j} = \phi_2 A$, $\hat{L} = \frac{A}{A_1}$. Again applying standard comparison theorem, we have

$$
\limsup_{t\to\infty}(M(t)) \leqslant \hat{\iota}, \text{ where } \hat{\iota} = \max\{M_0, \hat{L}\}.
$$

Next consider the function

$$
X = \ln(S+I) + \ln H + \ln M.
$$

Its time derivative along the solution of (2.4) gives

$$
\begin{aligned}
\frac{dX}{dt} &= \frac{1}{S+I}\frac{d(S+I)}{dt} + \frac{1}{H}\frac{dH}{dt} + \frac{1}{M}\frac{dM}{dt}\\
&= j\left(1-\frac{S+I}{L}\right) - \frac{q_1SH+\mu I+q_2IH}{S+I} + \phi_1\Big((q_1S+q_2I)(M-\tau)-c\Big)\\
&\quad + \phi_2(A-A_1M-A_2M^2-(q_1S+q_2I)H)\\
&\leqslant (j+\phi_2A) - \left(\frac{j}{L}(S+I)+q_3H+\phi_1q_3\iota M\right) \text{ (following Eq. (3.2))}\\
&\leqslant (j+\phi_2A) - \left(\frac{j}{L}\ln(S+I)+q_3\ln H+\phi_1q_3\iota\ln M\right) \text{ (since } S > \ln S,\ \forall\, S > 0)\\
&\leqslant s_4 - s_3X,
\end{aligned}
$$

where $s_3 = \min\{\frac{j}{L}, q_3, \phi_1q_3\iota\}$ and $s_4 = j + \phi_2A$. Then, following differential inequality theorem [11],

$$
\frac{dX}{dt} + s_3X \leqslant s_4
$$

provides

$$
0 < X(S, I, H, M) < \frac{s_4}{s_3} + \frac{X(S_0, I_0, H_0, M_0)}{e^{s_3t}}.
$$

Making $t \to \infty$, one obtains $0 < X(S, I, H, M) < \frac{s_4}{s_3}$. Hence, every solution of the system (2.4) starting at $\mathcal{I}$, belongs to $\mathcal{G}_{\mathcal{L}}$, where $\mathcal{G}_{\mathcal{L}} = \{(S, I, H, M) : 0 < (S+I) < \iota+\zeta_1,\ 0 < M < \hat{\iota}+\zeta_2,\ 0 < X(S, I, H, M) < \frac{s_4}{s_3}+\zeta$, for any positive $\zeta_1, \zeta_2, \zeta\}$, and $\mathcal{I} = (S_0, I_0, H_0, M_0) \in \mathbb{R}_+^{4,0}$. Therefore, the region $\mathcal{G}_{\mathcal{L}}$ is positively invariant with respect to the system (2.4). Thus, the lemma is proven. □

3.2. *Basic reproduction number*

The basic reproduction number (BRN), defined by the number of secondary cases arising from a single infected individual introduced into a group of susceptible individuals [2], is an essential measure of disease dynamics. The success of a pathogen depends on the value of BRN, $\mathcal{R}_0$. If $\mathcal{R}_0 < 1$, then the epidemic cannot grow, and the system eventually becomes disease-free [24].

The system (2.4) contains only one infection state, I. Let $\mathcal{F}$ and $\mathcal{V}$, respectively, represent the rate of appearance of new infection and the rate of transitions [24]. Then

$$\mathcal{F} = \left(fSI\right)_{1\times 1} \text{ and } \mathcal{V} = (\mu I + q_2 IH)_{1\times 1}.$$

At the infection-free equilibrium point $E_1 = (S_1, 0, H_1, M_1)$, the transmission matrix $\hat{F}$ and the transition matrix $\hat{V}$ associated with system (2.4) are given by

$$\hat{F} = \left[\frac{\partial \mathcal{F}}{\partial I}\right]_{E_1=(S_1,0,H_1,M_1)} = fS_1 \text{ and } \hat{V} = \left[\frac{\partial \mathcal{V}}{\partial I}\right]_{E_1=(S_1,0,H_1,M_1)} = \mu + q_2 H_1.$$

Then

$$\mathcal{K} = \hat{F}\hat{V}^{-1} = \left(\frac{fS_1}{\mu + q_2 H_1}\right)_{1\times 1},$$

where $\hat{F}\hat{V}^{-1}$ is called the next generation matrix. The basic reproduction number ($\mathcal{R}_0$), which is the spectral radius of the next generation matrix ($\mathcal{K}$) [23], is given by

$$\mathcal{R}_0 = \frac{fS_1}{\mu + q_2 H_1},$$

where $H_1 = \frac{j}{q_1 L}(L - S_1)$ and S_1 is the equilibrium value of susceptible fish at the disease-free state.

3.3. *Equilibrium points*

The equilibrium points of the system (2.4) are the solutions of the simultaneous equations

$$\begin{aligned}
*j(S+I)\left(1 - \frac{S+I}{L}\right) - fSI - q_1 SH = 0,\\
fSI - \mu I - q_2 IH = 0,\\
\phi_1\Big((q_1 S + q_2 I)(M - \tau) - c\Big)H = 0,\\
\phi_2 M\big(A - A_1 M - A_2 M^2 - q_1 SH - q_2 IH\big) = 0.
\end{aligned}$$

The system (2.4) has seven equilibrium points:

(i) The trivial equilibrium $E_0 = (0, 0, 0, 0)$, which always exists.

(ii) The disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$, where the equilibrium components are

$$H_1 = \frac{j}{q_1 L}(L - S_1), \quad M_1 = \tau + \frac{c}{q_1 S_1},$$

and S_1 is the positive root of the equation

$$S_1^4 + B_1 S_1^3 + B_2 S_1^2 + B_3 S_1 + B_4 = 0, \tag{3.3}$$

where

$$B_1 = -L < 0, \quad B_2 = \frac{L}{j}\Big\{(A - A_1\tau - A_2\tau^2)\Big\}, \quad B_3 = -\frac{Lc}{jq_1}(A_1 + A_2\tau L) < 0, B_4 = -\frac{A_2 c^2 L}{jq^2} < 0.$$

Since the number of sign change of the coefficients is exactly one under the restriction $A < A_1\tau + A_2\tau^2$, by Descartes' rule of sign, Eq. (3.3) has exactly one positive root. Note that H_1 is always positive as $S_1 < L$, and M_1 is also positive. Thus, the disease-free equilibrium point (E_1) uniquely exists if $A < A_1\tau + A_2\tau^2$.

(iii) The harvesting-free equilibrium has the form $E_2 = (S_2, I_2, 0, M_2)$, whose equilibrium components are given by $S_2 = \frac{\mu}{f}$, $I_2 = \frac{1}{2jf}\left(-(2j\mu + j\mu - jfL) + \sqrt{(2j\mu + j\mu - jfL)^2 - 4j^2\mu(\mu - fL)}\right)$ and $M_2 = \frac{1}{2A_2}(-A_1 + \sqrt{A_1^2 + 4A_1A_2})$. Since S_2 and M_2 are always positive, so E_2 exists if I_2 is positive and it holds whenever $fL > \mu$.

(iv) The harvesting-and disease-free equilibrium $E_3 = (S_3, 0, 0, M_3)$ always exists, where $S_3 = L > 0$ and $M_3 = \frac{1}{2A_2}(-A_1 + \sqrt{A_1^2 + 4A_1A_2}) = M_2 > 0$.

(v) The healthy and infected fish only equilibrium $E_4 = (S_4, I_4, 0, 0)$, whose state variables at the equilibrium level can be represented as $S_4 = \frac{\mu}{f}$ and

$$I_4 = \frac{1}{2jf}\left(-(2j\mu + j\mu - jfL) + \sqrt{(2j\mu + j\mu - jfL)^2 - 4j^2\mu(\mu - fL)}\right) = I_2.$$

This equilibrium exists if $fL > \mu$.

(vi) The only healthy fish equilibrium $E_5 = (S_5, 0, 0, 0)$ always exists with $S_5 = L$.

(vii) The coexisting equilibrium $E^* = (S^*, I^*, H^*, M^*)$, and the corresponding equilibrium components can be computed as

$$S^* = \frac{1}{f}(\mu + q_2H^*), \qquad M^* = \tau + \frac{c}{\frac{q_1}{f}(\mu + q_2H^*) + q_2I^*}.$$

Observe that both S^* and M^* are positive. The other two equilibrium components I^* and H^* are the positive roots of the equations

$$j\left(\frac{1}{f}(\mu + q_2H^*) + I^*\right)\left(1 - \frac{\frac{1}{f}(\mu + q_2H^*) + I^*}{L}\right) - (\mu + q_2H^*)I^* - \frac{q_1}{f}(\mu + q_2H^*)H^* = 0,$$

$$A - A_1\left[\tau + \frac{c}{\frac{q_1}{f}(\mu + q_2H^*) + q_2I^*}\right] - A_2\left[\tau + \frac{c}{\frac{q_1}{f}(\mu + q_2H^*) + q_2I^*}\right]^2 - \frac{q_1}{f}(\mu + q_2H^*)H^* + q_2I^*H^* = 0.$$

Our computational results for the considered parameter values show that the equilibrium E^* is unique.

3.4. *Stability of the equilibria*

Under what parametric conditions an equilibrium state will be stable is essential for population persistence and sustainable yield. The stability of an equilibrium point means whether the system will return to the equilibrium point over time or not if the equilibrium point is perturbed. One way of determining such stability is the linearization technique of the system around the equilibrium point [48]. The Jacobian matrix of the system (2.4) at any arbitrary equilibrium point $\hat{E} = (\hat{S}, \hat{I}, \hat{H}, \hat{M})$ reads

$$J(\hat{S}, \hat{I}, \hat{H}, \hat{M}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \tag{3.4}$$

where $a_{11} = -f\hat{I} - \hat{H}q_1 - j\left(\frac{2(\hat{I}+\hat{S})}{L} - 1\right)$, $a_{12} = -\hat{S}f - j\left(\frac{2(\hat{I}+\hat{S})}{L} - 1\right)$, $a_{13} = -\hat{S}q_1$, $a_{21} = \hat{I}f$, $a_{22} = \hat{S}f - \mu - \hat{H}q_2$, $a_{23} = -\hat{I}q_2$, $a_{31} = \hat{H}\phi_1 q_1(\hat{M} - \tau)$, $a_{32} = \hat{H}\phi_1 q_2(\hat{M} - \tau)$, $a_{33} = \phi_1\left((q_1\hat{S} + q_2\hat{I})(\hat{M} - \tau) - c\right)$, $a_{34} = \phi_1(\hat{H}\hat{I}q_2 + \hat{H}\hat{S}q_1)$, $a_{41} = -\hat{H}\hat{M}\phi_2 q_1$, $a_{42} = -\hat{H}\hat{M}\phi_2 q_2$, $a_{43} = -\hat{M}\phi_2\left(\hat{I}q_2 + \hat{S}q_1\right)$, $a_{44} = -\phi_2(A_2\hat{M}^2 + A_1\hat{M} - A + \hat{H}\hat{I}q_2 + \hat{H}\hat{S}q_1) - \hat{M}\phi_2(A_1 + 2A_2\hat{M})$. One can then prove the following stability theorem.

Theorem 3.2.

(i) *The equilibrium points* $E_0 = (0,0,0,0)$, $E_4 = (S_4, I_4, 0, 0)$, *and* $E_5 = (L, 0, 0, 0)$ *are always unstable.*

(ii) *The disease-free equilibrium* $E_1 = (S_1, 0, H_1, M_1)$ *is locally asymptotically stable if the conditions* $\mathcal{R}_0 < 1$, $C_1 > 0$, $C_3 > 0$, *and* $C_1C_2 - C_3 > 0$ *are satisfied, otherwise it is unstable, where* C_1, C_2, *and* C_3 *are given in* (3.8).

(iii) *If* $c > (I_2q_2 + S_2q_1)(M_2 - \tau)$ *and* $2(S_2 + I_2) > L$, *then the harvesting-free equilibrium* $E_2 = (S_2, I_2, 0, M_2)$ *is locally asymptotically stable, and unstable otherwise.*

(iv) *Whenever the conditions* $\mu > Lf$, $c > Lq_1(M_3 - \tau)$ *hold, the harvesting-and disease-free equilibrium* $E_3 = (S_3, 0, 0, M_3)$ *remains locally asymptotically stable, and unstable otherwise.*

(v) *A set of necessary and sufficient conditions for the stability of the coexisting equilibrium point* $E^* = (S^*, I^*, H^*, M^*)$ *is* $\{C_4 > 0, C_6 > 0, C_7 > 0, C_4C_5C_6 - (C_6^2 + C_4^2C_7) > 0\}$, *where* C_4, C_5, C_6, *and* C_7 *are given in* (3.11).

Proof.

(i) The variational matrix (3.4) at the trivial equilibrium point $E_0 = (0,0,0,0)$ reads

$$J_{E_0} = \begin{pmatrix} j & j & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & -c\phi_1 & 0 \\ 0 & 0 & 0 & A\phi_2 \end{pmatrix}. \tag{3.5}$$

Since two eigenvalues (j and $A\phi_2$) of the Jacobian matrix (3.5) are positive, the equilibrium point E_0 is always unstable. Similarly, a positive eigenvalue of the form $A\phi_2$ for both the equilibrium points $E_4 = (S_4, I_4, 0, 0)$ and $E_5 = (L, 0, 0, 0)$ makes them unstable.

(ii) At the disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$, the variational matrix (3.4) reads

$$J_{E_1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & 0 \\ 0 & b_{22} & 0 & 0 \\ b_{31} & b_{32} & 0 & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \tag{3.6}$$

where $b_{11} = -\frac{S_1 j}{L}$, $b_{12} = -S_1 f - j\left(\frac{2S_1}{L} - 1\right)$, $b_{13} = -S_1q_1$, $b_{22} = S_1 f - \mu - H_1q_2$, $b_{31} = H_1\phi_1q_1(M_1 - \tau)$, $b_{32} = H_1\phi_1q_2(M_1 - \tau)$, $b_{34} = H_1S_1\phi_1q_1$, $b_{41} = -H_1M_1\phi_2q_1$, $b_{42} = -H_1M_1\phi_2q_2$, $b_{43} = -M_1S_1\phi_2q_1$, $b_{44} = -M_1\phi_2(A_1 + 2A_2M_1)$. Its one eigenvalue is $S_1 f - \mu - H_1q_2$, which is negative whenever the basic reproduction number $\mathcal{R}_0 < 1$. The other three eigenvalues are the roots of the equation

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{3.7}$$

where

$$C_1 = -(b_{11} + b_{44}), \quad C_2 = -b_{13}b_{31} + b_{11}b_{44} - b_{34}b_{43}, \quad C_3 = -b_{13}b_{34}b_{41} + b_{13}b_{31}b_{44} + b_{11}b_{34}b_{43}. \tag{3.8}$$

Following Routh-Hurwitz criterion [40], the necessary and sufficient conditions for all roots of Eq. (3.7) to have negative real part are $C_1 > 0, C_3 > 0, C_1C_2 - C_3 > 0$. Therefore, the disease-free equilibrium $E_1 = (S_1, 0, H_1, M_1)$ is locally asymptotically stable under the condition $\mathcal{R}_0 < 1$, $C_1 > 0$, $C_3 > 0$, and $C_1C_2 - C_3 > 0$.

(iii) The Jacobian matrix at the harvesting-free equilibrium $E_2 = (S_2, I_2, 0, M_2)$ is

$$J_{E_2} = \begin{pmatrix} -I_2 f - j\left(\frac{2(I_2+S_2)}{L} - 1\right) & -S_2 f - j\left(\frac{2(I_2+S_2)}{L} - 1\right) & -S_2 q_1 & 0 \\ I_2 f & 0 & -I_2 q_2 & 0 \\ 0 & 0 & \begin{matrix}\phi_1((I_2q_2 + S_2q_1) \\ (M_2 - \tau) - c)\end{matrix} & 0 \\ 0 & 0 & \begin{matrix}-M_2\phi_2(I_2q_2 \\ +S_2q_1)\end{matrix} & \begin{matrix}-M_2\phi_2(A_1 \\ +2A_2M_2)\end{matrix} \end{pmatrix}. \tag{3.9}$$

Its two eigenvalues are $-M_2\,\phi_2\,(A_1+2\,A_2\,M_2)<0$ and $\phi_1\,((I_2q_2+S_2q_1)(M_2-\tau)-c)$. The latter eigenvalue is negative provided $c>(I_2q_2+S_2q_1)(M_2-\tau)$, i.e., the cost per unit of fishing effort greater than the corresponding earnings. The other two eigenvalues are the roots of the equation

$$\lambda_1^2+\left(I_2\,f+j\left(\frac{2(I_2+S_2)}{L}-1\right)\right)\lambda_1+\left(S_2\,f+j\left(\frac{2(I_2+S_2)}{L}-1\right)\right)I_2f=0. \tag{3.10}$$

Clearly, the roots of Eq. (3.10) will have negative real parts whenever $2(S_2+I_2)>L$. Thus, the equilibrium point $E_2=(S_2,I_2,0,M_2)$ is locally asymptotically stable under the conditions $c>(I_2q_2+S_2q_1)(M_2-\tau)$, $2(S_2+I_2)>L$.

(iv) The characteristic equation corresponding to the Jacobian matrix (3.4) at the harvesting-and disease-free equilibrium $E_3(S_3,0,0,M_3)$ can be written as

$$(\lambda_3+j)\{\lambda_3-(Lf-\mu)\}\{\lambda_3+\phi_1\,(c-Lq_1\,(M_3-\tau))\}\{\lambda_3+M_3\,\phi_2\,(A_1+2\,A_2\,M_3)\}=0.$$

Therefore, the eigenvalues are $-j$, $Lf-\mu$, $-\phi_1\,(c-Lq_1\,(M_3-\tau))$ and $-M_3\,\phi_2\,(A_1+2\,A_2\,M_3)$. Clearly, two eigenvalues $-j$ and $-M_3\,\phi_2\,(A_1+2\,A_2\,M_3)$ are always negative. The negativity of the remaining two is assured under the conditions $\mu>fL$ and $c>Lq_1\,(M_3-\tau)$. Recall that the existence condition of equilibrium points E_2 and E_4 is $\mu<fL$. Therefore, whenever the equilibrium point E_2 or E_4 exists, the steady state E_3 cannot be stable. The other condition $c>Lq_1\,(M_3-\tau)$ tells that the fishing cannot be profitable whenever E_3 is stable.

(v) Suppose an interior equilibrium $E^*=(S^*,I^*,H^*,M^*)$ of the system (2.4) exists. The Jacobian matrix in this case is evaluated as

$$J_{E^*}=\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0\\ c_{21} & 0 & c_{23} & 0\\ c_{31} & c_{32} & 0 & c_{34}\\ c_{41} & c_{42} & c_{43} & c_{44}\end{pmatrix},$$

where $c_{11}=-\frac{2j(I^*+S^*)}{L}$, $c_{12}=-S^*f-j\left(\frac{2(I^*+S^*)}{L}-1\right)$, $c_{13}=-S^*q_1$, $c_{21}=I^*f$, $c_{23}=-I^*q_2$, $c_{31}=H^*\phi_1q_1(M^*-\tau)$, $c_{32}=H^*\phi_1q_2(M^*-\tau)$, $c_{34}=\phi_1(H^*I^*q_2+H^*S^*q_1)$, $c_{41}=-H^*M^*\phi_2q_1$, $c_{42}=-H^*M^*\phi_2q_2$, $c_{43}=-M^*\phi_2\,(I^*q_2+S^*q_1)$, $c_{44}=-M^*\phi_2(A_1+2A_2M^*)$. The corresponding characteristic equation reads

$$\lambda_2{}^4+C_4\lambda_2{}^3+C_5\lambda_2{}^2+C_6\lambda_2+C_7=0,$$

where

$$\begin{aligned} C_4&=(-c_{11}-c_{44}),\\ C_5&=(c_{11}\,c_{44}-c_{13}\,c_{31}-c_{12}\,c_{21}-c_{23}\,c_{32}-c_{34}\,c_{43}),\\ C_6&=\big(c_{11}\,c_{23}\,c_{32}-c_{12}\,c_{23}\,c_{31}-c_{13}\,c_{21}\,c_{32}+c_{12}\,c_{21}\,c_{44}+c_{11}\,c_{34}\,c_{43}\\ &\quad+c_{13}\,c_{31}\,c_{44}-c_{13}\,c_{34}\,c_{41}+c_{23}\,c_{32}\,c_{44}-c_{23}\,c_{34}\,c_{42}\big),\\ C_7&=c_{11}\,c_{23}\,c_{34}\,c_{42}-c_{11}\,c_{23}\,c_{32}\,c_{44}+c_{12}\,c_{21}\,c_{34}\,c_{43}+c_{12}\,c_{23}\,c_{31}\,c_{44}\\ &\quad-c_{12}\,c_{23}\,c_{34}\,c_{41}+c_{13}\,c_{21}\,c_{32}\,c_{44}-c_{13}\,c_{21}\,c_{34}\,c_{42}. \end{aligned} \tag{3.11}$$

Following Routh-Hurwitz criterion [40], a set of necessary and sufficient conditions for the stability of the equilibrium point $E^*=(S^*,I^*,H^*,M^*)$ is

$$C_4>0,\quad C_6>0,\quad C_7>0,\quad C_4C_5C_6-(C_6^2+C_4^2C_7)>0.$$

This completes the proof of the theorem. □

3.5. *Bifurcation analysis*

Changes in the system dynamics for the variation of a system parameter may be well described through its bifurcation results. Considering the fishing tax τ as the control parameter, we investigate the occurrence of bifurcations in the system (2.4). One can prove the following theorem for the existence of bifurcations.

Theorem 3.3.

(i) *The system* (2.4) *undergoes a transcritical bifurcation at the disease-free equilibrium point* $E_1(S_1, 0, H_1, P_1)$ *if* τ *reaches the critical value* τ_1^{TC}, *where* τ_1^{TC} *is the positive root of the equation*

$$fS_1(\tau) - \mu - q_2H_1(\tau) = 0,$$

and the transversality condition $f \neq \frac{q_2v_3}{v_1}$ *holds.*

(ii) *The system* (2.4) *undergoes a transcritical bifurcation at the harvesting-free equilibrium point* $E_2(S_2, I_2, 0, P_2)$ *if* τ *arrives the threshold level* τ_2^{TC}, *where*

$$\tau_2^{TC} = M_2 - \frac{c}{I_2q_2 + S_2q_1},$$

and the transversality condition $\tau_2^{TC} \neq M_2 + \frac{q_1S_2+q_2I_2}{q_1w_1+q_2w_2}w_4$ *holds.*

Proof.

(i) From (3.6), one can observe that the Jacobian matrix leaves a zero eigenvalue if

$$fS_1(\tau) - \mu - q_2H_1(\tau) = 0. \tag{3.12}$$

Let $\tau = \tau_1^{TC}$ be a positive root of the Eq. (3.12). Then, at τ_1^{TC}, the eigenvector corresponding to the zero eigenvalue of $J_{E_1}(S_1, 0, H_1, M_1)$ and $J_{E_1}(S_1, 0, H_1, M_1)^T$ are

$$\zeta = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where $J_{E_1}(S_1, 0, H_1, M_1)^T$ is the transpose of $J_{E_1}(S_1, 0, H_1, M_1)$ and

$$\begin{aligned}
v_1 &= -\frac{b_{32}v_2 + b_{34}}{b_{31}}, \\
v_2 &= \frac{b_{13}b_{34}(b_{31}b_{44} - b_{41}b_{34}) + b_{11}b_{34}b_{31}b_{43}}{(b_{12}b_{31} - b_{11}b_{32})b_{31}b_{43} - b_{13}b_{31}(b_{42}b_{31} - b_{41}b_{32})}, \\
v_3 &= \frac{b_{11}b_{34}(b_{42}b_{31} - b_{41}b_{32}) - (b_{12}b_{31} - b_{11}b_{32})(b_{31}b_{44} - b_{41}b_{34})}{(b_{12}b_{31} - b_{11}b_{32})b_{31}b_{43} - b_{13}b_{31}(b_{42}b_{31} - b_{41}b_{32})}.
\end{aligned}$$

Now the three conditions of Sotomayor's theorem [60] for the existence of a degenerate transcritical bifurcation at $\tau = \tau_1^{TC}$ are

$$\begin{aligned}
\eta^T R_\tau\Big(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big) &= 0, \\
\eta^T DR_\tau\Big(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big)\zeta &= 0, \\
\eta^T D^2R\Big(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\Big)(\zeta, \zeta) &\neq 0.
\end{aligned} \tag{3.13}$$

Here $R_\tau = \left(\frac{dF_1}{d\tau}, \frac{dF_2}{d\tau}, \frac{dF_3}{d\tau}, \frac{dF_4}{d\tau}\right)^T$ and $DR_\tau\left(J_{E_1}(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\right)\zeta$ is the linear transformation formed by the matrix of partial derivatives of the components of R_τ with respect to the state variables (S, I, H, M). Similarly, one can define the other linear transformation $D^2R\left(J_{E_1}(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\right)(\zeta, \zeta)$. It is to be noted that the second condition of (3.13) needs to be non-zero for the appearance of non-degenerate transcritical bifurcation [60]. Now,

$$\eta^T R_\tau\left(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\right) = (0\ 1\ 0\ 0)\begin{pmatrix} 0 \\ 0 \\ \phi_1 q_1 S_1 H_1 \\ 0 \end{pmatrix}_{\tau=\tau_1^{TC}} = 0,$$

$$\eta^T DR_\tau\left(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\right)\zeta = (0\ 1\ 0\ 0)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \phi_1 q_1 H_1 & \phi_1 q_1 H_1 & \phi_1 q_1 S_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\tau=\tau_1^{TC}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix} = 0,$$

$$\begin{aligned}\eta^T D^2R\left(E_1(S_1, 0, H_1, M_1); \tau = \tau_1^{TC}\right)(\zeta, \zeta) &= (0\ 1\ 0\ 0)\begin{pmatrix} d_{11} & d_{12} & d_{13} & 0 \\ d_{21} & d_{22} & d_{23} & 0 \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}_{\tau=\tau_1^{TC}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix} \\ &= d_{21}v_1 + d_{22}v_2 + d_{23}v_3 = 2(fv_1v_2 - q_2v_2v_3),\end{aligned}$$

where $d_{11} = -fv_2 - \frac{2j}{L}(v_1 + v_2) - q_1v_3$, $d_{12} = -fv_1 - \frac{2j}{L}(v_1 + v_2)$, $d_{13} = -q_1v_1$, $d_{21} = fv_2$, $d_{22} = fv_1 - q_2v_3$, $d_{23} = -q_2v_2$, $d_{31} = \phi_1 q_1((M_1 - \tau_1^{TC})v_3 + H_1)$, $d_{32} = \phi_2 q_2((M_1 - \tau_1^{TC})v_3 + H_1)$, $d_{33} = \phi_1\{q_1S_1 + (q_1v_1 + q_2v_2)(M_1 - \tau_1^{TC})\}$, $d_{34} = \phi_1\{q_1S_1v_3 + q_1H_1(v_1 + v_2)\}$, $d_{41} = -\phi_2 q_1(H_1 + M_1v_3)$, $d_{42} = -\phi_2 q_2(H_1 + M_1v_3)$, $d_{43} = -\phi_2\{M_1(q_1v_1 + q_2v_2) + S_1q_1(v_3 + 1)\}$, $d_{44} = -\phi_2\{H_1(q_1v_1 + q_2v_2) + S_1q_1 + (2A_1 + 6A_2M_1)\}$. Thus, following Sotomayars theorem [60], whenever the control parameter τ reaches the critical value $\tau = \tau_1^{TC}$, a degenerate transcritical bifurcation point occurs if the condition $f \neq \frac{q_2v_3}{v_1}$ holds.

(ii) Proceeding similarly, one can show that the variational matrix (3.9), corresponding to the harvesting effort-free equilibrium point $E_2(S_2, I_2, 0, M_2)$, gives a zero eigenvalue at $\tau = M_2 - \frac{c}{I_2q_2 + S_2q_1} = \tau_2^{TC}$ (say). In this case, the eigenvectors of $J_{E_2}(S_2, I_2, 0, M_2)$ and $J_{E_2}(S_2, I_2, 0, M_2)^T$, corresponding to the zero eigenvalue at τ_2^{TC}, are

$$\hat{\zeta} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \text{ and } \hat{\eta} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} w_1 &= -\frac{q_2}{f}, \\ w_2 &= \frac{q_2\left(-fI_2 - j\left(\frac{2(S_2+I_2)}{L} - 1\right)\right)(A_1 + 2A_2M_2) + f(q_1S_2 + q_2I_2)\left(fS_2 + j\left(\frac{2(S_2+I_2)}{L} - 1\right)\right)}{q_1 f\phi_2 S_2 I_2 M_2(A_1 + 2A_2M_2)}, \\ w_3 &= 1,\ w_4 = \frac{q_1S_2 + q_2I_2}{A_1 + 2A_2M_2}. \end{aligned}$$

Similar calculations show that there exists a degenerate transcritical bifurcation point at $\tau = \tau_2^{TC}$ if $\tau_2^{TC} \neq M_2 + \frac{q_1S_2 + q_2I_2}{q_1w_1 + q_2w_2}w_4$. □

3.6. Computational results

To visualize the previous bifurcations, and the in-between stabilities, we have presented a bifurcation diagram in Fig. 1 with the variations in τ. It shows three distinct dynamic behaviours of the system when the tax is varied in some stipulated range $0 < \tau < 11$.

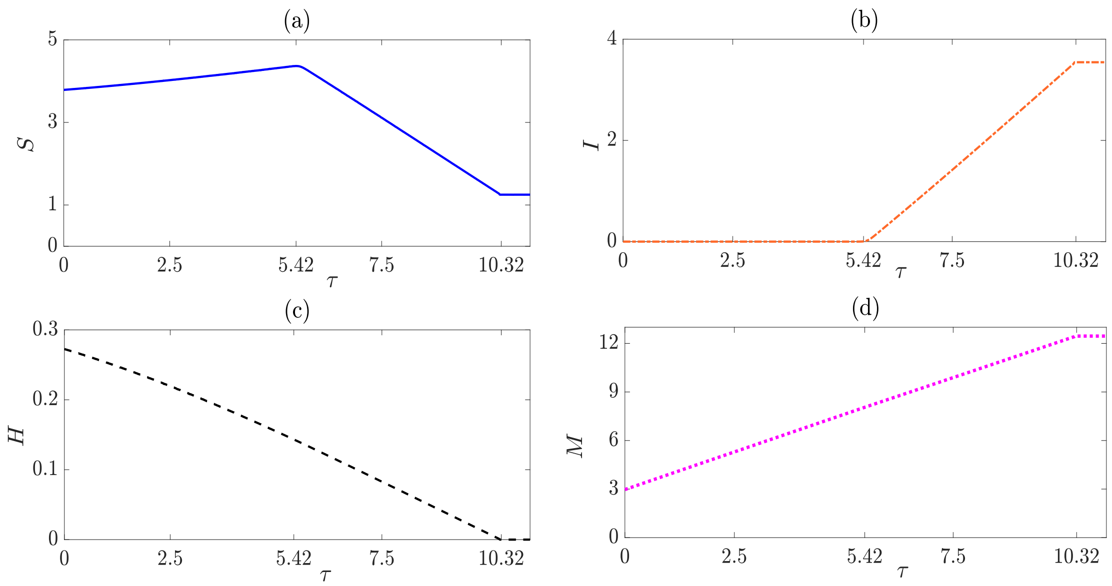

Figure 1: Bifurcation results of the system (2.4) when the tax, τ, is varied in the range $0 < \tau < 11$. We have plotted the maxima and minima of each state variables for each value of τ. This tax range is classified into three categories, low, intermediate and high, depending on the system's stabilities. The disease-free equilibrium (E_1) is stable when the tax is low $(0 < \tau < 5.42)$. The coexisting equilibrium (E^*) is stable in the intermediate tax, $5.42 < \tau < 10.32$. The harvesting-free equilibrium (E_2) is stable if the tax is high $(\tau > 10.32)$. Parameters are as in Table 1.

Solving Eq. (3.12), one gets the unique root as $\tau = \tau_1^{TC} = 5.42$. At this critical value, the eigenvector $\zeta = (v_1, v_2, v_3, 1)^T$ becomes $(26.30, -24.65, 0.01, 1)^T$ and therefore the transversality condition of Theorem 3.3 (i) is satisfied as $f = 0.04 \neq \frac{q v_3}{v_1} = 0.0004$. Therefore, a transcritical bifurcation arises at $\tau = \tau_1^{TC} = 5.42$, following Theorem 3.3 (i), where the disease-free equilibrium E_1 coalesces with the coexisting equilibrium E^* and exchanges their stability (see Fig. 1). At $\tau = \tau_2^{TC} = M_2 - \frac{c}{I_2 q_2 + S_2 q_1} = 10.32$, one can obtain the eigenvector as $\hat{\zeta} = (w_1, w_2, w_3, w_4)^T = (-22.50, 0.83, 1, 31.12)^T$. Also the transversality condition of Theorem 3.3 (ii) is satisfied as $\tau_2^{TC} = 10.32 \neq M_2 + \frac{q_1 S_2 + q_2 I_2}{q_1 w_1 + q_2 w_2} w_4 = 4.90$. Therefore, following Theorem 3.3 (ii), another shift of stability through a transcritical bifurcation occurs at $\tau = 10.32$, where the coexisting equilibrium E^* and the harvesting-free equilibrium (E_2) met. Notice that the market price (M) increases as the tax increases, while the harvesting effort (H) steadily decreases in the same range $0 < \tau < 10.32$. The disease is established through the appearance of the I population as the imposed tax exceeds the first transcritical value $\tau_1^{TC} = 5.42$. The infected fish population increases rapidly for further increase in τ, while a gradual decline occurs in the healthy fish population. As the regulatory tax crosses the higher transcritical value $\tau_2^{TC} = 10.32$, harvesting effort declines to zero. Thus, there exist three different dynamic regimes for the variation in τ: (i) the system remains disease-free for low tax $(0 < \tau < 5.42)$; (ii) the disease persists when tax is intermediate $(5.42 < \tau < 10.32)$; and (iii) harvesting is not possible if the imposed tax is high $(\tau > 10.32)$. The harvesting-and disease-free equilibrium, E_3, does not appear in the bifurcation analysis results because it is always unstable whenever the equilibrium point E_2 or E_4 exists.

The time series solutions (Fig. 2) of the system for three particular values of τ show the representative behaviour of the state variables for all τ in the considered range. At the lower value of the regulating tax (say $\tau = 2.5$) the required conditions of Theorem 3.2 (ii) are satisfied as $C_1 = 0.91 > 0, C_3 = 0.16 > 0$, $C_1 C_2 - C_3 = 0.28 > 0$. Here, the basic reproduction number is $\mathcal{R}_0 = 0.65 < 1$. Therefore, the system

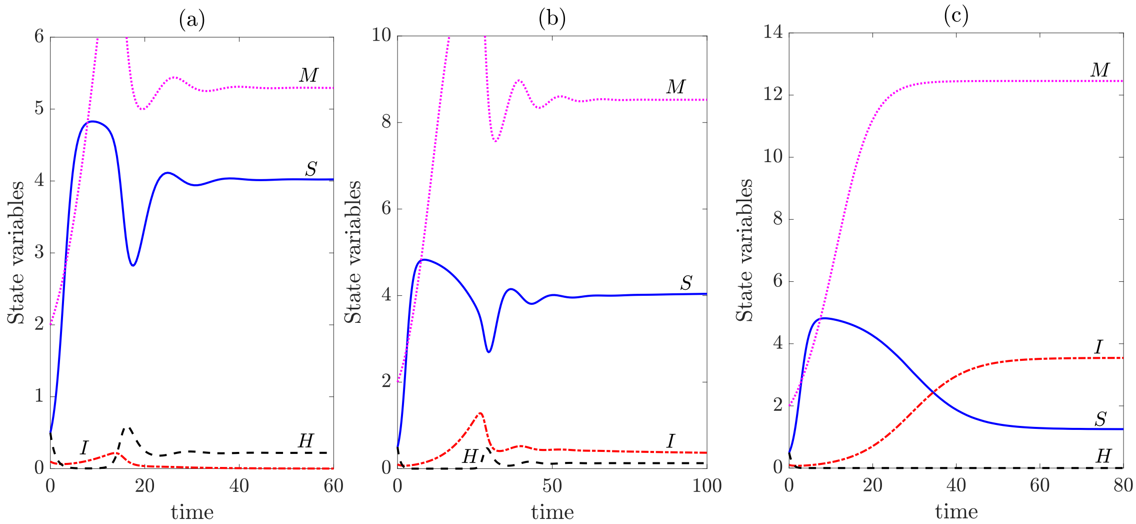

Figure 2: Time evolutions of the system (2.4) for some particular values of τ taken one from each region (see Fig. 1). (a) Stable behaviour of the infection-free equilibrium $E_1 = (4.02, 0, 0.21, 5.29)$ for $\tau = 2.5$. (b) The endemic equilibrium $E^* = (4.10, 0.31, 0.12, 8.52)$ is stable for $\tau = 6$. (c) For $\tau = 10.35$, harvesting effort becomes zero and the system stabilizes to the harvesting-free equilibrium $E_2 = (1.25, 3.54, 0, 12.45)$. In each case, the system started from the initial value $(0.5, 0.1, 0.5, 2)$. Parameters are as in Table 1.

stabilizes to the disease-free equilibrium $E_1 = (4.02, 0, 0.21, 5.29)$ (Fig. 2 (a)). Healthy fish stock in this state is high, at 4.02 units. Consequently, the price remains low ($M = 5.29$ units), and harvesting effort is high ($H = 0.21$ units) due to the availability of the fish stock. Intense harvesting reduces the infected fish, causing the elimination of infection from the system when the tax is low. If the imposed tax is moderate, say $\tau = 6$, the system converges to the endemic state E^*, by satisfying the set of necessary and sufficient conditions of Theorem 3.2 (v) as $C_4 = 0.91 > 0, C_6 = 0.17 > 0, C_7 = 0.002 > 0, C_4C_5C_6 - (C_6^2 + C_4^2C_7) = 0.03 > 0$. This gives the stable solutions of all the state variables with equilibrium population levels $S^* = 4.10, I^* = 0.31, H^* = 0.12, M^* = 8.52$ (Fig. 2 (b)). The infected fish can persist in the intermediate range of $5.42 < \tau < 10.32$. This is reasonable because increasing tax reduces harvesting and causes a compensatory increase in infected fish, which helps infection invade the host population. The total fish stock $(S^* + I^*)$ at E^* increases to 4.41 units from 4.02 units compared to the previous state. For higher tax, say $\tau = 10.35$ (> 10.32), the local stability condition given in Theorem 3.2 (iii) becomes $c - (I_2q_2 + S_2q_1)(M - \tau) = 0.21 > 0$ and $2(S_2 + I_2) - L = 4 > 0$. Therefore, following Theorem 3.2 (iii), the system converges to the harvesting-free equilibrium state $E_2 = (1.25, 3.54, 0, 12.45)$ (Fig. 2 (c)), where each state variable has positive value except the fishing effort, which is zero. Observe that the fish market price in this state becomes too high ($M = 12.45$ units) for an imbalance in the demand and supply. Interestingly, even though the available fish stock is maximum ($S_2 + I_2 = 4.795$ units) in this case, the demand diminishes to zero due to the high market price (see Fig. 1 (d)). Thus, there is a regime shift as τ crosses the upper transcritical value, where the system enters into a non-harvesting regime from the harvesting regime due to excessive fishing tax. Fishers opt out of fishing as harvesting is not economically viable at a higher tax ($\tau > 10.32$). Therefore, it is necessary to control the tax parameter sustainably, and the challenge for the regulating agency is to optimize this parameter for sustainable socio-economic benefits.

4. Optimal taxation policy

Here we explore the trade-off between the regulatory tax and the societal net benefit. The societal benefit (say, Θ) is defined here as the sum of net revenue from fish selling (say, Θ_1) and the income earned from the fishing tax (say, Θ_2), where

$$\begin{aligned}&\Theta_1(S,I,H,M,\tau) = \text{landed fish} \times (\text{market price minus fishing tax}) = H(q_1S+q_2I)(M-\tau),\\&\Theta_2(S,I,H,M,\tau) = \text{landed fish} \times \text{fishing tax} = H(q_1S+q_2I)\tau,\end{aligned} \tag{4.1}$$

and

$$\Theta(S,I,H,M,\tau) = \text{revenue from fishing } (\Theta_1) + \text{revenue from tax } (\Theta_2) = (q_1MS+q_2MI)H. \tag{4.2}$$

We find whether there exists an optimal value of the imposed tax so that the societal benefit is maximum. To maximize the societal benefit, the optimal taxation problem may be defined as

$$\mathfrak{J} = \int_0^\infty \Theta(S,I,H,M,\tau)e^{-\delta t}dt,$$

where δ indicates the annual discount rate and Θ is defined in Eq. (4.2). The control variable τ is subject to the constraints $0 \leqslant \tau < \tau_{max}$, where τ_{max} denote the upper limits of the imposed tax. By virtue of the Pontryagin's maximum principle [63], one can write the Hamiltonian of the system as

$$\begin{aligned}\Upsilon(S,I,H,M,\tau) = {} & H(q_1MS+q_2MI)e^{-\delta t} + \xi_1\left[j(S+I)\left(1-\frac{S+I}{L}\right) - fSI - q_1SH\right]\\&+\xi_2\Big[fSI-\mu I-q_2IH\Big]+\xi_3\Big[\phi_1\big((q_1S+q_2I)(M-\tau)-c\big)H\Big]\\&+\xi_4\left[\phi_2M\big(A-A_1M-A_2M^2-q_1SH-q_2IH\big)\right],\end{aligned}$$

subject to the system (2.4), where ξ_1, ξ_2, ξ_3, and ξ_4 are the adjoint variables. The optimal control variable τ has to satisfy the following conditions to maximize Υ [52]:

$$\frac{\partial\Upsilon}{\partial\tau}=0,\ \frac{d\xi_1}{dt}=-\frac{\partial\Upsilon}{\partial S},\ \frac{d\xi_2}{dt}=-\frac{\partial\Upsilon}{\partial I},\ \frac{d\xi_3}{dt}=-\frac{\partial\Upsilon}{\partial H},\ \frac{d\xi_4}{dt}=-\frac{\partial\Upsilon}{\partial M}.$$

At any arbitrary equilibrium point $(\hat{S},\hat{I},\hat{H},\hat{M})$, $\frac{\partial\Upsilon}{\partial\tau}=0$ gives $\xi_3\phi_1\big(-q_1\hat{S}-q_2\hat{I}\big)\hat{H}=0$. For the nontrivial solution, one must have

$$\xi_3 = 0.$$

Again, $\frac{d\xi_4}{dt} = -\left[\frac{\partial\Upsilon}{\partial M}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$ gives

$$\frac{d\xi_4}{dt} = D_2e^{-\delta t} + D_1\xi_4, \tag{4.3}$$

where $D_1 = -\phi_2\{A-2A_1\hat{M}-3A_2\hat{M}^2-q_1\hat{S}\hat{H}-q_2\hat{I}\hat{H}\}$ and $D_2 = -\{q_1\hat{S}\hat{H}+q_2\hat{I}\hat{H}\}$. Solving (4.3), one gets

$$\xi_4 = -\frac{D_2}{D_1+\delta}e^{-\delta t}.$$

Also, $\frac{d\xi_3}{dt} = -\left[\frac{\partial\Upsilon}{\partial H}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$ provides

$$\xi_1 = F_1e^{-\delta t} - F_2\xi_2,$$

where $F_1 = (q_1\hat{S}+q_2\hat{I})\left(\hat{M}+\frac{D_2\phi_2\hat{M}}{D_1+\delta}\right)\frac{1}{q_1\hat{S}}$ and $F_2 = \frac{q_2\hat{I}}{q_1\hat{S}}$. Putting the value of ξ_1 in $\frac{d\xi_2}{dt} = -\left[\frac{\partial\Upsilon}{\partial I}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$, one gets

$$\xi_2 = -\frac{D_4}{D_3+\delta}e^{-\delta t} \quad \text{and consequently} \quad \xi_1 = \left\{F_1+\frac{D_4F_2}{D_3+\delta}\right\}e^{-\delta t},$$

where $D_3 = F_2\left(j(1-\frac{2(\hat{S}+\hat{I})}{L}) - f\hat{S}\right) - f\hat{S} + \mu + q_2\hat{H}$, and $D_4 = -q_2\hat{H}\hat{M} - F_1\left(j(1-\frac{2(\hat{S}+\hat{I})}{L}) - f\hat{S}\right) - \frac{q_2\phi_2 D_2\hat{H}\hat{M}}{(D_1+\delta)}$.

Observe that each of these adjoint variables $(\xi_1, \xi_2, \xi_3, \xi_4)$ is bounded. Substituting the values of these adjoint variables in $\frac{d\xi_1}{dt} = -\left[\frac{\partial \Upsilon}{\partial S}\right]_{(\hat{S},\hat{I},\hat{H},\hat{M})}$, one gets the optimal tax equation as

$$\Gamma(\tau) = q_1\hat{M}\hat{H} + \left(j(1-\frac{2(\hat{S}+\hat{I})}{L}) - f\hat{I} - q_1\hat{H} - \delta\right)\left(F_1 + \frac{D_4F_2}{D_3+\delta}\right) - \frac{D_4 f\hat{I}}{D_3+\delta} + \frac{D_2\phi_2 q_1\hat{H}\hat{M}}{(D_1+\delta)} = 0 \tag{4.4}$$

for a suitable choice of the annual discount rate, δ. The positive values of τ for which $\Gamma(\tau) = 0$ are the possible optical candidates. The optimal value $\tau = \tau^c$ is the value for which Θ is maximum. If there are i number of equilibrium points with non-zero harvesting value, we will obtain i number of critical τ^c's. Then the optimal societal revenue, Θ^{max}, is given by

$$\Theta^{max} = \max_i \Theta(S, I, H, M, \tau_i^c).$$

To compute the optimum tax level and the corresponding societal revenue $\Theta(S, I, H, M, \tau)$ for the parameter values considered in Table 1 with an annual discount rate $\delta = 0.001$, we solve Eq. (4.4) at the disease-free and endemic equilibrium points, where harvesting has non-zero equilibrium value. We obtain two optimal values of τ, namely, $\tau_1^c = 4.44$M\$/metric ton at the infection-free equilibrium state, and $\tau_2^c = 9.22$ M\$/metric ton at the endemic equilibrium state (See Fig. 3 (a)). The societal benefit or the net revenue at these two optimal tax values are computed from (4.2) as $\Theta(\tau_1^c) = 4.096$ M\$/year and $\Theta(\tau_2^c) = 1.478$M\$/year. Thus, the maximum net revenue is $\Theta^{max} = \max(4.09, 1.478) = 4.09$M\$/year and the optimal tax is $\tau_1^c = 4.44$M\$/metric ton, which is obtained at the disease-free equilibrium state, E_1. Following similar calculations, one can get the optimal equation for the fishing tax revenue Θ_2 as

$$\begin{aligned}\Gamma_1(\tau) = \frac{1}{q_1\hat{S}}&\left\{-\delta + j\left\{1-\frac{2(\hat{S}+\hat{I})}{L}\right\} - f\hat{I} - q_1\hat{H}\right\}\left\{\frac{D_6 q_2\hat{I}}{D_5+\delta} + (q_1\hat{S}+q_2\hat{I})\hat{M} - c - \frac{\delta}{\phi_1} + \frac{D_2}{D_1+\delta}\right\}\\ &+ q_1\tau\hat{H} - \frac{D_6 f\hat{I}}{D_5+\delta} + q_1(\hat{M}-\tau)\hat{H} + \frac{D_2\tau\hat{H}}{D_1+\delta} = 0,\end{aligned} \tag{4.5}$$

where

$$\begin{aligned} D_5 &= \frac{q_2\hat{I}}{q_1\hat{S}}\left\{j\left(1-\frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{S}\right\} + \mu - f\hat{S} + q_2\hat{H},\\ D_6 &= -q_2\tau\hat{H} - \left\{(q_1\hat{S}+q_2\hat{I})\hat{M} - c - \frac{\delta}{\phi_1} + \frac{D_2}{D_1+\delta}\right\}\left\{j\left(1-\frac{2(\hat{S}+\hat{I})}{L}\right) - f\hat{S}\right\} - \frac{D_2\phi_2 q_2\hat{M}\hat{H}}{D_1+\delta}.\end{aligned}$$

The solution of Eq. (4.5) provides the optimal value of τ as $\tau_{\Theta_2}^{c1} = 1$ M\$/metric ton and $\tau_{\Theta_2}^{c2} = 5.68$ M\$/metric ton (See Fig. 3 (b)). The earnings from fishing tax at these two optimal tax values are computed from Eq. (4.1) as $\Theta_2(\tau_{\Theta_2}^{c1}) = 0.78$ M\$/year and $\Theta_2(\tau_{\Theta_2}^{c2}) = 2.717$ M\$/year. Thus, the maximum fishing tax revenue is $\Theta_2^{max} = 2.717$ M\$/year and the optimal tax is $\tau_{\Theta_2}^{c2} = 5.68$M\$/metric ton, which is obtained at the endemic equilibrium state, E^*. It is worth mentioning that the fisherman revenue (Θ_1) is a decreasing tax function, and it is maximum when $\tau = 0$.

In Table 2, we have presented the equilibrium values of the state variables and the revenues at equilibrium points E_1 and E^* for some particular discounts of τ. It shows that the societal income is maximum $(\Theta = 4.096)$ when $\tau = 4.44$. Fishermen's earnings from selling fishing are maximum $(\Theta_1 = 2.453)$ when $\tau = 0$ and the revenue from the fishing tax is maximum $(\Theta_2 = 2.717)$ when $\tau = 5.68$. It is interesting to note that the total equilibrium fish stock $(\hat{S}+\hat{I})$ is maximum (4.795) in the endemic state; however, the maximum societal revenue (4.096) is generated at the disease-free equilibrium state for the optimal tax $\tau = 4.44$.

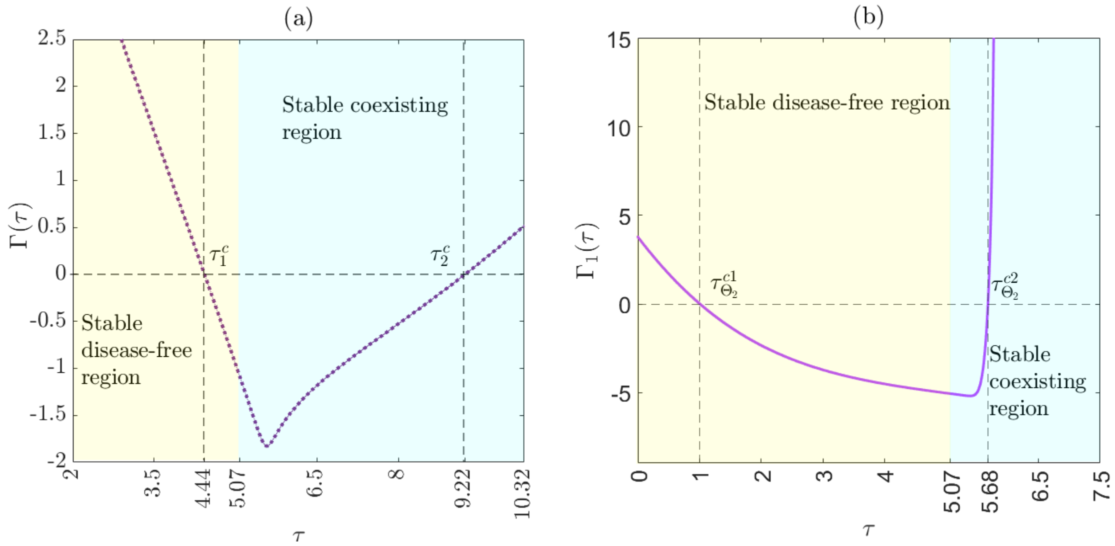

Figure 3: (a) Plot of the optimal tax equation (4.4) for Θ with respect to τ. It shows that there are two optimal values of τ, viz., $\tau_1^c = 4.44$, and $\tau_2^c = 9.22$, for which $\Gamma(\tau) = 0$. (b) Similar plot of (4.5) for Θ_2 shows that there exists two optimal values of τ, viz., $\tau_{\Theta_2}^{c1} = 1$, and $\tau_{\Theta_2}^{c2} = 5.68$. Here the annual discount rate is $\delta = 0.001$, and the other parameters are as in Table 1.

Table 2: This table evaluates the societal revenue $\Theta(\hat{S}, \hat{I}, \hat{H}, \hat{M})$, fisherman's revenue $\Theta_1(\hat{S}, \hat{I}, \hat{H}, \hat{M})$, and tax revenue $\Theta_2(\hat{S}, \hat{I}, \hat{H}, \hat{M})$ at the equilibrium states E_1 and E^*, where harvesting is possible, for some particular values of fishing tax with an annual discount rate $\delta = 0.001$. Observe that societal revenue is maximum (4.096 M\$/year) in the disease-free state (where $\hat{I} = 0$) for $\tau = 4.44$ M\$/metric ton. Tax revenue is maximum (2.717 M\$/year) in the endemic state for $\tau = 5.68$ M\$/metric ton. Fishers' revenue is maximum (2.453 M\$/year) when there is no fishing tax and gradually declines with increasing τ. The optimum values are written in boldface. The parameters are as in Fig. 3.

τ (M\$/MT*)	$\hat{S}$ (MT)	$\hat{I}$ (MT)	$\hat{S}+\hat{I}$ (MT)	$\hat{H}$ (SFU)	$\hat{M}$ (M\$/MT)	$\Theta_1(\hat{S}, \hat{I}, \hat{H}, \hat{M})$ (M\$/year)	$\Theta_2(\hat{S}, \hat{I}, \hat{H}, \hat{M})$ (M\$/year)	$\Theta = \Theta_1 + \Theta_2$ (M\$/year)
0	3.789	0	3.789	0.273	2.970	**2.453**	0	2.453
2.5	4.024	0	4.024	0.2195	5.296	1.976	1.767	3.743
4.44	4.244	0	4.244	0.1701	7.091	1.531	2.565	**4.096**
5	4.3123	0	4.3123	0.1549	7.609	1.394	2.672	4.066
5.07	4.320	0	4.320	0.153	7.674	1.377	2.681	4.058
5.68	4.305	0.0839	4.3889	0.1359	8.237	1.22	**2.717**	3.941
6	4.101	0.3118	4.4128	0.1267	8.527	1.140	2.707	3.847
8	2.780	1.791	4.571	0.0685	10.345	0.6132	2.0918	2.705
9.22	1.971	2.715	4.686	0.0321	11.459	0.289	1.189	1.478
10.32	1.250	3.545	4.795	0	12.454	0	0	0

*MT stands for metric ton.

The equilibrium revenue curves for varying taxes are plotted in Fig. 4. It shows that the fisherman's revenue (Θ_1) is maximum when there is no fishing tax. Here the societal benefit (Θ) coincided with the fishers' earnings. Otherwise, societal benefits exceed the fishers' incomes for the feasible range of τ. The societal benefit is always higher from the generated revenue from the fishing tax (Θ_1) in the same range. It is observable that the societal revenue gradually increases with τ and becomes maximum in the disease-free state for $\tau = 4.44$, and after that, it decreases to zero. Whereas the tax revenue increases till $\tau = 5.68$ and then declines to zero. The maximum tax earned at $\tau = 5.68$ when disease persists in the system. These results show the existence of a trade-off between the revenue earnings and the imposed tax.

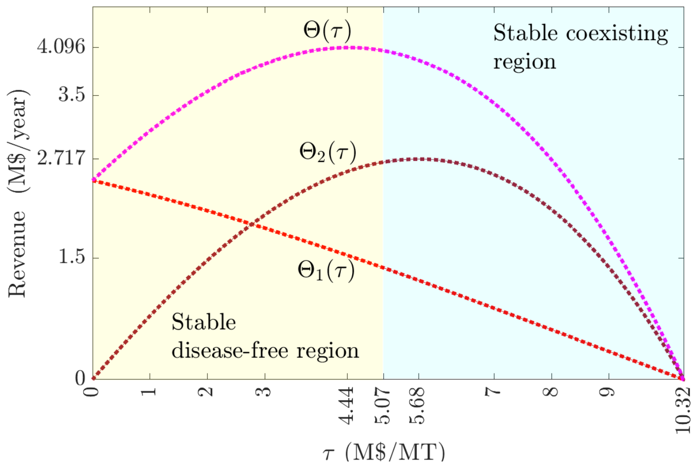

Figure 4: Equilibrium revenue curves are plotted against the tax. The societal revenue ($\Theta(\tau)$) is maximum at $\tau_1^c = 4.44$ M$/metric ton, and the corresponding maximum revenue is 4.096 M$/year. The maximum revenue generated from the imposed tax ($\Theta_2(\tau)$) is obtained at the optimal tax $\tau_{\Theta_2}^c = 5.68$ M$/metric ton, and the corresponding tax revenue is 2.717 M$/year. At $\tau = 0$, the fishers' revenue is maximum, and the corresponding earning is 2.453 M$/year. Here the annual discount rate is $\delta = 0.001$, and the other parameters are as in Table 1.

4.1. Sensitivity analysis

We estimated the changes (see Table 3) in the optimal societal revenue due to the changes in the parameter values. Table 3 shows only those parameters out of 13 parameters in the Table 1 which bring significant change in the result. While determining the sensitivity of a parameter, all other parameters remain fixed as in Table 1 with $\tau = 4.44$ M$/metric ton at which societal revenue is maximum (4.096 M$/year). This table shows that the maximal demand A is the most sensitive parameter. If the parameter A is enhanced by 50% or 25% from its default value 0.9 (see Table 1), then the optimal societal revenue will be increased by 83.86% or 40.18%, respectively. On the contrary, if it is decreased by 50% or 25%, the societal revenue decreases by 76.07% or 37.95%, respectively. It is observable that the stability region interchanges between disease-free and endemic states with the variation of most of the parameters. However, the scenario is completely different with the variation of j and A_1, where the stability region always remains disease-free. It is interesting to observe that the optimal societal revenue always decreases from its default value with any increment or decrement of the parameter τ. This implies that the value of τ (4.44 M$/metric ton) is optimal and the corresponding societal revenue (4.096 M$/year) is also optimal.

5. Discussion

The fishery has become one of the significant subsistences across the globe. According to the 2021 report of the Food and Agriculture Organization (FAO), about 38.98 million people are engaged in fisheries [71], justifying why most fisheries are under stress. Some governing agencies try to restrict harvesting by imposing a tax per unit of biomass of landed fish. Although taxation controls overfishing, an irrational tax policy may negatively affect fishery dynamics and revenue generation. It may help increase fishery-related infection and drastically reduce the amount of landed fish, causing a significant difference between the demand and supply of this globally accepted food item. A pronounced effect of this imbalance is the price hike of the fish stock, which may directly impact the fishery & related industries and employability. So the question is: how much taxation benefits a fishery in the presence of infection? Does there exist any trade-off? How does the intricacy of demand, tax, and disease play a role in fishery dynamics and revenue generation? We proposed a nonlinear bioeconomic harvesting model of a single-species fishery with infection, variable market price, and nonlinear demand to answer these questions and explore taxation's ecological and economic effects. To our knowledge, such a theoretical investigation is rare in the literature.

Table 3: Effect on optimal societal revenue due to the change in the key parameters and the corresponding changes in the stability state. The seven parameters are varied 25 or 50 per cent upside or downside from their default values mentioned in Table 1, and the corresponding changes in the optimal societal revenue are tabulated. Here a " + " sign indicates a shift in the upside, and a " − " sign suggests a change in the downside.

Parameters	Changes in parameters (%)	Changes in Θ (%)	Stability region	Parameters	Changes in parameters (%)	Changes in Θ (%)	Stability region
A	+50	83.86	Disease-free	A_1	+50	−6.09	Disease-free
	−50	−76.07	Endemic		−50	6.21	Disease-free
	+25	40.18	Disease-free		+25	−3.06	Disease-free
	−25	−37.95	Endemic		−25	3.09	Disease-free
A_2	+50	−20.81	Endemic	c	+50	−4.001	Endemic
	−50	22.85	Disease-free		−50	−4.47	Disease-free
	+25	−10.65	Disease-free		+25	−0.999	Disease-free
	−25	11.19	Disease-free		−25	−1.13	Disease-free
L	+50	−2.91	Disease-free	τ	+50	−12.59	Endemic
	−50	−24.00	Endemic		−50	−11.11	Disease-free
	+25	−1.07	Disease-free		+25	−3.03	Endemic
	−25	−2.84	Endemic		−25	−2.96	Disease-free
j	+50	−0.08	Disease-free				
	−50	−1.79	Disease-free				
	+25	−0.13	Disease-free				
	−25	−0.03	Disease-free				

We have considered a nonlinear quadratic market demand to represent the demand-price relation. Such a quadratic demand may be a more suitable demand function, compared to constant [10], linear [53], and saturated [8] types functions, when the demand of a particular commodity decreases sharply if its price is high.

Our system has seven equilibrium points, of which three are always unstable, and the remaining four may be stable or unstable depending on the parametric conditions. The bifurcation analysis for the tax parameter classified the system stability into three distinct dynamic regimes. It is revealed that the system remains disease-free if the regulatory tax is low, which promotes intensive harvesting. Such intense harvesting reduces the infected fish, causing the elimination of infection from the system. A reduction in the harvesting efforts due to increased tax helps the infection spread, and the disease can invade the fish population for an extended range of intermediate tax. Healthy fish density gradually decreases in this case with a complementary increase in the infected fish density. Since fish harvesting is relatively low in the medium range of tax, its supply reduces significantly, increasing the difference between demand and supply with the growing tax. Therefore, the fish price steadily increases following the open market theory. As the market price becomes too high, the demand gradually diminishes to zero (see Fig. 5).

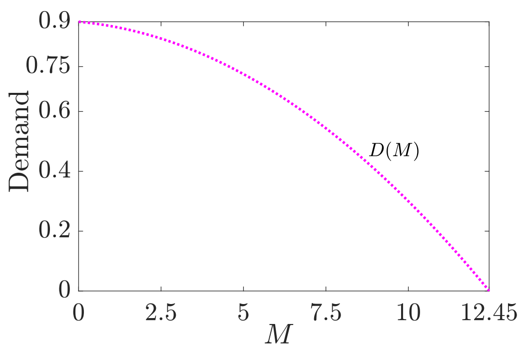

Figure 5: Quadratic demand curve $D(M) = A - A_1 - A_2M^2$ is plotted as a function of price, M, in the range $0 \leqslant M \leqslant 12.45$. The upper value of M is fixed from Fig. 2 (c), where the harvesting-free equilibrium E_2 is stable. It shows that demand decreases from its maximum when the price is zero to its minimum when it is high. The parameters are as in Table 1.

Thus, the fisheries experience a tax-induced functioning instability at the higher level of fishing tax. In such a case, fishing is no more economically viable, and the fishers opt out of fisheries due to a lack of demand and high fishing tax. The ecological and economic effect of such a non-harvesting regime shift is immense. Such a shift from a harvesting regime to a non-harvesting regime is not due to the scarcity of harvested stock but the need for better governance. Therefore, it is necessary to control the tax parameter sustainably, and the challenge for the regulatory agency is to optimize this parameter for maximal socio-economic benefits.

It is worth mentioning that the fisherman's income will be maximum if they do not pay any fishing tax. Indeed, their earnings will gradually decrease with the increasing tax. On the other hand, the regulatory authority earns more revenue by charging a higher fishing tax. Imposing a tax is beneficial because it controls harvesting and saves fishery from overexploitation. Secondly, the regulatory authority may use the tax revenue for various welfare measures for the people associated with the fishery, marine ecosystem, coastal management, and related value chains for sustainable development and economic prosperity. Therefore, an effective regulatory taxation policy may play a crucial role in the sustainable use of fisheries through a win-win solution. A low tax may help make the system infection-free, while infection may persist if the tax is high. A higher regulatory tax, however, may put an end to harvesting. It implies that there exists a trade-off. Consequently, an optimal taxation policy is necessary to make a balance among the harvesting intensity, infection spreading, market demand & supply, and revenue earnings. It is revealed that some optimum tax exists, where the societal income is maximum and occurs at the disease-free state for some lower optimal tax. However, the regulatory authority earns the maximum revenue for some higher optimal tax in the disease state. Fishers' income is maximized with no tax and steadily decreases to zero with increasing tax. Noticeably, the gap between demand and supply of fish widens with the increasing tax, causing a steady price increase in this globally accepted renewable food item. Thus, the higher regulatory tax causes an imbalance in the fish supply and price, which may severely impact fishery, fishery-related industries, and employability. Therefore, there should be an optimal tax policy for which the fishery sustains and maximizes societal revenue. The future of fishing thus depends on many interconnected factors, including infection control, ecosystem management, maintaining the demand-supply chain, and implementing a justifiable regulatory taxation policy through good governance. Indeed, this will help put a step forward in achieving the sustainable development goals by 2030 as set by the United Nations.

6. Conclusion

Our study reveals that fish disease may alter the system dynamics and reduce revenue generation. Taxation might control overfishing but may help disease spreading and price hiking if it is high. The system may experience a regime shift for a too high regulatory tax, where the system enters into a non-harvesting regime from the harvesting one. There exist some trade-offs between revenue generation and regulatory tax. The overall societal revenue is highest at the optimal tax level. However, the individual earnings in these heads are different at different tax levels.

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