J. Nonlinear Sci. Appl., 16 (2023), 79–89

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

The reciprocity gap functional method for an impedance inverse scattering problem in chiral media

Evagelia S. Athanasiadou

Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis GR-15784, Athens, Greece.

Abstract

A time-harmonic electromagnetic wave is scattered by a buried object. We assume that the scattering object has an impedance boundary surface and it is embedded in a piecewise homogeneous isotropic background chiral medium. Using a chiral reciprocity gap operator and appropriate density properties of chiral Herglotz wave functions we solve an inverse scattering problem for reconstruction of the shape of the scatterer from the knowledge of the tangential components of electric and magnetic fields, without requiring any a priori information of the physical properties. Furthermore, a characterization of the surface impedance of the scattering object is proved.

Keywords: Inverse scattering, reciprocity gap functional, chiral media, impedance boundary condition.

2020 MSC: 35Q60, 35R30, 78A46, 35P25.

1. Introduction

The chiral media are characterized by two constitutive relations in which the electric and magnetic fields are coupled via a material parameter, the chirality. In this work, Drude-Born-Fedorov constitutive relations are used, because they are symmetric under time reversality and duality transformation [16]. Chiral materials are those which exhibit optical activity in the sense that the plane of vibration of linearly polarized light is rotated on passage through an optically active medium. For details on the properties of chiral media we refer to the books [16–18].

In recent years various papers have been written on direct and inverse electromagnetic scattering problems in chiral media. Indicatively we refer to the papers [1, 4, 6, 7]. In an isotropic homogeneous chiral medium the electric and magnetic fields are composed of left-circularly polarized and right-circularly polarized components which can both propagate with different phase speeds. So, in the applications we can use the Bohren decomposition [16, 17] of electric and magnetic fields into suitable Beltrami fields, which satisfy simple differential equations of first order.

In [6] the direct electromagnetic scattering problem by a mixed impedance screen in chiral media is studied. Beltrami fields have been used for the uniqueness and a variational method has been employed for the existence of solution. An inverse scattering problem for the same scattering model has been studied

Email address: eathan@math.uoa.gr (Evagelia S. Athanasiadou)

doi: 10.22436/jnsa.016.02.01

Received: 2021-06-11 *Revised:* 2022-01-12 *Accepted:* 2022-02-27

in [7] where a modified linear sampling method based on a factorization of the chiral far-field operator has been used. In [1], Ammari and Nédélec have proved that the Silver-Müller radiation condition remains valid for chiral media. In [2], the chirality is defined as a property of the far-field operator.

In this paper we consider the inverse scattering problem of determining both the shape and the surface impedance of a buried object from the knowledge of the electric and magnetic fields measured on the surface of the earth. We use a qualitative method [8], which is a modified version of the classical linear sampling method. This method is based on the chiral reciprocity gap operator and was introduced by Colton and Haddar in [15] for acoustic waves and by Cakoni, Fares and Haddar in [12] for electromagnetic waves. Applications of this method we may find in medical imaging, ground imagery and target identification (see [11]). In linear elasticity a reciprocity gap functional has been used for solving an inverse mixed impedance scattering problem [5]. In [3] a reciprocity gap functional for chiral media has been defined in order an inverse scattering problem for a perfect conductor to be solved. In [9] the shape and the surface impedance of a buried coated scattering object have been determined, while in [13] the same method has been applied to solve an electromagnetic inverse scattering problem for an anisotropic dielectric that is partially coated by a thin layer of highly conducting material. Using this method, an inverse electromagnetic scattering problem for a perfectly conducting cavity with interior measurements has been solved in [20]. In [22] an interior inverse scattering problem for a cavity with an inhomogeneous medium inside has been studied. Also in [14], the boundary and the permittivity of the scattering object in radar imaging have been calculated. For more details on the linear sampling and reciprocity gap functional method we refer to the book [11].

In Section 2 we consider the basic equations of electromagnetic fields in chiral media and introduce the function spaces which will be used to formulate the impedance scattering problem. In Section 3 we define the reciprocity gap operator and prove that it is injective and has a dense range. Finally, in the Section 4 we prove the main theorem that characterizes the shape of the scattering object as well as giving an estimate of the surface impedance.

2. The scattering problem in chiral media

We consider the Drude-Born-Fedorov constitutive relations [16]:

$$\mathcal{D} = \varepsilon\left(\mathcal{E} + \beta \operatorname{curl}(\mathcal{E})\right), \quad \mathcal{B} = \mu\left(\mathcal{H} + \beta \operatorname{curl}(\mathcal{H})\right), \tag{2.1}$$

where $\mathcal{E}$, $\mathcal{H}$ are the electric and magnetic fields, $\mathcal{D}$, $\mathcal{B}$ are the electric and magnetic densities, respectively, β is the chirality measure, ε the electric permittivity and μ the magnetic permeability. In a source-free region we have that

$$\operatorname{curl}(\mathcal{E}) - i\omega\mathcal{B} = 0, \quad \operatorname{curl}(\mathcal{H}) + i\omega\mathcal{D} = 0, \tag{2.2}$$

where we have suppressed a time dependance of $e^{-i\omega t}$, $\omega > 0$ being the angular frequency. From (2.1) and (2.2) we get the following equations:

$$\operatorname{curl}(\mathcal{E}) = \beta\gamma^2\mathcal{E} + i\omega\mu\left(\frac{\gamma}{k}\right)^2\mathcal{H}, \qquad \operatorname{curl}(\mathcal{H}) = \beta\gamma^2\mathcal{H} - i\omega\varepsilon\left(\frac{\gamma}{k}\right)^2\mathcal{E}, \tag{2.3}$$

where $k^2 = \omega^2\varepsilon\mu$ and $\gamma^2 = k^2(1-\beta^2k^2)^{-1}$. We note that $\mathcal{E}$ and $\mathcal{H}$ are divergence-free fields and k is not a wave number but it is a shorthand notation without any particular physical significance. Moreover, it is valid that $|k\beta| < 1$ ([17, p.87]). From (2.3) we take

$$\operatorname{curl}(\operatorname{curl}(\mathcal{E})) - 2\beta\gamma^2\operatorname{curl}(\mathcal{E}) - \gamma^2\mathcal{E} = 0. \tag{2.4}$$

We note that the magnetic field satisfies the same equation. In a homogeneous isotropic chiral medium the electric and magnetic fields are composed of left-circularly polarized (LCP) and right-circularly polarized

(RCP) components with different wave numbers. To see this, we make use of the Bohren decomposition of $\mathcal{E}$ and $\mathcal{H}$ into suitable Beltrami fields Q_L and Q_R [17],

$$\mathcal{E} = Q_L + Q_R, \quad \mathcal{H} = -i\sqrt{\frac{\varepsilon}{\mu}}\,(Q_L - Q_R),$$

and hence

$$Q_L = \frac{1}{2}(\mathcal{E} + i\sqrt{\frac{\mu}{\varepsilon}}\mathcal{H}), \quad Q_R = \frac{1}{2}(\mathcal{E} - i\sqrt{\frac{\mu}{\varepsilon}}\mathcal{H}).$$

The Beltrami fields satisfy the equations

$$\mathrm{curl}(Q_L) = \gamma_L Q_L, \quad \mathrm{curl}(Q_R) = -\gamma_R Q_R, \tag{2.5}$$

as well as the equations

$$\mathrm{curl}(\mathrm{curl}(Q_L)) = \gamma_L^2 Q_L, \quad \mathrm{curl}(\mathrm{curl}(Q_R)) = \gamma_R^2 Q_R, \tag{2.6}$$

with

$$\gamma_L = k\,(1 - k\beta)^{-1}, \quad \gamma_R = k\,(1 + k\beta)^{-1},$$

being the wave numbers for the LCP and RCP Beltrami fields, respectively, and satisfying the following relations:

$$\gamma_L + \gamma_R = 2k^{-1}\gamma^2, \quad \gamma_L - \gamma_R = 2\beta\gamma^2, \quad \gamma_L\gamma_R = \gamma^2.$$

The equations (2.5) show that the homogeneous isotropic chiral media are circularly birefringent. For details on the physical background for chiral media again we refer to the books [16–18].

We assume that a scatterer D with C^2-boundary ∂D is embedded in a piecewise homogeneous isotropic chiral medium with $\mathbb{R}^3 \setminus \overline{D}$ to be connected. The boundary ∂D of D is a surface impedance which is described by the positive constant λ.

We consider a bounded domain Ω, which contains $\overline{D}$, with C^2- boundary $\partial\Omega$. We denote by $\nu = \nu(x)$ the outward unit normal vector at the point x of the corresponding surface. The medium $\Omega \setminus \overline{D}$ which will be referred to as the background medium is characterized by the chirality β_b, the electric permittivity ε_b and the magnetic permeability μ_b. Also, let β_0, ε_0, and μ_0 be the corresponding parameters in the exterior $\mathbb{R}^3 \setminus \overline{\Omega}$ of Ω. All the physical parameters are assumed to be positive constants.

We consider that the incident field is an electric dipole located at x_0 with polarization $p \in \mathbb{R}^3$ in a chiral medium. The point x_0 lies on an auxiliary close surface Λ contained in $\mathbb{R}^3 \setminus \overline{\Omega}$. In particular, the incident electric field is given by the formula [16, 17]

$$E_{x_0}(x, p, \gamma_0) = \frac{k_0}{2\gamma_0^2} p \cdot \left\{ \left(\gamma_{0L}\widetilde{I} + \frac{1}{\gamma_{0L}}\nabla\nabla + \nabla \times \widetilde{I} \right) \frac{e^{i\gamma_{0L}|x - x_0|}}{4\pi|x - x_0|} + \left(\gamma_{0R}\widetilde{I} + \frac{1}{\gamma_{0R}}\nabla\nabla - \nabla \times \widetilde{I} \right) \frac{e^{i\gamma_{0R}|x - x_0|}}{4\pi|x - x_0|} \right\},$$

where $\widetilde{I}$ is the identity dyadic in $\mathbb{R}^3$ and γ_{0L} and γ_{0R} are the LCP and RCP wave numbers, respectively in $\mathbb{R}^3 \setminus \overline{\Omega}$ with

$$\gamma_{0L} = k_0(1 - k_0\beta_0)^{-1}, \quad \gamma_{0R} = k_0(1 + k_0\beta_0)^{-1},$$

and $\gamma_0^2 = \gamma_{0L}\gamma_{0R}$, $k_0^2 \equiv k^2 = \omega^2\varepsilon_0\mu_0$.

The electric wave E^i which is incident on the scatterer D is given by

$$E^i(x) \equiv E^i_{x_0}(x, p) = E_{x_0}(x, p, \gamma_0) + E^{s,b}_{x_0}(x, p),$$

where $E^{s,b}_{x_0}(x, p)$ is the scattered field due to the background medium. Also, the wave E^i in $\Omega \setminus \overline{D}$ is given by

$$E^i(x) \equiv E^i_{x_0}(x, p) = p \cdot \widetilde{B}(x, x_0), \tag{2.7}$$

where $\widetilde{B}(x, x_0)$ is the dyadic Green's function of the chiral background medium. We define $\eta(x) = \eta_b = (\varepsilon_b\mu_b)(\varepsilon_0\mu_0)^{-1}$ for $x \in \Omega \setminus \overline{D}$ and $\eta(x) = 1$ for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$ as well as $\beta(x) = \beta_b$ for $x \in \Omega \setminus \overline{D}$ and $\beta(x) = \beta_0$ for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, then $\widetilde{B}(x, x_0)$ satisfies the equation

$$\left(k^{-2} - \eta(x)\beta^2(x)\right) \mathrm{curl}(\mathrm{curl}(\widetilde{B}(x, x_0))) - 2\beta(x)\eta(x)\mathrm{curl}(\widetilde{B}(x, x_0)) - \eta(x)\widetilde{B}(x, x_0) = \widetilde{I}\delta(x - x_0), \tag{2.8}$$

with respect to x. In order to formulate the general impedance boundary value problem in chiral media we need the following function spaces:

$$H(\mathrm{curl}, D) = \left\{u \in \left(L^2(D)\right)^3 : \ \mathrm{curl}(u) \in \left(L^2(D)\right)^3\right\},$$
$$L^2_t(\partial D) = \left\{u \in \left(L^2(\partial D)\right)^3 : \ \nu \cdot u = 0 \text{ on } \partial D\right\},$$
$$X(D, \partial D) = \left\{u \in H(\mathrm{curl}, D) : \ \nu \times u|_{\partial D} \in L^2_t(\partial D)\right\},$$

with the norm

$$\|u\|^2_{X(D,\partial D)} = \|u\|^2_{H(\mathrm{curl},D)} + \|\nu \times u\|^2_{L^2(\partial D)}.$$

Also, we define the space of solutions

$$\mathbb{H}(\Omega) = \left\{u \in H(\mathrm{curl}, \Omega) : u_T \in L^2_t(\partial D), \ \mathrm{curl}(u_T) \in L^2_t(\partial D), \ \mathrm{curl}(\mathrm{curl}\ u - 2\beta_b\gamma_b^2 \mathrm{curl}\ u - \gamma_b^2 u = 0\right\},$$

where $u_T = (\nu \times u) \times \nu$. For the traces $\nu \times u|_{\partial D}$ and $\nu \times (u \times \nu)|_{\partial D}$ of $u \in H(\mathrm{curl}, D)$ we have

$$H^{-\frac{1}{2}}_{\mathrm{div}}(\partial D) = \left\{u \in \left(H^{-\frac{1}{2}}(\partial D)\right)^3 : \ \nu \cdot u = 0, \ \mathrm{div}_{\partial D} u \in H^{-\frac{1}{2}}(\partial D)\right\},$$
$$H^{-\frac{1}{2}}_{\mathrm{curl}}(\partial D) = \left\{u \in \left(H^{-\frac{1}{2}}(\partial D)\right)^3 : \ \nu \cdot u = 0, \ \mathrm{curl}_{\partial D} u \in H^{-\frac{1}{2}}(\partial D)\right\},$$

where with $\mathrm{div}_{\partial D}$ and $\mathrm{curl}_{\partial D}$ we denote the surface div and the surface curl, respectively. Finally, for the exterior domain $\mathbb{R}^3 \setminus \overline{D}$ we define the spaces $H_{loc}(\mathrm{curl}, \mathbb{R}^3 \setminus \overline{D})$, $H_{loc}(\mathbb{R}^3 \setminus \overline{D}, \partial D)$ and $X_{loc}(\mathbb{R}^3 \setminus \overline{D}, \partial D)$ considering the domain $(\mathbb{R}^3 \setminus \overline{D}) \bigcap B_R$, where B_R is a ball of arbitrary radius R.

The exterior impedance boundary value problem in chiral media is formulated as follows: given $f \in L^2_t(\partial D)$ find $E^s \in X_{loc}(\mathbb{R}^3 \setminus \overline{D}, \partial D)$ such that:

$$\left(k^{-2} - \eta(x)\beta^2(x)\right) \mathrm{curlcurl}\ E^s - 2\beta(x)\eta(x)\mathrm{curl}\ E^s - \eta(x)E^s = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}, \tag{2.9}$$
$$\nu \times \mathrm{curl}\ E^s - ik_b^{-1}\gamma_b^2\lambda(\nu \times E^s) \times \nu - \beta_b\gamma_b^2 \nu \times E^s = f \text{ on } \partial D, \tag{2.10}$$
$$\hat{x} \times \mathrm{curl}\ E^s - \beta_0\gamma_0^2 \hat{x} \times E^s + ik_0^{-1}\gamma_0^2 E^s = o\left(\frac{1}{|x|}\right), \ |x| \to \infty \tag{2.11}$$
$$\text{uniformly in all directions } \hat{x} = \frac{x}{|x|} \in S^2.$$

If $f = -\nu \times \mathrm{curl}\ E^i + ik_b^{-1}\gamma_b^2\lambda(\nu \times E^i) \times \nu + \beta_b\gamma_b^2 \nu \times E^i$ on ∂D then the problem (2.9)-(2.11) describes the direct impedance scattering problem in chiral media and E^s is the corresponding scattered field which is connected to the total field E and the incident field E^i with the relation

$$E = E^i + E^s. \tag{2.12}$$

The direct scattering problem can be studied as in [6]. The uniqueness of solution has been proved via the Beltrami fields, while for the existence of solution the variational method has been employed using a Calderon type operator [19] for chiral media. The corresponding inverse scattering problem is the determination of the unknown boundary of D from the knowledge of the tangential components $\nu \times E$ and $\nu \times H$ on the boundary $\partial\Omega$ for all points $x_0 \in \Lambda$ as well as the evaluation of the surface impedance λ.

Let $z \in D$ and $E^z \in H(\mathrm{curl}, D)$. We consider the following chiral interior impedance boundary value problem corresponding to (2.9)-(2.12); given $f \in L^2_t(\partial D)$ find $E^z \in H(\mathrm{curl}, D)$ such that:

$$\mathrm{curl}\ \mathrm{curl}\ E^z - 2\beta_b\gamma_b^2 \mathrm{curl}\ E^z - \gamma_b^2 E^z = 0 \ \text{ in } D, \tag{2.13}$$

$$\nu \times \mathrm{curl}\ E^z - ik_b^{-1}\gamma_b^2\lambda(\nu \times E^z) \times \nu + \beta_b\gamma_b^2 \nu \times E^z = f \ \text{ on } \partial D. \tag{2.14}$$

The values of k for which the corresponding homogeneous boundary value interior problem admits a nontrivial solution will be referred to as chiral Maxwell eigenvalues for D.

In chiral media, a Stratton-Chu type exterior integral representation for a radiating solution of equation (2.4) is the following

$$\begin{aligned} E^s(r) = &-2\beta\gamma^2 \int_S \widetilde{B}(r, r') \cdot \left[\nu \times E^s(r')\right] ds(r') \\ &+ \int_S \left\{\widetilde{B}(r, r') \cdot \left[\nu \times \mathrm{curl}\ E^s(r')\right] + \left[\mathrm{curl}_r \widetilde{B}(r, r')\right] \cdot \left[\nu \times E^s(r')\right]\right\} ds(r'). \end{aligned} \tag{2.15}$$

3. The chiral reciprocity gap operator

Let $E = E_{x_0}(\cdot, p)$ be the solution of the scattering problem (2.9)-(2.12). Then for $W \in H(\mathrm{curl}, \Omega)$ we define the chiral reciprocity gap functional

$$\mathcal{R}(E, W) = \int_{\partial\Omega} [(\nu \times E) \cdot \mathrm{curl}\ W - (\nu \times W) \cdot \mathrm{curl}(\ E]\ ds - 2\beta_b\gamma_b^2 \int_{\partial\Omega} [(\nu \times E) \cdot W]\ ds. \tag{3.1}$$

We note that the integrals are interpreted in the sense of the duality between $H^{-\frac{1}{2}}_{\mathrm{div}}(\partial\Omega)$, $H^{-\frac{1}{2}}_{\mathrm{curl}}(\partial\Omega)$. In particular, if $W \in \mathbb{H}(\Omega) \subset H(\mathrm{curl}, \Omega)$, then we define the chiral reciprocity gap functional operator $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ given by

$$R(W)(x_0) = \mathcal{R}\left(E_{x_0}(\cdot, p(x_0)), W\right) p(x_0), \quad x_0 \in \Lambda. \tag{3.2}$$

The method is based on the solvability of an integral equation for the reciprocity gap functional, which contains an appropriate family of solutions in $\mathbb{H}(\Omega)$. For this family of solutions we use the chiral Herglotz wave functions. In [4] the electric $\mathcal{E}_g$ and magnetic $\mathcal{H}_g$ chiral Herglotz wave functions have been defined as

$$\mathcal{E}_g = \mathcal{E}_{g_L} + \mathcal{E}_{g_R}, \quad \mathcal{H}_g = -i\sqrt{\frac{\varepsilon}{\mu}}\left(\mathcal{E}_{g_L} - \mathcal{E}_{g_R}\right),$$

where the LCP and the RCP Beltrami Herglotz fields $\mathcal{E}_{g_L}$ and $\mathcal{E}_{g_R}$ with kernels g_L and g_R, respectively are given by

$$\mathcal{E}_{g_L}(x) = \int_{S^2} g_L(\hat{d}_L) e^{i\gamma_L \hat{d}_L \cdot x}\ ds(\hat{d}_L), \qquad \mathcal{E}_{g_R}(x) = \int_{S^2} g_R(\hat{d}_R) e^{i\gamma_R \hat{d}_R \cdot x}\ ds(\hat{d}_R), \tag{3.3}$$

with $\hat{d}_L$, $\hat{d}_R \in S^2$. For the kernels we have $g_A : S^2 \to T^2_A(S^2)$, $A = L, R$, where

$$\begin{aligned} T^2_L(S^2) &= \left\{b_L \in \left(L^2(S^2)\right)^3 : \nu \cdot b_L = 0,\ \nu \times b_L = -ib_L\right\}, \\ T^2_R(S^2) &= \left\{b_R \in \left(L^2(S^2)\right)^3 : \nu \cdot b_R = 0,\ \nu \times b_R = ib_R\right\}. \end{aligned}$$

Also, we define the following space

$$T^2_{LR}(S^2) = \left\{b = b_L + b_R : b_L \in T^2_L(S^2),\ b_R \in T^2_R(S^2)\right\},$$

with the inner product

$$< b, h >_{T^2_{LR}(S^2)} = (b_L, h_L)_{T^2_L(S^2)} + (b_R, h_R)_{T^2_R(S^2)},$$

where $b_A, h_A, A = L, R$, are the Beltrami fields of b and h, respectively and $(b_A, h_A)_{T^2_A(S^2)} = \int_{S^2} b_A \cdot \overline{h}_A \, ds$ [4].

We consider the electric dipole

$$E_z(x, q, \gamma_b) = \frac{k_b}{2\gamma_b^2} q \cdot \left\{ \left(\gamma_{b_L} \widetilde{I} + \frac{1}{\gamma_{b_L}} \nabla\nabla + \nabla \times \widetilde{I} \right) \frac{e^{i\gamma_{b_L}|x-z|}}{4\pi|x-z|} + \left(\gamma_{b_R} \widetilde{I} + \frac{1}{\gamma_{b_R}} \nabla\nabla - \nabla \times \widetilde{I} \right) \frac{e^{i\gamma_{b_R}|x-z|}}{4\pi|x-z|} \right\},$$

located at z with polarization $q \in \mathbb{R}^3$ corresponding to γ_b, where $\gamma_b^2 = \gamma_{b_L}\gamma_{b_R} = k_b^2 \left(1 - k_b^2 \gamma_b^2\right)^{-1}$, with $k_b^2 = \omega^2 \varepsilon_b \mu_b$. We investigate the solvability of the integral equation

$$\mathcal{R}(E, \mathcal{E}_g) = \mathcal{R}(E, E_z(\cdot, q, \gamma_b)), \tag{3.4}$$

with respect to g in $T^2_{LR}(S^2)$. The determination of D is based on the behavior of g for different sampling points $z \in \Omega$.

Lemma 3.1. *The operator* $R : \mathbb{H}(\Omega) \to L^2_t(\Lambda)$ *defined by* (3.2) *is injective.*

Proof. Let $RW = 0$. Then $\mathcal{R}(E_{x_0}(\cdot, p), W) = 0$ for all $x_0 \in \Lambda$ and $p \in \mathbb{R}^3$. In (3.2) we apply the second vector Green's theorem for the first integral and Gauss' theorem for the second integral in $\Omega \setminus \overline{D}$ for E and W. Taking into account that both E and W are solutions of (2.9) in $\Omega \setminus D$ and using the impedance boundary condition on ∂D we take

$$\begin{aligned} 0 &= \int_{\partial D} [(\nu \times E) \cdot \operatorname{curl} W - (\nu \times W) \cdot \operatorname{curl} E] \, ds - 2\beta_b \gamma_b^2 \int_{\partial D} (\nu \times E) \cdot W ds \\ &= \int_{\partial D} E \cdot \left[\nu \times \operatorname{curl} W - i k_b^{-1} \gamma_b^2 \lambda (\nu \times W) \times \nu - \beta_b \gamma_b^2 (\nu \times W) \right] ds. \end{aligned} \tag{3.5}$$

We denote by $\check{E}$ the unique solution of the following boundary value problem:

$$\left(k^{-2} - \beta(x)^2 \eta(x)\right) \operatorname{curl}\operatorname{curl} \check{E} - 2\beta(x)\eta(x) \operatorname{curl} \check{E} - \eta(x)\check{E} = 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}, \tag{3.6}$$

$$\nu \times \operatorname{curl} \left(\check{E} - W\right) = i k_b^{-1} \gamma_b^2 \lambda \left[\nu \times \left(\check{E} - W\right)\right] \times \nu + \beta_b \gamma_b^2 \nu \times \left(\check{E} - W\right) \text{ on } \partial D, \tag{3.7}$$

$$\hat{x} \times \operatorname{curl} \check{E} - \beta_0 \gamma_0^2 \hat{x} \times \check{E} + i k_0^{-1} \gamma_0^2 \check{E} = o\left(\frac{1}{|x|}\right), \; |x| \to \infty,$$

uniformly in all directions of $\frac{x}{|x|} \in S^2$.

Substituting the tangential component of curl W from (3.7) into (3.5) we have that

$$0 = -\int_{\partial D} E \cdot \left[\nu \times \operatorname{curl} \check{E} - i k_b^{-1} \gamma_b^2 \lambda (\nu \times \check{E}) \times \nu - \beta_b \gamma_b^2 (\nu \times \check{E}) \right] ds. \tag{3.8}$$

In view of (2.12) and (2.7) the total electric field E is given by

$$E = p \cdot \widetilde{B}(\cdot, x_0) + E^s. \tag{3.9}$$

From (3.8) and (3.9) we get

$$\begin{aligned} 0 = &\int_{\partial D} \left[\left(\nu \times \left(p \cdot \widetilde{B}(\cdot, x_0) + E^s \right) \right) \cdot \operatorname{curl} \check{E} - (\nu \times \check{E}) \cdot \operatorname{curl} \left(p \cdot \widetilde{B}(\cdot, x_0) + E^s \right) \right] ds \\ &- 2\beta_b \gamma_b^2 \int_{\partial D} \left[\left(\nu \times \left(p \cdot \widetilde{B}(\cdot, x_0) + E^s \right) \right) \cdot \check{E} \right] ds. \end{aligned}$$

Taking into account that $\breve{E}$ and E^s are both radiating solutions to equation (3.6) we have

$$-p \cdot \left\{ \int_{\partial D} \left[\widetilde{B}(\cdot, x_0) \cdot (\nu \times \operatorname{curl} \breve{E}) + (\operatorname{curl} \widetilde{B}(\cdot, x_0)) \cdot (\nu \times \breve{E}) \right] ds - 2\beta_b \gamma_b^2 \int_{\partial D} \widetilde{B}(\cdot, x_0) \cdot (\nu \times \breve{E}) ds \right\} = 0$$

and from the Stratton-Chu type formula (2.15) for chiral media we get

$$p \cdot \breve{E}(x_0) = 0,$$

for arbitrary polarization p. Therefore $\nu \times \breve{E}(x_0) = 0$ for $x_0 \in \Lambda$. Then, by the uniqueness of the electromagnetic scattering problem for a perfectly conducting obstacle in a chiral environment [1], we conclude that $\breve{E} = 0$ outside the surface Λ. By the unique continuation we have $\breve{E} = 0$ outside D and therefore

$$\nu \times \operatorname{curl} W - ik_b^{-1} \gamma_b^2 \lambda (\nu \times W) \times \nu - \beta_b \gamma_b^2 \nu \times W = 0 \text{ on } \partial D.$$

From the uniqueness of the interior impedance chiral electromagnetic boundary value problem it implies that $W = 0$. □

Lemma 3.2. *The operator $R : \mathbb{H}(\Omega) \to L_t^2(\Lambda)$ defined by (3.2) has dense range.*

Proof. Assume that $q \in L_t^2(\Lambda)$ such that $(RW, q)_{L_t^2(\Lambda)} = 0$ for all $W \in \mathbb{H}(\Omega)$. In view of the bilinearity of functional $\mathcal{R}$ and the definition of operator R, we have that

$$(RW, q)_{L_t^2(\Lambda)} = \int_\Lambda \mathcal{R}\left(E_{x_0}(\cdot, \alpha(x_0)), W\right) ds = 0, \tag{3.10}$$

where $\alpha = (p \cdot q)p$. Setting

$$\mathcal{E}(x) = \int_\Lambda E_{x_0}\left(x, \alpha(x_0)\right) ds(x_0),$$

then from (3.10) we have that

$$\mathcal{R}(\mathcal{E}, W) = 0.$$

Applying the Green's and Gauss' theorems for W, $\mathcal{E}$ in $\Omega \setminus \overline{D}$ as in Lemma 3.2 and taking into account the boundary condition on ∂D, we conclude that

$$\mathcal{R}(\mathcal{E}, W) = -\int_{\partial D} \mathcal{E} \cdot \left[\nu \times \operatorname{curl} W - ik_b^{-1} \gamma_b^2 \lambda (\nu \times W) \times \nu - \beta_b \gamma_b^2 (\nu \times W) \right] ds = 0,$$

for all $W \in \mathbb{H}(\Omega)$. The Beltrami Herglotz fields $\mathcal{E}_{g_L}$ and $\mathcal{E}_{g_R}$ given by (3.3) solve the equation (2.4) as well as (2.5) and (2.6), respectively. In addition taking into account that the interior problem (2.13)-(2.14) is well-posed we can conclude that the set $\{\nu \times \operatorname{curl} W - ik_b^{-1} \gamma_b^2 \lambda (\nu \times W) \times \nu - \beta_b \gamma_b^2 (\nu \times W) \mid_{\partial D}\}$ is dense in $L^2(\partial D)$ (see [4, 10]). Therefore, $\nu \times \mathcal{E} = 0$ and $\nu \times \operatorname{curl} \mathcal{E} = 0$ on ∂D. Hence $\mathcal{E}$ has zero Cauchy data on ∂D and therefore $\mathcal{E} = 0$ in the domain between Λ and ∂D. Finally, taking into account the jump relations [1] of curl $\mathcal{E}$ across Λ we arrive at $\alpha = 0$ on Λ. Therefore $(p \cdot q)p = 0$ for all $p \in L_t^2(\Lambda)$ and hence $q = 0$. □

4. Determination of the shape and surface impedance

The determination of the shape of the unknown scattering object is based on the solvability of the integral equation (3.4). In particular we have the following theorem.

Theorem 4.1. *If $\mathcal{R}$ is the reciprocity gap functional corresponding to (2.9)-(2.12), then we have following.*

(i) *Let $z \in D$. Then for a given $\epsilon > 0$ there exists a $g_z^\epsilon \in T_{LR}^2(S^2)$ such that*

$$\|\mathcal{R}(E, \mathcal{E}_{g_z^\epsilon}) - \mathcal{R}(E, E_z(\cdot, q, \gamma_b))\|_{L^2(\Lambda)} < \epsilon$$

and the chiral Herglotz wave function $\mathcal{E}_{g_z^\epsilon}$ converges to the solution of the interior boundary value problem in

$X(D, \partial D)$ *as* $\epsilon \to 0$. *Moreover,*

$$\lim_{\text{dist}(z,\partial D)\to 0} \|\mathcal{E}_{g_z^\epsilon}\|_{X(D,\partial D)} = \infty, \quad \lim_{\text{dist}(z,\partial D)\to 0} \|g_z^\epsilon\|_{T^2_{LR}(S^2)} = \infty.$$

(ii) *Let* $z \in \mathbb{R}^3 \setminus \overline{D}$. *Then for a given* $\epsilon > 0$, *if* $g_z^\epsilon \in T^2_{LR}(S^2)$ *satisfies*

$$\|\mathcal{R}(E, \mathcal{E}_{g_z^\epsilon}) - \mathcal{R}(E, E_z(\cdot, q, \gamma_b))\|_{L^2(\Lambda)} < \epsilon,$$

we have that

$$\lim_{\epsilon\to 0} \|\mathcal{E}_{g_z^\epsilon}\|_{X(D,\partial D)} = \infty, \quad \lim_{\epsilon\to 0} \|g_z^\epsilon\|_{T^2_{LR}(S^2)} = \infty.$$

Proof.

(i) Let $z \in D$. From the definition (3.1), taking into account that E is the total field and W and $E_z(\cdot, q, \gamma_b)$ are solutions to equation (2.9) in $\Omega \setminus \overline{D}$ and using the boundary condition on D, we have that

$$\mathcal{R}(E, W) - \mathcal{R}(E, E_z(\cdot, q, \gamma_b)) = -\int_{\partial D} [\nu \times W - \nu \times E_z(\cdot, q, \gamma_b)] \cdot \text{curl } E \, ds.$$

Taking into account that the chiral Herglotz functions are dense with respect to the $H(\text{curl}, D)$ norm in $\mathbb{H}(\Omega)$ and using the trace theorem it follows that for every $\epsilon > 0$ there exists a chiral electric Herglotz function $\mathcal{E}_{g_z^\epsilon}$ such that $\nu \times \text{curl } \mathcal{E}_{g_z^\epsilon} - ik_b^{-1}\gamma_b^2(\nu \times \mathcal{E}_{g_z^\epsilon}) \times \nu - \beta_b\gamma_b^2\nu \times \mathcal{E}_{g_z^\epsilon}$ approximates $\nu \times \text{curl } E_z(\cdot, q, \gamma_b) - ik_b^{-1}\gamma_b^2(\nu \times E_z(\cdot, q, \gamma_b)) \times \nu - \beta_b\gamma_b^2\nu \times E_z(\cdot, q, \gamma_b)$ with respect to $L^2_t(\partial D)$. Also g_z^ϵ is an approximate solution to (3.4) and $\mathcal{E}_{g_z^\epsilon}$ converges to the solution of the chiral interior boundary value problem (2.13)-(2.14). Moreover, since $E_z(\cdot, q, \gamma_b)$ with respect to the $X(D, \partial D)$ norm blows up as z approaches the boundary ∂D from inside, we obtain that for a fixed $\epsilon > 0$,

$$\lim_{\text{dist}(z,\partial D)\to 0} \|\mathcal{E}_{g_z^\epsilon}\|_{X(D,\partial D)} = \infty \quad \text{and} \quad \lim_{\text{dist}(z,\partial D)\to 0} \|g_z^\epsilon\|_{T^2_{LR}(S^2)} = \infty.$$

(ii) Let $z \in \Omega \setminus \overline{D}$. From (3.1) substituting $E = E^i + E^s$ and (3.9) we take

$$\mathcal{R}(E, E_z(\cdot, q, \gamma_b)) = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_{\partial\Omega} \left[\left(\nu \times E^s_{x_0}(x,p)\right) \cdot \text{curl } E_z(x,q,\gamma_b) - (\nu \times E_z(x,q,\gamma_b)) \cdot \text{curl } E^s_{x_0}(x,p)\right] \, ds(x) \\ &\quad - 2\beta_b\gamma_b^2 \int_{\partial\Omega} \left(\nu \times E^s_{x_0}(x,p)\right) \cdot E_z(x,q,\gamma_b) \, ds(x), \\ I_2 &= \int_{\partial\Omega} \left[\left(\nu \times E^i_{x_0}(x,p)\right) \cdot \text{curl } E_z(x,q,\gamma_b) - (\nu \times E_z(x,q,\gamma_b)) \cdot \text{curl } E^i_{x_0}(x,p)\right] \, ds(x) \\ &\quad - 2\beta_b\gamma_b^2 \int_{\partial\Omega} \left(\nu \times E^i_{x_0}(x,p)\right) \cdot E_z(x,q,\gamma_b) \, ds(x), \end{aligned}$$

By making use of the reciprocity properties [16],

$$\widetilde{B}(x, x_0) = \left[\widetilde{B}(x_0, x)\right]^\top, \quad \text{curl}_x \widetilde{B}(x, x_0) = \left[\text{curl}_{x_0} \widetilde{B}(x_0, x)\right]^\top,$$

where $\top$ denotes transposition, we can consider that the background dyadic Green's function solves (2.8) with respect to x_0. Hence $E^s_{x_0}(x,p)$ satisfies the same equation with respect to x_0. Therefore the integral I_1 gives a solution $W(x_0)$ of (2.8). The function $E_z(x, q, \gamma_b)$ is the fundamental solution of

$$\text{curl curl } E - 2\beta_b\gamma_b^2 \text{curl } E - \gamma_b^2 E = 0 \tag{4.1}$$

and $E^i_{x_0}(x,p) = p \cdot \widetilde{B}(x,x_0)$, $x \in \Omega \setminus \overline{D}$, is a solution of (4.1). Hence I_2 is an integral representation Stratton-Chu type in chiral media (2.15) for $-p \cdot \widetilde{B}(z,x_0)$, $z \in \Omega \setminus \overline{D}$. Let $\mathcal{E}_{g^\epsilon_z}$ be a chiral electric Herglotz function such that

$$\|\mathcal{R}(E,\mathcal{E}_{g^\epsilon_z}) - \mathcal{R}(E,E_z(\cdot,q,\gamma_b))\|_{L^2(\Lambda)} < \epsilon. \tag{4.2}$$

From the definition (3.1) and the boundary condition we obtain

$$\mathcal{R}(E,\mathcal{E}_{g^\epsilon_z}) = -\int_{\partial D} E \cdot \left[\nu \times \operatorname{curl} \mathcal{E}_{g^\epsilon_z} - ik_b^{-1}\gamma_b^2 \lambda(\nu \times \mathcal{E}_{g^\epsilon_z}) \times \nu - \beta_b\gamma_b^2(\nu \times \mathcal{E}_{g^\epsilon_z})\right] ds.$$

Therefore

$$\begin{aligned} &\mathcal{R}(E,\mathcal{E}_{g^\epsilon_z}) - \mathcal{R}(E,E_z(\cdot,q,\gamma_b)) \\ &\quad = -W(x_0) + p \cdot \widetilde{B}(x,x_0) - \int_{\partial D} E \cdot \left[\nu \times \operatorname{curl} \mathcal{E}_{g^\epsilon_z} - ik_b^{-1}\gamma_b^2 \lambda(\nu \times \mathcal{E}_{g^\epsilon_z}) \times \nu - \beta_b\gamma_b^2(\nu \times \mathcal{E}_{g^\epsilon_z})\right] ds. \end{aligned} \tag{4.3}$$

We assume that $\|\mathcal{E}_{g^\epsilon_z}\|_{X(D,\partial D)} < c$ with c being a positive constant independent of ϵ. Applying the trace theorem we take that $\nu \times \mathcal{E}_{g^\epsilon_z}$ is also bounded in the $H^{-\frac{1}{2}}_{\mathrm{div}}(\partial D)$ norm. Therefore there exists a weakly convergent subfamily converging to a function $V \in H^{-\frac{1}{2}}_{\mathrm{div}}(\partial D)$ as $\epsilon \to 0$. For $x_0 \in \Lambda$ we set

$$U(x_0) = -\int_{\partial D} E \cdot \left[\nu \times \operatorname{curl} V - ik_b^{-1}\gamma_b^2 \lambda(\nu \times V) \times \nu - \beta_b\gamma_b^2(\nu \times V)\right] ds. \tag{4.4}$$

From (4.2), (4.3), and (4.4) we obtain

$$U(x_0) = W(x_0) + p \cdot \widetilde{B}(z,x_0), \ x_0 \in \Lambda. \tag{4.5}$$

Taking into account that the functions $U(x_0)$ and $W(x_0)$ are radiating solutions of (2.8) and using the unique continuation principle we conclude that (4.5) holds true in $\mathbb{R}^3 \setminus \left(\overline{D} \cup \{z\}\right)$. If we now let $x_0 \to z$, then we arrive at a contradiction. □

Remark 4.2. The determination of the boundary ∂D of the scatterer is based on the integral equation (3.4) which contains chiral Herglotz functions in $\mathbb{H}(\Omega)$. In particular, if $\mathcal{E}_{g^\epsilon_z}$ is a solution of (3.4) then the boundary ∂D of the scatterer is reconstructed from points z with $\lim_{\epsilon\to 0} \|g^\epsilon_z\|_{T^2_{LR}(S^2)} = \infty$. It is obvious that the boundary ∂D cannot be found from $\lim_{\epsilon\to 0} \|\mathcal{E}_{g^\epsilon_z}\|_{X(D,\partial D)} = \infty$ since the corresponding norm is defined on the unknown scatterer D. Alternatively one can use instead of the chiral Herglotz functions appropriate potentials, [1, 12].

Finally, assuming that the shape of the scatterer is known, we will establish an expression for the surface impedance λ. In particular we prove the following theorem.

Theorem 4.3. *Let E^z be the solution of (2.13)-(2.14) for a fix point $z \in D$. Then, the surface impedance λ is given by*

$$\lambda = \frac{k_b}{2\gamma_b^2} \frac{\operatorname{Im}(q \cdot E^z(z)) + I_z(\Omega,q,\gamma_b)}{\int_{\partial D} |\nu \times (E^z - E_z(\cdot,q,\gamma_b))|^2 ds}, \tag{4.6}$$

where the integral

$$\begin{aligned} I_z(\Omega,q,\gamma_b) = &-i\int_{\partial\Omega} \left[(\nu \times E_z(\cdot,q,\gamma_b)) \cdot \operatorname{curl} \overline{E_z(\cdot,q,\gamma_b)} - (\nu \times \overline{E_z(\cdot,q,\gamma_b)}) \cdot \operatorname{curl} E_z(\cdot,q,\gamma_b)\right] ds \\ &+ 2i\beta_b\gamma_b^2 \int_{\partial\Omega} (\nu \times E_z(\cdot,q,\gamma_b)) \cdot \overline{E_z(\cdot,q,\gamma_b)}\, ds \end{aligned}$$

is depended on z, Ω, and q.

Proof. Let $z \in D$. We consider the unique solution E^z of the interior mixed boundary value problem (2.13)-(2.14) and we define the function

$$U^z(x) = E^z(x) + E_z(x, q, \gamma_b), \ x \in D.$$

Taking into account the boundary conditions

$$\nu \times \text{curl } U^z = ik_b^{-1}\gamma_b^2\lambda(\nu \times U^z) \times \nu + \beta_b\gamma_b^2\nu \times U^z \text{ on } \partial D,$$

the integral

$$I = \int_{\partial D} \left[(\nu \times U^z) \cdot \text{curl } \overline{U^z} - (\nu \times \overline{U^z}) \cdot \text{curl } U^z\right] ds - 2\beta_b\gamma_b^2 \int_{\partial D} (\nu \times U^z) \cdot \overline{U^z} ds$$

gives

$$I = 2ik_b^{-1}\gamma_b^2\lambda \int_{\partial D} |\nu \times U^z|^2 ds.$$

Furthermore, in view of the bilinearity of the integral I we have

$$I = I_1 + I_2 + I_3 + I_4, \tag{4.7}$$

where

$$\begin{aligned}
I_1 &= \int_{\partial D} \left[(\nu \times E^z) \cdot \text{curl } \overline{E^z} - (\nu \times \overline{E^z}) \cdot \text{curl } E^z\right] ds - 2\beta_b\gamma_b^2 \int_{\partial D} (\nu \times E^z) \cdot \overline{E^z}\, ds, \\
I_2 &= \int_{\partial D} \left[(\nu \times E^z) \cdot \text{curl } \overline{E_z(\cdot, q, \gamma_b)} - (\nu \times \overline{E_z(\cdot, q, \gamma_b)}) \cdot \text{curl } E^z\right] ds - 2\beta_b\gamma_b^2 \int_{\partial D} (\nu \times E^z) \cdot \overline{E_z(\cdot, q, \gamma_b)}\, ds, \\
I_3 &= \int_{\partial D} \left[(\nu \times E_z(\cdot, q, \gamma_b)) \cdot \text{curl } \overline{E^z} - (\nu \times \overline{E^z}) \cdot \text{curl } E_z(\cdot, q, \gamma_b)\right] ds - 2\beta_b\gamma_b^2 \int_{\partial D} (\nu \times E_z(\cdot, q, \gamma_b)) \cdot \overline{E^z}\, ds, \\
I_4 &= \int_{\partial D} \left[(\nu \times E_z(\cdot, q, \gamma_b)) \cdot \text{curl } \overline{E_z(\cdot, q, \gamma_b)} - (\nu \times \overline{E_z(\cdot, q, \gamma_b)}) \cdot \text{curl } E_z(\cdot, q, \gamma_b)\right] ds \\
&\quad - 2\beta_b\gamma_b^2 \int_{\partial D} (\nu \times E_z(\cdot, q, \gamma_b)) \cdot \overline{E_z(\cdot, q, \gamma_b)}\, ds.
\end{aligned}$$

In the integral I_1, we apply the second vector Green's theorem for the first integral and the Gauss' theorem for the second integral for E^z and $\overline{E^z}$ in D and taking into account that E^z, $\overline{E^z}$ are solutions of (2.13) we get $I_1 = 0$. A similar application in $\Omega \setminus \overline{D}$ for $E_z(\cdot, q, \gamma_b)$ and $\overline{E_z(\cdot, q, \gamma_b)}$ gives

$$\begin{aligned}
I_4 &= \int_{\partial \Omega} \left[(\nu \times E_z(\cdot, q, \gamma_b)) \cdot \text{curl } \overline{E_z(\cdot, q, \gamma_b)} - (\nu \times \overline{E_z(\cdot, q, \gamma_b)}) \cdot \text{curl } E_z(\cdot, q, \gamma_b)\right] ds \\
&\quad - 2\beta_b\gamma_b^2 \int_{\partial \Omega} (\nu \times E_z(\cdot, q, \gamma_b)) \cdot \overline{E_z(\cdot, q, \gamma_b)}\, ds.
\end{aligned}$$

Taking into account that $E_z = q \cdot \widetilde{B}(\cdot, z)$ and using the representation (2.15) we get $I_2 = -q \cdot E^z$ and $I_3 = q \cdot \overline{E^z}$. Substituting the values of the integrals in (4.7) we obtain (4.6). □

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