J. Nonlinear Sci. Appl., 16 (2023), 51–59

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

On the analytic and approximate solutions for the fractional nonlinear Schrödinger equations

Chenkuan Li[a,*], Kamsing Nonlaopon[b], Andrii Hrytsenko[a], Joshua Beaudin[a]

[a]*Department of Mathematics and Computer Science, Brandon University, Brandon, Manitoba, R7A 6A9, Canada.*

[b]*Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand.*

Abstract

In this work, we are devoted to the following fractional nonlinear Schrödinger equation with the initial conditions in the Caputo sense for $1 < \alpha \leqslant 2$:

$$\begin{cases} i\,{}_{c}\dfrac{\partial^{\alpha}}{\partial\theta^{\alpha}}W(\theta,\sigma) + \beta_1 \dfrac{\partial^2}{\partial\sigma^2}W(\theta,\sigma) \\ \qquad\qquad +\gamma(\theta,\sigma)W(\theta,\sigma) + \beta_2|W(\theta,\sigma)|^2 W(\theta,\sigma) + \beta_3 W^2(\theta,\sigma) = 0, \\ W(0,\sigma) = \phi_1(\sigma), \quad W'_{\theta}(0,\sigma) = \phi_2(\sigma), \end{cases} \tag{1}$$

where $\theta > 0, \sigma \in \mathbb{R}$, $\gamma(\theta,\sigma)$ is a continuous function and $\beta_1, \beta_2, \beta_3$ are constants. Our analysis for deriving analytic and approximate solutions to the Schrödinger equation relies on the Adomian decomposition method and fractional calculus. Several illustrative examples are presented to demonstrate the solution constructions. Finally, the variant and symmetric system of the fractional nonlinear Schrödinger equations are studied.

Keywords: Adomian's decomposition method, Fractional nonlinear Schrödinger equation, Fractional calculus, Approximate solution.

2020 MSC: 35C20, 26A33.

1. Introduction and Preliminaries

Nonlinear partial differential equations play an important role to construct the physical models of natural phenomena and dynamical processes in many scientific areas, such as physics, fluid mechanics, geophysics, plasma physics and optical fibres [1, 2, 3, 4]. In physics, the nonlinear Schrödinger equation [5] is defined as

$$i\frac{\partial}{\partial\theta}W + \frac{1}{2}\frac{\partial^2}{\partial\sigma^2}W - \nu|W|^2 W = 0,$$

*Corresponding author

Email addresses: lic@brandonu.ca (Chenkuan Li), nkamsi@kku.ac.th (Kamsing Nonlaopon), hrytsea32@gmail.com (Andrii Hrytsenko), beaudinj@brandonu.ca (Joshua Beaudin)

doi: 10.22436/jnsa.016.01.05

Received: 2023-02-01 *Revised:* 2023-03-10 *Accepted:* 2023-03-14

which has original application to the diffusion of light in planar wave guides and nonlinear optical fibers. In 2008, Rida et al. [6] investigated the following fractional nonlinear Schrödinger equation using Adomian's decomposition method:

$$i\frac{\partial^\alpha}{\partial\theta^\alpha}W+\beta\frac{\partial^2}{\partial\sigma^2}W+V(\sigma)W+\gamma|W|^2W=0,$$

where $\theta>0, \sigma\in\mathbb{R}$ and $0<\alpha<1$.

The operator I_θ^α is the partial Riemann-Liouville fractional integral of order $\alpha>0$ w.r.t θ with initial point zero [7],

$$(I_\theta^\alpha W)(\theta,\sigma)=\frac{1}{\Gamma(\alpha)}\int_0^\theta(\theta-\zeta)^{\alpha-1}W(\zeta,\sigma)d\zeta,\ \ \theta>0$$

and $\frac{{}_c\partial^\alpha}{\partial\theta^\alpha}$ is the partial Caputo fractional derivative of order α with respect to θ

$$\left(\frac{{}_c\partial^\alpha}{\partial\theta^\alpha}W\right)(\theta,\sigma)=\frac{1}{\Gamma(2-\alpha)}\int_0^\theta(\theta-s)^{1-\alpha}W_s^{(2)}(s,\sigma)ds,\quad 1<\alpha\leqslant 2.$$

It follows from [8] that

$$\begin{aligned}I_\theta^\alpha\left(\frac{{}_c\partial^\alpha}{\partial\theta^\alpha}W\right)(\theta,\sigma)&=W(\theta,\sigma)-W(0,\sigma)-W'_\theta(0,\sigma)\theta\\&=W(\theta,\sigma)-\varphi_1(\sigma)-\varphi_2(\sigma)\,\theta.\end{aligned}$$

2. Mathematical Analysis

Let $\varphi_1(\sigma)$ and $\varphi_2(\sigma)$ be smooth functions. We apply the integral operator I_θ^α to both sides of (1) to get

$$\begin{aligned}&W(\theta,\sigma)-\varphi_1(\sigma)-\varphi_2(\sigma)\,\theta\\&=iI_\theta^\alpha[\beta_1L_\sigma W(\theta,\sigma)+\gamma(\theta,\sigma)W(\theta,\sigma)+\beta_2|W(\theta,\sigma)|^2W(\theta,\sigma)+\beta_3W^2(\theta,\sigma)],\end{aligned}\tag{2.1}$$

where

$$L_\sigma=\frac{\partial^2}{\partial\sigma^2}.$$

The Adomian decomposition method assumes a series solution for $W(\theta,\sigma)$ given by

$$W(\theta,\sigma)=\sum_{k=0}^{\infty}W_k(\theta,\sigma),\tag{2.2}$$

and the nonlinear term $g(W)=\beta_2|W(\theta,\sigma)|^2W(\theta,\sigma)+\beta_3W^2(\theta,\sigma)$ is decomposed as

$$g(W)=\sum_{k=0}^{\infty}A_k(W_0,W_1,\dots,W_k).\tag{2.3}$$

Substituting (2.2) and (2.3) into (2.1) deduces that

$$\begin{aligned}\sum_{k=0}^{\infty}W_k(\theta,\sigma)=\varphi_1(\sigma)+\varphi_2(\sigma)\,\theta+iI_\theta^\alpha\Bigg[&\beta_1L_\sigma\left(\sum_{k=0}^{\infty}W_k(\theta,\sigma)\right)+\gamma(\theta,\sigma)\left(\sum_{k=0}^{\infty}W_k(\theta,\sigma)\right)\\&+\sum_{k=0}^{\infty}A_k(W_0,W_1,\dots,W_k)\Bigg].\end{aligned}$$

Then we have the following recursions:

$$\begin{aligned} W_0 &= \phi_1(\sigma) + \phi_2(\sigma)\,\theta, \\ W_{k+1} &= i\beta_1 I_\theta^\alpha L_\sigma W_k + i I_\theta^\alpha[\gamma(\theta,\sigma) W_k] + i I_\theta^\alpha A_k \end{aligned} \tag{2.4}$$

for all $k = 0, 1, 2, \ldots$, and A_k are Adomain's polynomials, which are computed as

$$\begin{aligned} A_0 &= \beta_2 |W_0|^2 W_0 + \beta_3 W_0^2, \\ A_1 &= 2\beta_2 |W_0|^2 W_1 + \beta_2 W_0^2 \bar{W}_1 + 2\beta_3 W_0 W_1, \\ A_2 &= \beta_2[2|W_0|^2 W_2 + \bar{W}_0 W_1^2 + 2|W_1|^2 W_0 + W_0^2 \bar{W}_2] + \beta_3[W_1^2 + 2W_0W_2], \\ A_3 &= \beta_2[2|W_0|^2 W_3 + 2\bar{W}_0 W_1 W_2 + 2W_0 \bar{W}_1 W_2 + |W_1|^2 W_1 + 2W_0 W_1 \bar{W}_2 + W_0^2 \bar{W}_3] \\ &\quad + \beta_3[2W_1W_2 + 2W_0W_3], \\ &\ \ \vdots \\ A_n &= \beta_2 \sum_{j=0}^{n} \left(\sum_{i=0}^{j} W_i W_{j-i} \right) \bar{W}_{n-j} + \beta_3 \sum_{i=0}^{n} W_i W_{n-i}, \end{aligned} \tag{2.5}$$

for $n = 0, 1, 2, \ldots$, based on

$$\begin{aligned} &\beta_2(W_0 + W_1 + W_2 + \cdots)(\bar{W}_0 + \bar{W}_1 + \bar{W}_2 + \cdots)(W_0 + W_1 + W_2 + \cdots) \\ &+ \beta_3(W_0 + W_1 + W_2 + \cdots)(W_0 + W_1 + W_2 + \cdots) \\ &= A_0 + A_1 + A_2 + A_3 + \cdots \end{aligned}$$

and the rule for indices.

We would like to add that there is another expression for A_n:

$$A_n = \beta_2 \sum_{i+j+r=n} W_i \bar{W}_j W_r + \beta_3 \sum_{i+j=n} W_i W_j,$$

where the sums are over all distinct (i, j, r) and (i, j) respectively for all $i, j, r = 0, 1, \cdots, n$. However,

$$A_n = \beta_2 \sum_{j=0}^{n} \left(\sum_{i=0}^{j} W_i W_{j-i} \right) \bar{W}_{n-j} + \beta_3 \sum_{i=0}^{n} W_i W_{n-i}$$

is better in terms of computation efficiency, since we have the $\sum_{i=0}^{n} W_i W_{n-i}$ in the first term for $j = n$, which is the second one.

3. Examples

Example 3.1. *Consider the following time-fractional nonlinear Schrödinger equation for* $1 < \alpha \leqslant 2$*:*

$$i\frac{\partial^\alpha}{\partial\theta^\alpha} W + \frac{\partial^2}{\partial\sigma^2} W + W^2 = 0$$

subject to the initial conditions

$$W(0, \sigma) = \sigma, \quad W'_\theta(0, \sigma) = 0.$$

Clearly,

$$I_\theta^\alpha \theta^s = \frac{\Gamma(s+1)}{\Gamma(\alpha+s+1)} \theta^{\alpha+s}, \quad s > -1.$$

From recurrences (2.4) and (2.5), we come to

$$\begin{aligned}
W_0 &= \sigma,\\
W_1 &= iI_\theta^\alpha L_\sigma(\sigma) + iI_\theta^\alpha \sigma^2 = \frac{i\sigma^2}{\Gamma(\alpha+1)}\theta^\alpha,\\
W_2 &= iI_\theta^\alpha L_\sigma\left(\frac{i\sigma^2}{\Gamma(\alpha+1)}\theta^\alpha\right) + iI_\theta^\alpha\left(2\sigma\frac{i\sigma^2}{\Gamma(\alpha+1)}\theta^\alpha\right)\\
&= -\frac{2}{\Gamma(2\alpha+1)}\theta^{2\alpha} - \frac{2\sigma^3}{\Gamma(2\alpha+1)}\theta^{2\alpha},\\
W_3 &= iI_\theta^\alpha L_\sigma\left(-\frac{2}{\Gamma(2\alpha+1)}\theta^{2\alpha} - \frac{2\sigma^3}{\Gamma(2\alpha+1)}\theta^{2\alpha}\right)\\
&\quad + iI_\theta^\alpha\left(-\frac{\sigma^4}{\Gamma^2(\alpha+1)}\theta^{2\alpha} - \frac{4\sigma}{\Gamma(2\alpha+1)}\theta^{2\alpha} - \frac{4\sigma^4}{\Gamma(2\alpha+1)}\theta^{2\alpha}\right),\\
&= -i\frac{12\sigma}{\Gamma(3\alpha+1)}\theta^{3\alpha} - i\frac{\sigma^4\,\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\,\Gamma(3\alpha+1)}\theta^{3\alpha}\\
&\quad - i\frac{4\sigma}{\Gamma(3\alpha+1)}\theta^{3\alpha} - i\frac{4\sigma^4}{\Gamma(3\alpha+1)}\theta^{3\alpha},
\end{aligned}$$

which claims that

$$\begin{aligned}
W &\approx \sigma + i\sigma^2\frac{\theta^\alpha}{\Gamma(\alpha+1)} - (2+2\sigma^3)\frac{\theta^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad - i\left(16\sigma + \frac{\sigma^4\,\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + 4\sigma^4\right)\frac{\theta^{3\alpha}}{\Gamma(3\alpha+1)}.
\end{aligned}$$

In particular for $\alpha = 2$,

$$W \approx \sigma + \frac{i\sigma^2\theta^2}{2} - \frac{(2+2\sigma^3)\theta^4}{4!} - i(16\sigma + 10\sigma^4)\frac{\theta^6}{6!},$$

and for $\alpha = 1.5$ we have

$$W \approx \sigma + i4\sigma^2\frac{\theta^{1.5}}{3\sqrt{\pi}} - (2+2\sigma^3)\frac{\theta^3}{3!} - i\left(16\sigma + \frac{32}{3\pi}\sigma^4 + 4\sigma^4\right)\frac{32}{945\sqrt{\pi}}\theta^{4.5},$$

using

$$\Gamma(1.5+1) = \frac{3}{4}\sqrt{\pi}, \quad \textit{and} \quad \Gamma(4.5+1) = \frac{945}{32}\sqrt{\pi}.$$

Example 3.2. *Consider the following time-fractional nonlinear Schrödinger equation for $1 < \alpha \leqslant 2$:*

$$i\frac{\partial^\alpha}{\partial\theta^\alpha}W + 2\frac{\partial^2}{\partial\sigma^2}W + \theta W + |W|^2 W = 0$$

subject to the initial conditions

$$W(0,\sigma) = 0, \quad W'_\theta(0,\sigma) = e^\sigma.$$

From recurrences (2.4) and (2.5), we infer that

$$
\begin{aligned}
W_0 &= e^{\sigma}\theta,\\
W_1 &= i2I_{\theta}^{\alpha}L_{\sigma}\left(e^{\sigma}\theta\right) + iI_{\theta}^{\alpha}\left(\theta^2 e^{\sigma}\right) + iI_{\theta}^{\alpha}\left(e^{3\sigma}\theta^3\right)\\
&= i\frac{2e^{\sigma}}{\Gamma(\alpha+2)}\theta^{\alpha+1} + i\frac{2e^{\sigma}}{\Gamma(\alpha+3)}\theta^{\alpha+2} + i\frac{6e^{3\sigma}}{\Gamma(\alpha+4)}\theta^{\alpha+3},\\
W_2 &= -2I_{\theta}^{\alpha}L_{\sigma}\left(\frac{2e^{\sigma}}{\Gamma(\alpha+2)}\theta^{\alpha+1} + \frac{2e^{\sigma}}{\Gamma(\alpha+3)}\theta^{\alpha+2} + \frac{6e^{3\sigma}}{\Gamma(\alpha+4)}\theta^{\alpha+3}\right)\\
&\quad - I_{\theta}^{\alpha}\left(\frac{2e^{\sigma}}{\Gamma(\alpha+2)}\theta^{\alpha+2} + \frac{2e^{\sigma}}{\Gamma(\alpha+3)}\theta^{\alpha+3} + \frac{6e^{3\sigma}}{\Gamma(\alpha+4)}\theta^{\alpha+4}\right)\\
&\quad - I_{\theta}^{\alpha}\left(\frac{2e^{3\sigma}}{\Gamma(\alpha+2)}\theta^{\alpha+3} + \frac{2e^{3\sigma}}{\Gamma(\alpha+3)}\theta^{\alpha+4} + \frac{6e^{5\sigma}}{\Gamma(\alpha+4)}\theta^{\alpha+5}\right)\\
&= -\frac{4e^{\sigma}}{\Gamma(2\alpha+2)}\theta^{2\alpha+1} - \frac{4e^{\sigma}}{\Gamma(2\alpha+3)}\theta^{2\alpha+2} - \frac{108e^{3\sigma}}{\Gamma(2\alpha+4)}\theta^{2\alpha+3}\\
&\quad - \frac{2e^{\sigma}(\alpha+2)}{\Gamma(2\alpha+3)}\theta^{2\alpha+2} - \frac{2e^{\sigma}(\alpha+3)}{\Gamma(2\alpha+4)}\theta^{2\alpha+3} - \frac{6e^{3\sigma}(\alpha+4)}{\Gamma(2\alpha+5)}\theta^{2\alpha+4}\\
&\quad - \frac{2e^{3\sigma}(\alpha+3)(\alpha+2)}{\Gamma(2\alpha+4)}\theta^{2\alpha+3} - \frac{2e^{3\sigma}(\alpha+4)(\alpha+3)}{\Gamma(2\alpha+5)}\theta^{2\alpha+4}\\
&\quad - \frac{6e^{5\sigma}(\alpha+5)(\alpha+4)}{\Gamma(2\alpha+6)}\theta^{2\alpha+5},
\end{aligned}
$$

which derives that

$$
\begin{aligned}
W &\approx e^{\sigma}\theta + i\frac{2e^{\sigma}}{\Gamma(\alpha+2)}\theta^{\alpha+1} + i\frac{2e^{\sigma}}{\Gamma(\alpha+3)}\theta^{\alpha+2} + i\frac{6e^{3\sigma}}{\Gamma(\alpha+4)}\theta^{\alpha+3}\\
&\quad - \frac{4e^{\sigma}}{\Gamma(2\alpha+2)}\theta^{2\alpha+1} - \frac{2e^{\sigma}(\alpha+4)}{\Gamma(2\alpha+3)}\theta^{2\alpha+2}\\
&\quad - \left[2e^{\sigma}(\alpha+3) + 2e^{3\sigma}(\alpha^2+5\alpha+60)\right]\frac{\theta^{2\alpha+3}}{\Gamma(2\alpha+4)}\\
&\quad - 2e^{3\sigma}(\alpha+4)(\alpha+6)\frac{\theta^{2\alpha+4}}{\Gamma(2\alpha+5)} - \frac{6e^{5\sigma}(\alpha+5)(\alpha+4)}{\Gamma(2\alpha+6)}\theta^{2\alpha+5}.
\end{aligned}
$$

In particular for $\alpha = 2$,

$$
\begin{aligned}
W &\approx e^{\sigma}\theta + i\frac{2e^{\sigma}}{3!}\theta^3 + i\frac{2e^{\sigma}}{4!}\theta^4 + i\frac{6e^{3\sigma}}{5!}\theta^5 - \frac{4e^{\sigma}}{5!}\theta^5\\
&\quad - \frac{12e^{\sigma}}{6!}\theta^6 - [10e^{\sigma} + 148e^{3\sigma}]\frac{\theta^7}{7!} - 96e^{3\sigma}\frac{\theta^8}{8!} - 252e^{5\sigma}\frac{\theta^9}{9!}.
\end{aligned}
$$

4. The Variant and System of the Schrödinger Equations

Following a similar procedure, we are able to study the following generalized fractional nonlinear Schrödinger equation in the Caputo sense for $1 < \alpha_1 \leqslant 2$ and $0 < \alpha_2 < \alpha_1$:

$$
\begin{cases}
i\frac{{}_c\partial^{\alpha_1}}{\partial\theta^{\alpha_1}}W(\theta,\sigma) + \lambda\frac{{}_c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}}W(\theta,\sigma) + \beta_1\triangle_{\sigma}W(\theta,\sigma)\\
\qquad +\gamma(\theta,\sigma)W(\theta,\sigma) + \beta_2|W(\theta,\sigma)|^2 W(\theta,\sigma) + \beta_3 W^2(\theta,\sigma) = 0,\\
W(0,\sigma) = \phi_1(\sigma), \quad W'_{\theta}(0,\sigma) = \phi_2(\sigma),
\end{cases}
\tag{4.1}
$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and

$$\triangle_\sigma = \frac{\partial^2}{\partial\sigma_1^2} + \frac{\partial^2}{\partial\sigma_2^2} + \frac{\partial^2}{\partial\sigma_3^2}$$

is the Laplace operator and λ is a constant.

Indeed for $0 < \alpha_2 \leqslant 1$,

$$\begin{aligned}\lambda I_\theta^{\alpha_1} \frac{{}^c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}} W(\theta,\sigma) &= \lambda I_\theta^{\alpha_1-\alpha_2} I_\theta^{\alpha_2} \frac{{}^c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}} W(\theta,\sigma) = \lambda I_\theta^{\alpha_1-\alpha_2}(W(\theta,\sigma) - \phi_1(\sigma)) \\ &= \lambda I_\theta^{\alpha_1-\alpha_2} W(\theta,\sigma) - \lambda\phi_1(\sigma)\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}.\end{aligned}$$

As for $1 < \alpha_2 < \alpha_1$,

$$\begin{aligned}\lambda I_\theta^{\alpha_1} \frac{{}^c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}} W(\theta,\sigma) &= \lambda I_\theta^{\alpha_1-\alpha_2} I_\theta^{\alpha_2} \frac{{}^c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}} W(\theta,\sigma) = \lambda I_\theta^{\alpha_1-\alpha_2}(W(\theta,\sigma) - \phi_1(\sigma) - \phi_2(\sigma)\theta) \\ &= \lambda I_\theta^{\alpha_1-\alpha_2} W(\theta,\sigma) - \lambda\phi_1(\sigma)\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} - \lambda\phi_2(\sigma)\frac{\theta^{\alpha_1-\alpha_2+1}}{\Gamma(\alpha_1-\alpha_2+2)}.\end{aligned}$$

Then we have the recurrence for $0 < \alpha_2 \leqslant 1$:

$$\begin{aligned}W_0 &= \phi_1(\sigma) + \phi_2(\sigma)\,\theta - i\lambda\phi_1(\sigma)\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}, \\ W_{k+1} &= i\lambda I_\theta^{\alpha_1-\alpha_2} W_k(\theta,\sigma) + i\beta_1 I_\theta^{\alpha_1}\triangle_\sigma W_k + iI_\theta^{\alpha_1}[\gamma(\theta,\sigma)W_k] + iI_\theta^{\alpha_1}A_k\end{aligned} \tag{4.2}$$

for all $k = 0, 1, 2, \ldots$.

The recurrence for $1 < \alpha_2 < \alpha_1$ can be obtained as

$$\begin{aligned}W_0 &= \phi_1(\sigma) + \phi_2(\sigma)\,\theta - i\lambda\phi_1(\sigma)\frac{\theta^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} - i\lambda\phi_2(\sigma)\frac{\theta^{\alpha_1-\alpha_2+1}}{\Gamma(\alpha_1-\alpha_2+2)}, \\ W_{k+1} &= i\lambda I_\theta^{\alpha_1-\alpha_2} W_k(\theta,\sigma) + i\beta_1 I_\theta^{\alpha_1}\triangle_\sigma W_k + iI_\theta^{\alpha_1}[\gamma(\theta,\sigma)W_k] + iI_\theta^{\alpha_1}A_k\end{aligned} \tag{4.3}$$

for all $k = 0, 1, 2, \ldots$.

Example 4.1. *Consider the following time-fractional nonlinear Schrödinger equation for $\sigma = (\sigma_1, \sigma_2, \sigma_3)$:*

$$i\frac{\partial^{1.5}}{\partial\theta^{1.5}}W + \frac{\partial^{0.5}}{\partial\theta^{0.5}}W + 2\triangle_\sigma W + |W|^2 W = 0$$

subject to the initial conditions

$$W(0,\sigma) = \sigma_2, \quad W'_\theta(0,\sigma) = \sigma_1.$$

From the recurrence (4.2), we have for $\alpha_2 = 0.5$,

$$\begin{aligned}W_0 &= \sigma_2 + \sigma_1\theta - i\sigma_2\theta = \sigma_2 + (\sigma_1 - i\sigma_2)\theta, \\ W_1 &= iI_\theta W_0 + i2I_\theta^{1.5}\triangle_\sigma W_0 + iI_\theta^{1.5}A_0 \\ &= iI_\theta W_0 + i2I_\theta^{1.5}\triangle_\sigma W_0 + iI_\theta^{1.5}\left(|W_0|^2 W_0\right) \\ &= i\sigma_2\theta + (\sigma_2 + i\sigma_1)\frac{\theta^2}{2} + iI_\theta^{1.5}\left((\sigma_2+\sigma_1\theta)^2 + \sigma_2^2\theta^2)(\sigma_2 + \sigma_1\theta - i\sigma_2\theta)\right).\end{aligned}$$

Clearly,

$$\begin{aligned}&((\sigma_2+\sigma_1\theta)^2 + \sigma_2^2\theta^2)(\sigma_2 + \sigma_1\theta - i\sigma_2\theta) \\ &= \sigma_2^3 + (3\sigma_1\sigma_2^2 - i\sigma_2^3)\theta + (2\sigma_1^2\sigma_2 - i2\sigma_1\sigma_2^2 + \sigma_2(\sigma_1^2+\sigma_2^2))\theta^2 + (\sigma_1^2+\sigma_2^2)(\sigma_1 - i\sigma_2)\theta^3.\end{aligned}$$

Hence,

$$
\begin{aligned}
& iI_\theta^{1.5}\left((\sigma_2+\sigma_1\theta)^2+\sigma_2^2\theta^2)(\sigma_2+\sigma_1\theta-i\sigma_2\theta)\right) \\
& = i\sigma_2^3\frac{\theta^{1.5}}{\Gamma(1.5+1)}+(\sigma_2^3+i3\sigma_1\sigma_2^2)\frac{\theta^{2.5}}{\Gamma(3.5)}+2(2\sigma_1\sigma_2^2+i2\sigma_1^2\sigma_2+i\sigma_2(\sigma_1^2+\sigma_2^2))\frac{\theta^{3.5}}{\Gamma(4.5)} \\
& \quad +6(\sigma_2+i\sigma_1)(\sigma_1^2+\sigma_2^2)\frac{\theta^{4.5}}{\Gamma(5.5)}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
W & \approx \sigma_2+\sigma_1\theta+i\sigma_2^3\frac{\theta^{1.5}}{\Gamma(2.5)}+(\sigma_2+i\sigma_1)\frac{\theta^2}{2} \\
& \quad +(\sigma_2^3+i3\sigma_1\sigma_2^2)\frac{\theta^{2.5}}{\Gamma(3.5)}+2(2\sigma_1\sigma_2^2+i2\sigma_1^2\sigma_2+i\sigma_2(\sigma_1^2+\sigma_2^2))\frac{\theta^{3.5}}{\Gamma(4.5)} \\
& \quad +6(\sigma_2+i\sigma_1)(\sigma_1^2+\sigma_2^2)\frac{\theta^{4.5}}{\Gamma(5.5)} \\
& = \sigma_2+\sigma_1\theta+i4\sigma_2^3\frac{\theta^{1.5}}{3\sqrt{\pi}}+(\sigma_2+i\sigma_1)\frac{\theta^2}{2} \\
& \quad +(\sigma_2^3+i3\sigma_1\sigma_2^2)\frac{8\,\theta^{2.5}}{15\sqrt{\pi}}+2(2\sigma_1\sigma_2^2+i2\sigma_1^2\sigma_2+i\sigma_2(\sigma_1^2+\sigma_2^2))\frac{16\,\theta^{3.5}}{105\sqrt{\pi}} \\
& \quad +6(\sigma_2+i\sigma_1)(\sigma_1^2+\sigma_2^2)\frac{945\,\theta^{4.5}}{32\sqrt{\pi}}.
\end{aligned}
$$

Obviously the initial conditions

$$W(0,\sigma)=\sigma_2, \quad W'_\theta(0,\sigma)=\sigma_1$$

are satisfied.

To end off this paper, we would like to mention that the following system of fractional nonlinear Schrödinger equations in symmetry can also be solved numerically and analytically based on the Adomian decomposition method and fractional calculus for $0<\alpha_1,\alpha_2\leqslant 1$:

$$
\begin{cases}
i\frac{{}^c\partial^{\alpha_1}}{\partial\theta^{\alpha_1}}W(\theta,\sigma)+\beta_1\frac{\partial^2}{\partial\sigma^2}W(\theta,\sigma)+\beta_2 U\frac{\partial}{\partial\sigma}U(\theta,\sigma)=0, \\
i\frac{{}^c\partial^{\alpha_2}}{\partial\theta^{\alpha_2}}U(\theta,\sigma)+\gamma_1\frac{\partial^2}{\partial\sigma^2}U(\theta,\sigma)+\gamma_2 W\frac{\partial}{\partial\sigma}W(\theta,\sigma)=0, \\
W(0,\sigma)=\phi_1(\sigma), \quad U(0,\sigma)=\phi_2(\sigma),
\end{cases}
$$

where β_i,γ_i for $i=1,2$ are constants.

Let

$$
\begin{aligned}
W &= W_0+W_1+W_2+\cdots, \\
U &= U_0+U_1+U_2+\cdots,
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_2 W\frac{\partial}{\partial\sigma}W(\theta,\sigma) &= \sum_{k=0}^{\infty}B_k(W_0,W_1,\cdots,W_k), \\
\beta_2 U\frac{\partial}{\partial\sigma}U(\theta,\sigma) &= \sum_{k=0}^{\infty}C_k(U_0,U_1,\cdots,U_k).
\end{aligned}
$$

Then we derive that

$$\begin{aligned}
B_0 &= \gamma_2 W_0 W'_{0\sigma},\\
B_1 &= \gamma_2 W_0 W'_{1\sigma} + \gamma_2 W'_{0\sigma} W_1,\\
&\vdots\\
B_n &= \gamma_2 \sum_{i=0}^{n} W_i W'_{(n-i)\sigma},
\end{aligned}$$

and

$$\begin{aligned}
C_0 &= \beta_2 U_0 U'_{0\sigma},\\
C_1 &= \beta_2 U_0 U'_{1\sigma} + \beta_2 U'_{0\sigma} U_1,\\
&\vdots\\
C_n &= \beta_2 \sum_{i=0}^{n} U_i U'_{(n-i)\sigma}.
\end{aligned}$$

The following double recurrences can be obtained by a similar approach indicated above:

$$\begin{aligned}
W_0 &= \phi_1(\sigma),\\
U_0 &= \phi_2(\sigma),\\
W_{k+1} &= i\beta_1 I_\theta^{\alpha_1} L_\sigma W_k + i I_\theta^{\alpha_1} C_k,\\
U_{k+1} &= i\gamma_1 I_\theta^{\alpha_2} L_\sigma U_k + i I_\theta^{\alpha_2} B_k.
\end{aligned}$$

Remark 4.2. We construct the computational methods for analytic and approximate solutions to the Schrödinger nonlinear problems which have a wide range of applications in physics. Clearly, the order of convergence is a topic to be examined in the future. On the other hand, we have no information about exact solutions to the examples shown and hence we are unable to provide numerical tables on approximate precisions.

5. Conclusion

We have studied several Cauchy problems of the fractional nonlinear Schrödinger equations and derived analytic and approximate solutions using Adomian's decomposition methods. Several examples were presented to demonstrate the algorithms for constructing solutions. This approach clearly works for other types of nonlinear partial differential equations with initial conditions, as mentioned in Section 4.

Acknowledgment

The authors thank the Natural Sciences and Engineering Research Council of Canada for support under Grant No. 2019-03907.

References

[1] A. Atangana, *Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system*, Chaos Solitons Fractals, **102** (2017), 396–406. 1

[2] A. R. Seadawy, D. Lu, C. Yue, *Travelling wave solutions of the generalized nonlinear fifth-order KdV water wave equations and its stability*, J Taibah Univ Sci, **11** (2017), 623–33. 1

[3] M. A. Helal, A. R. Seadawy, M. H. Zekry, *Stability analysis of solitary wave solutions for the fourth-order nonlinear Boussinesq water wave equation*, Appl Math Comput, **232** (2014), 1094—103. 1

[4] S. Vlase, M. Marin, A. Ochsner, *Scutaru ML. Motion equation for a flexible onedimensional element used in the dynamical analysis of a multibody system*, Contin Mech Thermodyn, **31** (2019), 715–24. 1

[5] B. Malomed, *Nonlinear Schrödinger equations. In Scott, Alwyn (ed.), Encyclopedia of Nonlinear Science*, New York: Routledge, (2005) pp. 639–643. 1

[6] S. Z. Rida, H. M. El-Sherbinyb, A. A. M. Arafa, *On the solution of the fractional nonlinear Schrödinger equation*, Phys. lett. A, **372** (2008), 553–558. 1

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, North-Holland (2006). 1

[8] C. Li, R. Saadati, R. Srivastava, J. Beaudin, *On the boundary value problem of nonlinear fractional integro-differential equations*, Mathematics (2022), 10, 1971. 1