Deferred Nörlund statistical convergence in probability, mean and distribution for sequences of random variables

Kuldip Raj*, Swati Jasrotia
School of Mathematics Shri Mata Vaishno Devi University, Katra-182320, J & K, India.

Abstract
We introduce and study deferred Nörlund statistical convergence in probability, mean of order \( r \), distribution and study the interrelation among them. Based upon the proposed method to illustrate the findings, we present new Korovkin-type theorems for the sequence of random variables via deferred Nörlund statistically convergence and present compelling examples to demonstrate the effectiveness of the results. ©2017 All rights reserved.

Keywords: Probability convergence, Deferred Nörlund, Mean convergence, Distribution convergence, Statistical convergence.

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1. Introduction and Preliminaries

Fast [6] and also Schoenberg [22] studied the concept of statistical convergence and continued by Rath–Tripathy [21] and Gadjiev–Orhan [8].

Suppose that \((x_m)\) and \((y_m)\) are the sequences of non-negative integers fulfilling
\[
x_m < y_m, \quad \forall \ m \in \mathbb{N} \quad \text{and} \quad \lim_{x \to \infty} y_m = \infty.
\] (1.1)

Further, let \((e_m)\) and \((g_m)\) be two sequences of non-negative real numbers such that
\[
\mathcal{E}_m = \sum_{n=x_m+1}^{y_m} e_n \quad \text{and} \quad \mathcal{F}_m = \sum_{n=x_m+1}^{y_m} g_n.
\] (1.2)

The convolution of (1.2) is defined as
\[
\mathcal{R}_m = \sum_{\nu=x_m+1}^{y_m} e_{\nu} g_{y_m-\nu}.
\]

As introduced by Srivastava et al. in [23], the deferred Nörlund (DN) mean is defined as
\[
t_m = \frac{1}{\mathcal{R}_m} \sum_{n=x_m+1}^{y_m} e_{y_m-n} g_n y_n.
\]

*Corresponding author
Email addresses: kuldipraj68@gmail.com (Kuldip Raj), swatijasrotia12@gmail.com (Swati Jasrotia)
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Suppose that \((x_m)\) and \((y_m)\) are the sequences fulfilling conditions (1.1) and \((e_m), (g_m)\) are sequences satisfying (1.2). A sequence \((Y_m)\) is called as deferred Nörlund statistically convergent to \(Y\) if \(\forall \varepsilon > 0\), the set
\[
\{n : n \leq R_m, \text{ and } |e_{y_m-n}g_n|Y_m - Y| \geq \varepsilon\}
\]
has zero deferred Nörlund density, i.e. if
\[
\lim_{m \to \infty} \frac{1}{R_m} \left| \left\{ n : n \leq R_m, \text{ and } |e_{y_m-n}g_n|Y_m - Y| \geq \varepsilon \right\} \right| = 0.
\]
We write it as
\[
\text{St}_{\text{DN}} \lim_{m \to \infty} Y_m = Y.
\]

Suppose that \((x_m)\) and \((y_m)\) are the sequences fulfilling conditions (1.1) and \((e_m), (g_m)\) are sequences satisfying (1.2). A sequence \((Y_m)\) is called as deferred Nörlund statistically probability (or \(\text{St}_{\text{DNP}}\)) convergent to a random variable \(Y\), if \(\forall \varepsilon > 0\) and \(\delta > 0\), the set
\[
\{n : n \leq R_m, \text{ and } |e_{y_m-n}g_nP(|Y_m - Y| \geq \varepsilon)| \geq \delta\}
\]
has DN-density zero, i.e.,
\[
\lim_{m \to \infty} \frac{1}{R_m} \left| \left\{ n : n \leq R_m, \text{ and } |e_{y_m-n}g_nP(|Y_m - Y| \geq \varepsilon)| \geq \delta \right\} \right| = 0
\]
or
\[
\lim_{m \to \infty} \frac{1}{R_m} \left| \left\{ n : n \leq R_m, \text{ and } 1 - |e_{y_m-n}g_nP(|Y_m - Y| \leq \varepsilon)| \geq \delta \right\} \right| = 0,
\]
and it is denoted as
\[
\text{St}_{\text{DNP}} \lim_{m \to \infty} e_{y_m-n}g_nP(|Y_m - Y| \geq \varepsilon) = 0
\]
or
\[
\text{St}_{\text{DNP}} \lim_{m \to \infty} e_{y_m-n}g_nP(|Y_m - Y| \leq \varepsilon) = 1.
\]

2. Deferred Nörlund statistically probability convergence

In this section, we study deferred Nörlund statistically probability convergence, for a historical review and basic concept we refer \cite{3, 4, 11, 15, 20, 12, 18, 9, 5, 13, 26}.

**Theorem 2.1.** Suppose that \((Y_m)\) and \((Z_m)\) are sequences of random variables and consider two random variables \(Y\) and \(Z\). Then the following assertions are satisfied

1. \(\text{St}_{\text{DNP}} Y_m \to Y \text{ and } \text{St}_{\text{DNP}} Y_m \to Z \Rightarrow P(Y = Z) = 1,\)
2. \(\text{St}_{\text{DNP}} Y_m \to y \Rightarrow \text{St}_{\text{DNP}} Y_m^2 = y^2,\)
3. \(\text{St}_{\text{DNP}} Y_m \to y \text{ and } \text{St}_{\text{DNP}} Z_m \to z \Rightarrow \text{St}_{\text{DNP}} Y_m Z_m \to yz,\)
4. \(\text{St}_{\text{DNP}} Y_m \to y \text{ and } \text{St}_{\text{DNP}} Z_m \to z \Rightarrow \text{St}_{\text{DNP}} \frac{Y_m}{Z_m} \to \frac{y}{z}, z \neq 0,\)
5. \( \text{St}_{DNP}Y_m \rightarrow Y \) and \( \text{St}_{DNP}Z_m \rightarrow Z \Rightarrow \text{St}_{DNP}Y_m Z_m \rightarrow YZ \),

6. if \( \text{St}_{DNP}Y_m \rightarrow Y \forall \varepsilon, \delta > 0 \), then \( \exists \alpha \in \mathbb{N} \) s.t.
   \[
   d(\{n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - Y| \geq \varepsilon) \geq \delta\}) = 0.
   \]

**Proof.** Let \( \varepsilon \) and \( \delta \) be positively small real numbers. Also consider \((x_m)\) and \((y_m)\) are the sequences fulfilling conditions (1.1) and \((e_m), (g_m)\) are sequences satisfying (1.2).

1. Suppose that
   \[a \in \left\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Y_m - Y| \geq \frac{\varepsilon}{2}\right) < \frac{\delta}{2}\right\} \cap \left\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Y_m - Z| \geq \frac{\varepsilon}{2}\right) < \frac{\delta}{2}\right\}, \]
   (as the limit density of both the sets is 1). Then,
   \[e_{y_m-n}g_nP\left(|Z - Y| \geq \varepsilon\right) \leq e_{y_m-n}g_nP\left(|Y_a - Y| \geq \frac{\varepsilon}{2}\right) + e_{y_m-n}g_nP\left(|Y_a - Z| \geq \frac{\varepsilon}{2}\right) < \delta.\]
   It means \( P(Y = Z) = 1 \).

2. If \( \text{St}_{DNP}Y_m \rightarrow 0 \), then \( \text{St}_{DNP}Y_m^2 \rightarrow 0 \). Here, we see that \( a \in \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m - 0| \geq \varepsilon) > \delta\} = a \in \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP(|Y_m^2 - 0| \geq \varepsilon > \delta\}. \) Now, take \( Y_m^2 = (Y_m - y)^2 + 2y(Y_m - y) + y^2 \). Thus, \( \text{St}_{DNP}Y_m^2 \rightarrow Y^2 \).

3. Suppose that \( \text{St}_{DNP}Y_m \rightarrow y \) and \( \text{St}_{DNP}Z_m \rightarrow z \). As \( \text{St}_{DNP}Y_m Z_m = \text{St}_{DNP}\frac{1}{2}((Y_m + Z_m)^2 - (Y_m - Z_m)^2) = \frac{1}{2}(y + z)^2 - (y - z)^2 = yz \).

4. Suppose that \( R \) and \( S \) be two events correspond \( |Z_m - z| < |z| \) and \( \frac{1}{Z_m - z} \geq \varepsilon \). We have
   \[|\mathcal{Z}_m - z| = \frac{|Z_m - z|}{|zZ_m|} = \frac{|Z_m - z|}{|z| \cdot |z + (Z_m - z)|} \leq \frac{|Z_m - z|}{|z| \cdot (|z| - |Z_m - z|)}.\]
   If the events \( R \) and \( S \) occurs at same time, then
   \[|Z_m - z| \geq \varepsilon |z|^2 \frac{1}{1 + \varepsilon |z|}.\]
   Further, let \( \varepsilon_0 = \varepsilon |z|^2/(1 + \varepsilon |z|) \) and \( A \) be the event such that \( |Z_m - z| \geq \varepsilon_0 \). Thus,
   \[\text{RS} \subseteq A \Rightarrow P(S) \leq P(A) + P(R^c).\]
   Thus,
   \[\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|\mathcal{Z}_m - z| \geq \varepsilon_0\right) \geq \delta\} \subseteq \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Z_m - z| \geq \varepsilon_0\right) \geq \frac{\delta}{2}\} \cup \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_nP\left(|Z_m - z| \geq |z|\right) \geq \frac{\delta}{2}\}.
   \]
   Therefore, \( \text{St}_{DNP}\frac{1}{Z_m} \rightarrow \frac{1}{z} \). Hence, we write \( \text{St}_{DNP}Y_m \rightarrow \frac{Y}{Z_m}, \ z \neq 0 \).

5. Suppose that \( \text{St}_{DNP}Y_m \rightarrow Y \) and \( X \) be a random variable such that \( Y_mX \rightarrow YX \). Since \( X \) is a random variable such that \( \forall \varepsilon > 0, \exists \delta > 0 \) and \( e_{y_m-n}g_nP(|X| > \delta) \leq \frac{\varepsilon}{2}. \) Next, \( \forall \varepsilon' > 0, \)
   \[e_{y_m-n}g_nP(|Y_mX - YX| \geq \varepsilon') = e_{y_m-n}g_nP\left(|Y_m - Y|X \geq \varepsilon', |Z| > \delta\right) + e_{y_m-n}g_nP\left(|Y_m - Y|X \geq \varepsilon', |Z| \leq \delta\right) \leq \frac{\varepsilon}{2} + e_{y_m-n}g_nP\left(|Y_m - Y| \geq \frac{\varepsilon'}{\delta}\right).\]
Which implies, \( \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) \} \subseteq \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - y| \geq \frac{\varepsilon}{\delta} \}) \geq \frac{\varepsilon}{2} \} \). Therefore,

\[
\text{St}_{\text{DNP}}(Y_m - Y)(Z_m - Z) \rightarrow 0.
\]

Thus,

\[
\text{St}_{\text{DNP}}(Y_m Z_m) \rightarrow YZ.
\]

6. Suppose that \((x_m)\) and \((y_m)\) be two non-negative sequences such that

\[
e_{y_m-n}g_n P(\{Y_m - Y| \geq \frac{\varepsilon}{2}\}) < \frac{\delta}{2}
\]

and

\[
\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) \geq \delta \} \subseteq \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) < \frac{\delta}{2} \} = 1.
\]

Now,

\[
\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) \geq \delta \} \subseteq \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) \geq \delta \} \subseteq \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) < \frac{\delta}{2} \} = 1.
\]

Which implies that

\[
d(\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{Y_m - Y| \geq \varepsilon \}) \geq \delta \}) = 0.
\]

\[\square\]

**Theorem 2.2.** Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is uniform continuous on \( \mathbb{R} \) and \( \text{St}_{\text{DNP}}(Y_m) \rightarrow Y \). Then \( \text{St}_{\text{DNP}} f(Y_m) \rightarrow f(Y) \).

**Proof.** Let us consider a random variable \( Y \) such that for each \( \delta > 0 \), \( \exists \beta \in \mathbb{R} \) such that \( P(Y > \beta) \leq \delta/2 \).

Since, \( f \) is uniformly continuous on \( [\beta, \beta] \forall \varepsilon > 0, \exists \delta_0 \) such that

\[|f(y_m) - f(y)| < \varepsilon \text{ whenever } |y_m - y| < \delta_0.\]

Thus,

\[P(|f(Y_m) - f(Y)| \geq \varepsilon) \leq P(|Y_m - Y| \geq \delta_0) + P(|Y > \beta|) \leq P(|Y_m - Y| \geq \delta_0) + \delta/2.\]

However, from the definition of \( \text{St}_{\text{DNP}} \)-convergence, we have

\[
\{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{f(Y_m) - f(Y)| \geq \varepsilon \}) \geq \delta \}
\]

\[
\subseteq \{ n : n \leq \mathcal{R}_m \text{ and } e_{y_m-n}g_n P(\{|Y_m - Y| \geq \delta_0\}) < \frac{\delta}{2} \}.
\]

\[\square\]

3. Deferred Nörlund statistical mean convergence

**Definition 3.1.** Suppose that \( r \geq 1 \) be a fixed number. A sequence \((Y_m)\) is \( r \)th mean convergent to \( Y \), if

\[
\lim_{m \rightarrow \infty} E(|Y_m - Y|^r) = 0.
\]

**Definition 3.2.** A sequence \((Y_m)\) is statistically \( r \)th mean convergent \((\text{MC})\) to a random variable \( Y \), where \( Y : S \rightarrow \mathbb{R} \) if,

\[
\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{n \leq m} E(|Y_m - Y|^r) = 0
\]

for any \( \varepsilon > 0 \). We write it as

\[
\text{St}_{\text{MC}} \lim_{m \rightarrow \infty} E(|Y_m - Y|^r) = 0.
\]
Definition 3.3. Suppose that \((x_m)\) and \((y_m)\) are the sequences fulfilling conditions (1.1) and \((e_m), (g_m)\) are sequences satisfying (1.2). A sequence \((Y_m)\) is said to be deferred Nörlund statistically \(r\)-th mean convergent to \(Y (Y : S \rightarrow \mathbb{R})\), if for \(\varepsilon > 0\),
\[
\lim_{m \to \infty} \frac{1}{R_m} \left| \left\{ n : n \leq R_m \text{ and } e_{y_m-n} g_n E(|Y_m - Y|^r \geq \varepsilon) \right\} \right| = 0.
\]
It is denoted as
\[
\text{St}_{DNM} \lim_{m \to \infty} E(|Y_m - Y|^r) = 0.
\]

Theorem 3.4. Let \(\text{St}_{DNM} \lim_{m \to \infty} E(|Y_m - Y|^r) = 0\) for \(r \geq 1\), then \(\text{St}_{DNM} \lim_{m \to \infty} P(|Y_m - Y| \geq \varepsilon) = 0\).

Proof. For every \(\varepsilon > 0\), we have from Markov’s inequality
\[
\text{St}_{DNM} \lim_{m \to \infty} P(|Y_m - Y| \geq \varepsilon) = \text{St}_{DNM} \lim_{m \to \infty} P(|Y_m - Y|^r \geq \varepsilon^r) \quad (r \geq 1)
\leq \text{St}_{DNM} \lim_{m \to \infty} \frac{E(|Y_m - Y|^r)}{\varepsilon^r} = 0.
\]
From definition of statistically deferred Nörlund mean convergence
\[
\text{St}_{DNM} \lim_{m \to \infty} E(|Y_m - Y|^r) = 0,
\]
it implies that
\[
\text{St}_{DNP} \lim_{m \to \infty} P(|Y_m - Y| \geq \varepsilon) = 0.
\]

We now present an example to show that a sequence of random variables is statistically probability convergent but not statistically \(r\)-th mean convergent.

Example 1: Suppose that \(x_m = 2m - 1, y_m = 4m - 1\). Also, suppose that \(e_{y_m-m} = 2m\) and \(g_m = 1\). Further, consider a sequence \((z_m)\) of random variables such that
\[
Y_m = \begin{cases} 
m, \text{ with probability } \frac{1}{\sqrt{m}} \\
0, \text{ with probability } 1 - \frac{1}{\sqrt{m}}.
\end{cases}
\]
Then the statistically deferred Nörlund convergence of \(Y_m\) is given as
\[
\lim_{m \to \infty} \frac{1}{2m} \left| \left\{ n : n \leq R_m \text{ and } 2mP(|Y_m - 0| \geq \varepsilon) \right\} \right| = \lim_{m \to \infty} P(Y_m = m)
= \lim_{m \to \infty} \frac{1}{\sqrt{m}}
= 0.
\]
However, statistically deferred Nörlund mean convergence, for \(r \geq 1\), is
\[
\lim_{m \to \infty} \frac{1}{2m} \left| \left\{ n : n \leq R_m \text{ and } 2mE(|Y_m - 0|^r) \right\} \right| = \lim_{m \to \infty} \left( m^r \left( \frac{1}{\sqrt{m}} \right) + 0 \left( 1 - \frac{1}{\sqrt{m}} \right) \right)
= \lim_{m \to \infty} m^{r-1/2}
= \infty.
\]
This implies that the sequence \((Y_m)\) is \(\text{St}_{DNP}\)-convergent but not \(\text{St}_{DNM}\)-convergent.
4. Statistical distribution convergence via Deferred Nörlund

**Definition 4.1.** The sequence of random variables \((Y_m)\) is said to be distribution convergent (or convergent in distribution) to \(Y\), if
\[
\lim_{m \to \infty} F_{Y_m}(y) = F_Y(y)
\]
for all \(y \in \mathbb{R}\) at which \(F_Y(y)\) is continuous.

Thoroughout the paper \((F_{Y_m}(y))\) is the sequence of distribution functions of \((Y_m)\) and \(F_Y(y)\) is the distribution function of \(Y\).

**Definition 4.2.** The sequence \((F_{Y_m}(y))\) is called as statistically distribution convergent (or \(\text{St}_{DC}\)), if there exists \(F_Y(y)\) of random variable \(Y\) such that for each \(\varepsilon > 0\),
\[
\lim_{m \to \infty} \frac{1}{|\mathbb{R}_m|} \left| \left\{ n : n \leq m \text{ and } |F_{Y_m}(y) - F_Y(y)| \geq \varepsilon \right\} \right| = 0.
\]
We may write this as
\[
\text{St}_{DC} \lim_{m \to \infty} F_{Y_m}(y) = F_Y(y).
\]

**Definition 4.3.** The sequence \((F_{Y_m}(y))\) of distribution functions is called as deferred Nörlund statistically distribution convergent (or \(\text{St}_{DNDc}\)), if there exists \(F_Y(y)\) of \(Y\) such that for each \(\varepsilon > 0\),
\[
\lim_{m \to \infty} \frac{1}{|\mathbb{R}_m|} \left| \left\{ n : n \leq m \text{ and } e_{y_m - n g_m} |F_{Y_m}(y) - F_Y(y)| \geq \varepsilon \right\} \right| = 0.
\]
In this case, we say
\[
\text{St}_{DNDc} \lim_{m \to \infty} F_{Y_m}(y) = F_Y(y).
\]

**Theorem 4.4.** Suppose that \(\text{St}_{DNP} \lim_{m \to \infty} P(|Y_m - Y| \geq \varepsilon) = 0\), then
\[
\text{St}_{DNDc} \lim_{m \to \infty} F_{Y_m}(y) = F_Y(y).
\]

**Proof.** Suppose that \((F_{Y_m}(y))\) is distribution functions of \((Y_m)\), and \(F_Y(y)\) be the distribution function of \(Y\). For \(i, j \in \mathbb{R}\) such that \(i < j\), we have
\[
(Y \leq i) = (Y_m \leq j, Y \leq i) + (Y_m \geq j, Y \leq i).
\]
Further,
\[
(Y_m \leq j, Y \leq i) \subseteq (Y_m \leq j),
\]
which implies that
\[
(Y \leq i) \subseteq (Y_m \leq j) + (Y_m \geq j, Y \leq i). \tag{4.1}
\]
Let us take the probability to left hand side and right hand side of equation (4.1)
\[
P(Y \leq i) \leq P((Y_m \leq j) + (Y_m \geq j, Y \leq i)) \leq P(Y_m \leq j) + P(Y_m \geq j, Y \leq i).
\]
It means that
\[
F_{Y_m}(j) \geq F_Y(i) - P(Y_m \geq j, Y \leq i). \tag{4.2}
\]
If \(Y_m \geq j, Y \leq i\), then \(Y_m \geq j, -Y \geq -i\), so that \(Y_m - Y \geq j - i\), that is,
\[
(Y_m \geq j, Y \leq i) \subseteq (Y_m - Y \geq j - i) \subseteq |Y_m - Y| > j - i.
\]
This means
\[
P(Y_m \geq j, Y \leq i) \leq P(|Y_m - Y| > j - i).
\]
As we know that $i < j$ and $\text{St}_{\text{DNP}} Y_m \to Y$, we obtain
\[
\text{St}_{\text{DNP}} \lim_{m \to \infty} P(Y_m \geq j, Y \leq i) = 0.
\]
From (4.2) we get
\[
\text{St}_{\text{DNDC}} \lim_{m \to \infty} F_{Y_m}(j) \geq F_Y(i).
\]
Similarly, if $j < a$ for any real constant $a$, then
\[
(Y \leq j) = (Y \leq a, Y_m \leq j) + (Y > a, Y_m \leq j).
\]
Consequently,
\[
F_{Y_m}(j) \leq F_Y(a) + P(Y > a, Y_m \leq j)
\]
and
\[
\text{St}_{\text{DNDC}} \lim_{m \to \infty} P(Y > a, Y_m \leq j) = 0.
\]
Therefore, we get
\[
\text{St}_{\text{DNDC}} \lim_{m \to \infty} F_{Y_m}(j) \leq F_Y(a).
\]
Thus, with $i < j < a$, we have
\[
\text{St}_{\text{DNDC}} \lim_{m \to \infty} F_{Y_m}(j) = F_Y(i).
\]
\[
\square
\]

**Example 2:** Consider the random variables $(Y_m, Y)$ of two dimensions as $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ such that
\[
(Y_m, Y) = \begin{cases} 
0, & [P(Y_m = 0, Y = 0) = 0 = P(Y_m = 1, Y = 1)] \\
\frac{1}{2}, & [P(Y_m = 1, Y = 0) = 0 = P(Y_m = 0, Y = 1)].
\end{cases}
\]
The distribution function of $Y_m$ is given by $Y_m = (\lambda_1 = 0, 1)$, with probability mass function
\[
(p_{y_m, \lambda_1}) = P(Y_m = \lambda_1), \text{ where } p_{y_m, 0} = \frac{1}{2} = p_{y_m, 1}
\]
and for $Y = \lambda_2(\lambda_2 = 0, 1)$, with probability mass function
\[
(p_{y_m, \lambda_2}) = P(Y_m = \lambda_2), \text{ where } p_{y, 0} = \frac{1}{2} = p_{y, 1}.
\]
If $(F_{Y_m}(y))$ is distribution functions of $(Y_m)$ and $F_Y(y)$ is the distribution function of $Y$, then
\[
F_Y(y) = \lim_{m \to \infty} F_{Y_m}(y) = \begin{cases} 
0, & (y < 0) \\
\frac{1}{2}, & (0 \leq y < 1). \\
1, & (z > 1).
\end{cases}
\]
Thus, we get
\[
\text{St}_{\text{DNDC}} \lim_{m \to \infty} F_{Y_m}(y) = F_Y(y), \text{ where } x_m = 2m - 1, y_m = 4m - 1, e_{y_{m-m}} = 2m \text{ and } g_m = 1.
\]
But, it is not $\text{St}_{\text{PC}}$ for the sequence of random variables, i.e.
\[
\text{St}_{\text{DNPC}} \lim_{m \to \infty} P(|Y_m - Y| \geq \varepsilon) \neq 0, \text{ where } x_m = 2m - 1, y_m = 4m - 1, e_{y_{m-m}} = 2m \text{ and } g_m = 1.
\]
5. Applications

The hypothesis of the Korovkin-type theorems have been studied by several researchers in various field in
different ways such as in, summability theory, functional analysis and probability theory. Korovkin-type
approximation theorems have been investigated by many mathematicians under various background, in-
volving function spaces, Banach spaces, and so on. Recently, Mohiuddine and Alamri studied Korovkin
and Voronovskaya type approximation theorems in [14]. Further, Hazarika et al. [10] studied Korovkin
approximation theorem for Bernstein operator of rough statistical convergence of triple sequences. For
detailed study on Korovkin approximation theorem one may refer [2], [16], [17], [19], [27].

By $C(Y)$, we denote the space of all continuous probability functions defined on a compact subset $Z \subset \mathbb{R}$.
The space $C(Y)$ is a Banach space with respect to the norm

$$\|f\|_\infty = \sup_{z \in Y} |f(z)|, \quad f \in C(Y).$$

We say that $\mathcal{Y}$ is a positive linear operator of sequence of random variables if

$$\mathcal{Y}(f,z) \geq 0 \text{ whenever } f \geq 0.$$ 

Throughout, $\mathcal{Y}_n : C(Y) \to C(Y)$ be a sequence of random variables of positive linear operators.

**Theorem 5.1** ([25]). Let $\mathcal{Y}_n : C(Y) \to C(Y)$. Then for all $f \in C(Y)$, we have

$$\text{St}_{\text{DNP}} \lim_{n \to \infty} \|\mathcal{Y}_n(f,z) - f(z)\|_\infty = 0$$

iff

$$\text{St}_{\text{DNP}} \lim_{n \to \infty} \|\mathcal{Y}_n(1,z) - 1\|_\infty = 0, \quad (5.1)$$

$$\text{St}_{\text{DNP}} \lim_{n \to \infty} \|\mathcal{Y}_n(z,z) - z\|_\infty = 0, \quad (5.2)$$

$$\text{St}_{\text{DNP}} \lim_{n \to \infty} \|\mathcal{Y}_n(z^2,z) - z^2\|_\infty = 0. \quad (5.3)$$

**Theorem 5.2.** Let $\mathcal{Y}_n : C(Y) \to C(Y)$. Then for all $f \in C(Y)$, we have

$$\text{St}_{\text{DNM}} \lim_{n \to \infty} \|\mathcal{Y}_n(f,z) - f(z)\|_\infty = 0$$

iff

$$\text{St}_{\text{DNM}} \lim_{n \to \infty} \|\mathcal{Y}_n(1,z) - 1\|_\infty = 0,$$

$$\text{St}_{\text{DNM}} \lim_{n \to \infty} \|\mathcal{Y}_n(z,z) - z\|_\infty = 0,$$

$$\text{St}_{\text{DNM}} \lim_{n \to \infty} \|\mathcal{Y}_n(z^2,z) - z^2\|_\infty = 0.$$

**Theorem 5.3.** Let $\mathcal{Y}_n : C(Y) \to C(Y)$. Then for all $f \in C(Y)$, we have

$$\text{St}_{\text{DNDC}} \lim_{n \to \infty} \|\mathcal{Y}_n(f,z) - f(z)\|_\infty = 0$$

iff

$$\text{St}_{\text{DNDC}} \lim_{n \to \infty} \|\mathcal{Y}_n(1,z) - 1\|_\infty = 0,$$

$$\text{St}_{\text{DNDC}} \lim_{n \to \infty} \|\mathcal{Y}_n(z,z) - z\|_\infty = 0,$$

$$\text{St}_{\text{DNDC}} \lim_{n \to \infty} \|\mathcal{Y}_n(z^2,z) - z^2\|_\infty = 0.$$
**Example 3:** Let $M_m(f, y)$ be a Meyer-König and Zeller operators on $C[0, 1]$ and $Z = [0, 1]$ as defined in [1] as

$$M_m(f, y) = (1 - y)^{m+1} \sum_{t=0}^{\infty} f\left(\frac{t}{t+m+1}\right)\left(\frac{m+t}{t}\right)y^t.$$  

Further, let us consider a sequence of operators $Y_n : C[0, 1] \to C[0, 1]$ and $(Y_n)$ as defined in example 2.4 such that

$$Y_n(f, y) = [1 + F_{Y_m}(y)]M_n(f), \quad (f \in C[0, 1]),$$  \hspace{1cm} (5.4)

where $(F_{Y_m}(y))$ is defined in Example 2. Now we observe that

$$Y_n(1, y) = [1 + F_{Y_m}(y)] \cdot 1 = [1 + F_{Y_m}(y)],$$

$$Y_n(w, z) = [1 + F_{Y_m}(y)] \cdot y = [1 + F_{Y_m}(y)] \cdot y$$

and

$$Y_n(u^2, z) = [1 + F_{Y_m}(y)] \cdot \left\{ y^2\left(\frac{m+2}{m+1}\right) + \frac{y}{m+1}\right\}.$$  

Therefore, we have

$$\text{StDND} \lim_{n \to \infty} \|Y_n(1, y) - 1\|_\infty = 0,$$

$$\text{StDND} \lim_{n \to \infty} \|Y_n(y, y) - y\|_\infty = 0,$$

$$\text{StDND} \lim_{n \to \infty} \|Y_n(y^2, y) - y^2\|_\infty = 0.$$  

Hence, $Y_n(f, y)$ fulfills (5.1), (5.2) and (5.3). Thus, from Theorem 5.3

$$\text{StDND} \lim_{n \to \infty} \|Y_n(f, y) - f\|_\infty = 0.$$  

Hence, it is (DNDC)–convergent. However, $(Y_m)$ is neither (DN)–statistical convergent nor (DN)– convergent. Thus, we can exhibit that the work in [23] does not hold for our operators described in (5.4). Hence, our Theorem 5.3 is stronger than the theorem proved in [23].

**6. Conclusion**

Upon prior analysis, our interest is to modify the studies of Srivastava et al. [24] and introduce various aspects of statistical convergence for the sequences of random variables and sequences of real numbers via deferred Norlund summability mean. We first study various results presenting the connection by using fundamental limit concepts of sequences of random variables. As an applications of our findings, we present new Krorvkin-type approximation results and also demonstrated the effectiveness of the findings. As a future work one can obtain the corresponding results of the present paper using deferred Euler summability mean.

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References

[6] H. Fast, Sur la convergence statistique, Colloquium Mathematicae, 2(1951), 241-244. 1