J. Nonlinear Sci. Appl., 16 (2023), 30–40

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Applications of statistical Riemann and Lebesgue integrability of sequence of functions

Kuldip Raj, Sonali Sharma

School of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J & K, India.

Abstract

In the present work, we propose to investigate statistical Riemann integrability, statistical Riemann summability, statistical Lebesgue integrability and statistical Lebesgue summability by means of deferred Nörlund and deferred Riesz mean. We discuss some fundamental theorems connecting these concepts with examples. As an application to our newly formed sequences, we introduce Korovkin-type approximation theorems with relevant example for positive linear operators by using Meyer-König and Zeller operators to exhibit the effectiveness of our findings.

Keywords: Statistical convergence, Riemann integral, Lebesgue integral, deferred Riesz, deferred Nörlund mean, Korovkin-type approximation theorem.

2020 MSC: 40A05, 40A30, 40G15.

1. Introduction and preliminaries

The perception of statistical convergence for a real sequence was established by Fast [4] and Stienhauss [22] independently. Later, it was restudied by Schoenberg [18]. Further, statistical convergence was investigated by sequence space point of view and summability theory by Connor [2], Fridy [5], Salat [16] and many others. For advance developments in the field of statistical convergence and their equivalent topics, one should refer to [6, 10, 11, 13, 19]. Consider $K \subset \mathbb{N}$, be such that

$$K_m = \{n : n \leqslant m \text{ and } n \in K\}.$$

The natural density $d(K)$ is given as

$$d(K) = \lim_{m\to\infty} \frac{|K_m|}{m} = a,$$

where a is finite real number and $|K_m|$ denotes the cardinality of the enclosed set. A sequence (z_m) is said to be statistically convergent to z_0 if, for every $\epsilon > 0$,

$$K_\epsilon = \{n : n \in \mathbb{N} \text{ and } |z_n - z_0| \geqslant \epsilon\}$$

Email addresses: kuldipraj68@gmail.com (Kuldip Raj), sonalisharma8082@gmail.com (Sonali Sharma)

doi: 10.22436/jnsa.016.01.03

Received: 2022-12-23 *Revised:* 2023-03-01 *Accepted:* 2023-03-05

has zero natural density which implies

$$d(K_\epsilon) = \lim_{m\to\infty} \frac{|K_\epsilon|}{m} = 0.$$

We may write it as

$$S \lim_{m\to\infty} z_m = z_0.$$

Consider $\mathcal{L} = [p_1, p_2] \subset \mathbb{R}$ (set of all real numbers) be a closed and bounded interval. A finite collection of points $P = \{l_0, l_1, l_2, \dots, l_m\}$ is called a partition of $[p_1, p_2]$, where

$$p_1 = l_0 < l_1 < l_2 < \cdots < l_m = p_2.$$

The interval $[p_1, p_2]$ is classified into subintervals as:

$$\mathcal{L}_1 = [l_0, l_1],\ \mathcal{L}_2 = [l_1, l_2], \dots, \mathcal{L}_m = [l_{m-1}, l_m].$$

Let γ_k be any arbitrary point from every subintervals $(\mathcal{L})_{k=1}^m$. These points are called tags and the subintervals linked with these tags are known as tagged partition of $\mathcal{L}$. It is written as

$$\mathcal{I} = \{([l_{k-1}, l_k; \gamma_k]) \text{ and } k = 1, 2, \cdots, m\}.$$

A sequence of functions $(\mathfrak{g}_m)_{m\in\mathbb{N}}$ s.t. $\mathfrak{g}_m : [p_1, p_2] \to \mathbb{R}$ linked with the tagged partition $\mathcal{I}$, the Riemann sum $\delta(\mathfrak{g}_m, \mathcal{I})$ is defined as

$$\delta(\mathfrak{g}_m, \mathcal{I}) = \sum_{k=1}^{m} \mathfrak{g}(\gamma_k)(l_k - l_{k-1}).$$

A sequence $(\mathfrak{g}_m)$ of functions is Riemann integrable (R.I.) to a function $\mathfrak{g}$ over $[p_1, p_2]$ if for every $\epsilon > 0$, $\exists\ \nu > 0$ s.t.

$$|\delta(\mathfrak{g}_m, \mathcal{I}) - \mathfrak{g}| < \epsilon,$$

where $\mathcal{I}$ denotes the tagged partition of $[p_1, p_2]$ and $\|\mathcal{I}\| < \nu$. Consider a finite measurable space (Z, Σ, ρ) and $(\mathfrak{g}_m)$ be the sequence of m.f. (measurable functions) with

$$\mathfrak{g}_m = \sum_{k=1}^{m} c_k \chi_{C_k},$$

where $C_k = [\alpha : \mathfrak{g}_m(\alpha) = c_k]$ and (c_k) are different values of $(\mathfrak{g}_m)$. Then, Lebesgue integral (L.I.) of $(\mathfrak{g}_m)$ w.r.t. measure ρ is given as

$$\int_Z \mathfrak{g}_m d\rho = \sum_{k=1}^{m} c_k \rho(C_k).$$

The sequence $(\mathfrak{g}_m)$ of m.f. is said to be L.I. to a m.f. $\mathfrak{g}$ if for every $\epsilon > 0$, we have

$$\left| \int_Z \mathfrak{g}_m d\rho - \mathfrak{g} \right| < \epsilon.$$

We may write it as $\mathfrak{g}_m \in L(Z, \rho)$. To know more about statistical Riemann integral and statistical Lebesgue integral, see [7, 20].

Assume that (r_m) and (s_m) be sequences of non-negative integers satisfying the following criteria:

(i) $r_m < s_m$, $(\forall\ m \in \mathbb{N})$;
(ii) $\lim_{m\to\infty} s_m = \infty$.

Now, suppose (u_m) and (v_m) are two sequences of non-negative real numbers s.t.

$$U_m = \sum_{j=r_m+1}^{s_m} u_j \quad \text{and} \quad V_m = \sum_{j=r_m+1}^{s_m} v_j.$$

The convolution of the above sequences is defined as:

$$\mathcal{P}_m = (U * V)_m = \sum_{k=r_m+1}^{s_m} u_k v_{s_m-k}.$$

Then, the deferred Nörlund mean ω_m is defined as

$$\omega_m = \frac{1}{\mathcal{P}_m} \sum_{j=r_m+1}^{s_m} u_{s_m-j} v_j z_j.$$

To know more about deferred Nörlund mean, see [3, 21]. Suppose $\{q_m\}$ be the sequence of non-negative real numbers and

$$\vartheta_m = \sum_{j=r_m+1}^{s_m} q_j,$$

where the sequences $\{r_m\}$ and $\{s_m\}$ satisfy above conditions (i) and (ii). Then, deferred Riesz mean t_m is defined as

$$t_m = \frac{1}{\vartheta_m} \sum_{j=r_m+1}^{s_m} q_j z_j.$$

For detailed study on deferred Riesz mean see [15]. Now, we define the product of deferred Riesz mean and deferred Nörlund mean as follows:

$$\psi_m = (\omega t)_m = \frac{1}{\mathcal{P}_m \vartheta_m} \sum_{j=r_m+1}^{s_m} u_{s_m-j} v_j q_j z_j.$$

Also, the sequence $\{\psi_m\}$ is said to be summable to z if

$$\lim_{m\to\infty} \psi_m = z.$$

A sequence $\{z_m\}$ is said to be deferred Nörlund and deferred Riesz statistically convergent to z if $\forall\ \epsilon > 0$, the set

$$\{m : m \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m-j} v_j q_j |z_m - z| \geqslant \epsilon\}$$

has natural density zero, i.e.,

$$\lim_{m\to\infty} \frac{1}{\mathcal{P}_m \vartheta_m} |\{m : m \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m-j} v_j q_j |z_m - z| \geqslant \epsilon\}| = 0.$$

It is written as $S - \lim z_m = z$ or $z_m \to z$ (ψ_m-statistically convergent) as $m \to \infty$.

Essentially motivated by the above mentioned investigations, here we investigate and study the concept of statistical version of Riemann summability, Riemann integrability, Lebesgue integrability and Lebesgue summability by means of deferred Nörlund and deferred Riesz mean. First, we discuss some fundamental theorems connecting these concepts with examples. As an application, we introduce Korovkin-type approximation theorems by means of deferred Nörlund and deferred Riesz mean with algebraic test functions and present example to illustrate the findings.

Throughout the text we represent SMF as sequence of measurable finctions.

2. Riemann integrability via deferred Nörlund and deferred Riesz mean

Here, we define the Riemann sum of a sequence of functions $\delta(\mathfrak{g}_m, \mathcal{J})$ by using deferred Nörlund and deferred Riesz summability mean of the form as

$$\mathcal{K}(\delta(\mathfrak{g}_m, \mathcal{J})) = \frac{1}{\mathcal{P}_m \vartheta_m} \sum_{i=r_m+1}^{s_m} u_{s_m - i} \nu_i q_i \delta(\mathfrak{g}_i, \mathcal{J}). \tag{2.1}$$

We now discuss some definitions related to statistical Riemann integrability and statistical Riemann summability by means of deferred Nörlund and deferred Riesz mean $(D(NR))$.

Definition 2.1. A sequence of functions $(\mathfrak{g}_m)_{m \in \mathbb{N}}$ is known as deferred Nörlund and deferred Riesz statistical Riemann integrable to $\mathfrak{g}$ on $[p_1, p_2]$, if $\forall\ \epsilon > 0$, $\exists\ \nu > 0$, and a tagged partition $\mathcal{J}$ of $[p_1, p_2]$ with $\|\mathcal{J}\| < \nu$ such that

$$\left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m - n} \nu_n q_n |\delta(\mathfrak{g}_n, \mathcal{J}) - \mathfrak{g}| \geqslant \epsilon \right\}$$

has zero natural density, i.e., $\forall \epsilon > 0$,

$$\lim_{m \to \infty} \frac{1}{\mathcal{P}_m \vartheta_m} \left| \left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m - n} \nu_n q_n |\delta(\mathfrak{g}_n, \mathcal{J}) - \mathfrak{g}| \geqslant \epsilon \right\} \right| = 0.$$

It can be written as

$$D(NR)\mathfrak{R}_S \lim_{m \to \infty} \delta(\mathfrak{g}_m, \mathcal{J}) = \mathfrak{g}.$$

Definition 2.2. A sequence of functions $(\mathfrak{g}_m)_{m \in \mathbb{N}}$ is known as deferred Nörlund and deferred Riesz statistically Riemann summable to $\mathfrak{g}$ on $[p_1, p_2]$, if $\forall\ \epsilon > 0$, $\exists\ \nu > 0$, and a tagged partition $\mathcal{J}$ of $[p_1, p_2]$ with $\|\mathcal{J}\| < \nu$,

$$\left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } \mathcal{K}|\delta(\mathfrak{g}_m, \mathcal{J}) - \mathfrak{g}| \geqslant \epsilon \right\}$$

has zero natural density, i.e., $\forall \epsilon > 0$,

$$\lim_{m \to \infty} \frac{1}{m} \left| \left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } \mathcal{K}|\delta(\mathfrak{g}_m, \mathcal{J}) - \mathfrak{g}| \geqslant \epsilon \right\} = 0.\right.$$

It can be written as $S_{D(NR)\mathfrak{R}} \lim_{m \to \infty} \delta(\mathfrak{g}_m, \mathcal{J}) = \mathfrak{g}$.

Theorem 2.3. *If a sequence $(\mathfrak{g}_m)_{m \in \mathbb{N}}$ of functions is deferred Nörlund and deferred Riesz integrable to $\mathfrak{g}$ on $[p_1, p_2]$, then it is deferred Nörlund and deferred Riesz statistical Riemann summable to $\mathfrak{g}$ on $[p_1, p_2]$.*

Proof. As $(\mathfrak{g}_m)_{m \in \mathbb{N}}$ is deferred Nörlund and deferred Riesz statistically Riemann integrable to $\mathfrak{g}$ on $[p_1, p_2]$, by Definition 2.1, we have

$$\lim_{m \to \infty} \frac{1}{\mathcal{P}_m \vartheta_m} \left| \left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m - n} \nu_n q_n |\delta(\mathfrak{g}_n, \mathcal{J}) - \mathfrak{g}| \geqslant \epsilon \right\} \right| = 0.$$

Now, we consider the following two sets

$$\mathcal{E} = \left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m - n} \nu_n q_n |\delta(\mathfrak{g}_n, \mathcal{J}) - \mathfrak{g}| \geqslant \epsilon \right\}$$

and

$$\mathcal{E}^c = \left\{ n : n \leqslant \mathcal{P}_m \vartheta_m \text{ and } u_{s_m - n} \nu_n q_n |\delta(\mathfrak{g}_n, \mathcal{J}) - \mathfrak{g}| < \epsilon \right\}.$$

Then, we have

$$
\begin{aligned}
\left|\mathcal{K}(\delta(\mathfrak{g}_m,\mathcal{I}))-\mathfrak{g}\right| &= \left|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}v_i q_i \delta(\mathfrak{g}_i,\mathcal{I})-\mathfrak{g}\right| \\
&\leqslant \left|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}v_i q_i [\delta(\mathfrak{g}_i,\mathcal{I})-\mathfrak{g}]\right| + \left|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}v_i q_i \mathfrak{g}-\mathfrak{g}\right| \\
&\leqslant \frac{1}{\mathcal{P}_m\vartheta_m}\sum_{\substack{i=r_m+1\\(n\in\mathcal{E})}}^{s_m} u_{s_m-i}v_i q_i \left|\delta(\mathfrak{g}_i,\mathcal{I})-\mathfrak{g}\right| + \frac{1}{\mathcal{P}_m\vartheta_m}\sum_{\substack{i=r_m+1\\(n\in\mathcal{E}^c)}}^{s_m} u_{s_m-i}v_i q_i \left|\delta(\mathfrak{g}_i,\mathcal{I})-\mathfrak{g}\right| \\
&\quad + |\mathfrak{g}|\left|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}v_i q_i - 1\right| \leqslant \frac{1}{\mathcal{P}_m\vartheta_m}|\mathcal{E}| + \frac{1}{\mathcal{P}_m\vartheta_m}|\mathcal{E}^c|.
\end{aligned}
$$

This implies that $\left|\mathcal{K}(\delta(\mathfrak{g}_m,\mathcal{I}))-\mathfrak{g}\right| < \epsilon$. Thus, the $(\mathfrak{g}_m)$ is deferred Nörlund and deferred Riesz statistical Riemann summable. □

Converse of the above result is not true. Let us prove this by an example.

Example 2.4. Suppose $r_m = 4m$, $s_m = 2m$, $u_{s_m-i} = 2m$, $v_i = 1$, $q_i = 1$ and suppose $\mathfrak{g}_m : [0,1] \to \mathbb{R}$ be the function of the form

$$
\mathfrak{g}_m(z) = \begin{cases} 0, & (z \in \mathbb{Q}\cap[0,1];\ \text{if } m \text{ is even}),\\ 1, & (z \in \mathbb{R}-\mathbb{Q}\cap[0,1];\ \text{if } m \text{ is odd}). \end{cases}
$$

The given sequence of functions $(\mathfrak{g}_m)$ shows that it is neither Riemann integrable nor deferred Nörlund and deferred Riesz statistical Riemann integrable. But from (2.1), we have

$$
\mathcal{K}(\delta(\mathfrak{g}_m,\mathcal{I})) = \frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}v_i q_i \delta(\mathfrak{g}_i,\mathcal{I}) = \frac{1}{2m\times 2m}\sum_{i=2m+1}^{4m}\delta(\mathfrak{g}_i,\mathcal{I}) = \frac{1}{2}.
$$

Therefore, $(\mathfrak{g}_m)$ has deferred Nörlund and deferred Riesz Riemann sum $\frac{1}{2}$ corresponding to tagged partition $\mathcal{I}$. So, $(\mathfrak{g}_m)$ is deferred Nörlund and deferred Riesz Riemann summable. But it is not deferred Nörlund and deferred Riesz statistical Riemann integrable.

3. Lebesgue integrability via deferred Nörlund and deferred Riesz mean

Now, let us present Lebesgue sum by means of deferred Nörlund and deferred Riesz summability mean for sequence of m.f. $(\mathfrak{g}_n)$ as

$$
\mathcal{W}(L(Z,\rho)) = \frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}v_i q_i c_i \rho(C_i). \tag{3.1}
$$

Definition 3.1. A SMF $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is known as deferred Nörlund and deferred Riesz statistically L.I. to m.f. $\mathfrak{g}$ on Z, if $\forall\ \epsilon > 0$,

$$
\left\{n : n \leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}v_n q_n\left|c_n\rho(C_n)-\mathfrak{g}\right| \geqslant \epsilon\right\}
$$

has zero natural density, i.e., $\forall\epsilon > 0$,

$$
\lim_{m\to\infty}\frac{1}{\mathcal{P}_m\vartheta_m}\left|\left\{n : n \leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}v_n q_n\left|c_n\rho(C_n)-\mathfrak{g}\right| \geqslant \epsilon\right\}\right| = 0.
$$

It can be written as

$$
D(NR)L_S \lim_{m\to\infty}\mathfrak{g}_m = \mathfrak{g}.
$$

Definition 3.2. A SMF $(\mathfrak{g}_m)_{m\in\mathbb{N}}$ is known as deferred Nörlund and deferred Riesz statistically Lebesgue summable to m.f. $\mathfrak{g}$ on Z, if $\forall\ \epsilon > 0$,

$$\Big\{n : n \leqslant m \text{ and } |\mathcal{W}(L(Z,\rho)) - \mathfrak{g}| \geqslant \epsilon\Big\}$$

has zero natural density, i.e., $\forall\epsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m}\Big|\Big\{n : n \leqslant m \text{ and } |\mathcal{W}(L(Z,\rho)) - \mathfrak{g}| \geqslant \epsilon\Big\}\Big| = 0.$$

It can be written as

$$S_{D(NR)L}\lim_{m\to\infty} \mathfrak{g}_m = \mathfrak{g}.$$

Theorem 3.3. *If a sequence $(\mathfrak{g}_m)_{m\in\mathbb{N}}$ of m.f. is deferred Nörlund and deferred Riesz statistically L.I. to measurable $\mathfrak{g}$ on Z, then it is deferred Nörlund and deferred Riesz statistical Lebesgue summable to same m.f. $\mathfrak{g}$ on Z.*

Proof. As $(\mathfrak{g}_m)_{m\in\mathbb{N}}$ is deferred Nörlund and deferred Riesz statistically L.I. to $\mathfrak{g}$ over Z, by Definition 3.1,

$$\lim_{m\to\infty} \frac{1}{\mathcal{P}_m\vartheta_m}\Big|\Big\{n : n \leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}\nu_n q_n\Big|c_n\rho(C_n) - \mathfrak{g}\Big| \geqslant \epsilon\Big\}\Big| = 0.$$

Now, we consider the following two sets

$$\mathcal{F} = \Big\{n : n \leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}\nu_n q_n\Big|c_n\rho(C_n) - \mathfrak{g}\Big| \geqslant \epsilon\Big\}$$

and

$$\mathcal{F}^c = \Big\{n : n \leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}\nu_n q_n\Big|c_n\rho(C_n) - \mathfrak{g}\Big| < \epsilon\Big\}.$$

Thus, we have

$$\begin{aligned}\Big|\mathcal{W}(L(Z,\rho)) - \mathfrak{g}\Big| &= \Big|\mathcal{P}_m\vartheta_m \sum_{i=r_m+1}^{s_m} u_{s_m-i}\nu_i q_i(c_i\rho(C_i)) - \mathfrak{g}\Big|\\ &\leqslant \Big|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}\nu_i q_i[(c_i\rho(C_i)) - \mathfrak{g}]\Big| + \Big|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}\nu_i q_i\mathfrak{g} - \mathfrak{g}\Big|\\ &\leqslant \frac{1}{\mathcal{P}_m\vartheta_m}\sum_{\substack{i=r_m+1\\(n\in\mathcal{F})}}^{s_m} u_{s_m-i}\nu_i q_i\Big|(c_i\rho(C_i)) - \mathfrak{g}\Big| + \frac{1}{\mathcal{P}_m\vartheta_m}\sum_{\substack{i=r_m+1\\(n\in\mathcal{F}^c)}}^{s_m} u_{s_m-i}\nu_i q_i\Big|(c_i\rho(C_i)) - \mathfrak{g}\Big|\\ &\quad + |\mathfrak{g}|\Big|\frac{1}{\mathcal{P}_m\vartheta_m}\sum_{i=r_m+1}^{s_m} u_{s_m-i}\nu_i q_i - 1\Big| \leqslant |\frac{1}{\mathcal{P}_m\vartheta_m}|\mathcal{F}| + |\frac{1}{\mathcal{P}_m\vartheta_m}|\mathcal{F}^c|.\end{aligned}$$

Thus, $\Big|\mathcal{W}(L(Z,\rho)) - \mathfrak{g}\Big| < \epsilon$. Hence the $(\mathfrak{g}_m)$ of measurable function is deferred Nörlund and deferred Riesz statistical Lebesgue summable to measurable function $\mathfrak{g}$ on Z. □

To show the converse of the above result is not true we present an example.

Example 3.4. Consider $s_m = 4m$, $r_m = 2m$, $u_{s_m - i} = 2m$, $q_i = 1$, $\nu_i = 1$, $s = 1$ and suppose $\mathfrak{g}_m : [0,1] \to \mathbb{R}$ be the function of the form

$$\mathfrak{g}_m(z) = \begin{cases} 0, & \text{(if } m \text{ is even)}, \\ 1, & \text{(if } m \text{ is odd)}, \end{cases}$$

where $z \in [0,1]$. The given sequence of functions $(\mathfrak{g}_m)$ shows that it is neither Lebesgue integrable nor deferred Nörlund and deferred Riesz statistical Lebesgue integrable. But from equation (3.1), we have

$$\mathcal{W}(L(Z,\rho)) = \frac{1}{\mathcal{P}_m \vartheta_m} \sum_{i=r_m+1}^{s_m} u_{s_m - i} \nu_i q_i (c_i \rho(C_i)) = \frac{1}{2m \times 2m} \sum_{i=2m+1}^{4m} c_i \rho(C_i) = \frac{1}{2}.$$

Therefore, $(\mathfrak{g}_m)$ has deferred Nörlund and deferred Riesz Lebesgue sum $\frac{1}{4m}$. So, $(\mathfrak{g}_m)$ is deferred Nörlund and deferred Riesz statistical Lebesgue summable. But it is not deferred Nörlund and deferred Riesz statistical Lebesgue integrable.

4. Korovkin-type approximation theorems

Korovkin-type approximation theorems have been investigated by many mathematicians under various fields such as function spaces, Banach spaces, and so on. Recently, Mohiuddine and Alamri [8] studied Korovkin and Voronovskaya-type approximation theorems. For detailed study on Korovkin approximation theorem one may refer to [9, 12, 14, 17]. By $C[0,1]$, we mean the space of all real valued continuous functions on [0,1]. The space $C[0,1]$ is a Banach space with the norm

$$||\mathfrak{g}||_\infty = \sup_{\tau \in [0,1]} |\mathfrak{g}(\tau)|, \ \ \forall\, \mathfrak{g} \in C[0,1].$$

Assume that $\mathfrak{F} : C[0,1] \to C[0,1]$ is a positive linear operator (PLO), that is, $\mathfrak{F}(\mathfrak{g}) \geqslant 0$ whenever $\mathfrak{g} \geqslant 0$. By $\mathfrak{F}(\mathfrak{g};\tau)$ we represent the value of $\mathfrak{F}\mathfrak{g}$ at a point τ.

Theorem 4.1. *Suppose* $\mathfrak{F}_l$ $(l \in \mathbb{N})$ *be a sequence of PLOs from* $C[0,1]$ *into itself. Then,* $\forall\, \mathfrak{g} \in C[0,1]$,

$$D(NR)\mathfrak{R}_S \lim_{l \to \infty} \|\mathfrak{F}_l(\mathfrak{g};\tau) - \mathfrak{g}(\tau)\|_\infty = 0 \tag{4.1}$$

iff

$$D(NR)\mathfrak{R}_S \lim_{l \to \infty} \|\mathfrak{F}_l(\mathfrak{g}_i;\tau) - \mathfrak{g}_i(\tau)\|_\infty = 0, \tag{4.2}$$

where

$$\mathfrak{g}_0(\tau) = 1, \quad \mathfrak{g}_1(\tau) = \tau, \ \ \textit{and} \ \ \mathfrak{g}_2(\tau) = \tau^2.$$

Proof. The condition (4.1) implies (4.2) by using the factor that each of the functions $\mathfrak{g}_i(\tau) = \tau^e \in C[0,1]$ $(e = 0,1,2)$ is continuous. Assume that (4.2) holds and suppose $\mathfrak{g} \in C[0,1]$, then $\exists$ a constant G, s.t.,

$$|\mathfrak{g}(\tau)| \leqslant G \quad (\forall\, \tau \in [0,1]),$$

which implies

$$|\mathfrak{g}(s) - \mathfrak{g}(\tau)| \leqslant 2G \quad (\forall\, \tau, s \in [0,1]).$$

For given $\epsilon > 0$, $\exists\, \delta > 0$, s.t.,

$$|\mathfrak{g}(s) - \mathfrak{g}(\tau)| < \epsilon, \tag{4.3}$$

whenever

$$|s - \tau| < \beta.$$

Choose $\psi = \psi(s,\tau) = (s-\tau)^2$. If $|s - \tau| \geqslant \delta$, then

$$|\mathfrak{g}(s) - \mathfrak{g}(\tau)| < \frac{2G}{\beta^2}\psi(s,\tau). \tag{4.4}$$

From (4.3) and (4.4), we get

$$|\mathfrak{g}(s)-\mathfrak{g}(\tau)|<\epsilon+\frac{2G}{\beta^2}\psi(s,\tau), \qquad -\epsilon-\frac{2G}{\beta^2}\psi(s,\tau)\leqslant \mathfrak{g}(s)-\mathfrak{g}(\tau)\leqslant \epsilon+\frac{2G}{\beta^2}\psi(s,\tau).$$

By the monotonicity and linearity of the operator $\mathfrak{F}_\iota(1,\tau)$, we have

$$\mathfrak{F}_\iota(1,\tau)(-\epsilon-\frac{2G}{\beta^2}\psi(s,\tau))\leqslant \mathfrak{F}_\iota(1,\tau)[\mathfrak{g}(s)-\mathfrak{g}(\tau)]\leqslant \mathfrak{F}_\iota(1,\tau)(\epsilon+\frac{2G}{\beta^2}\psi(s,\tau)).$$

Suppose that τ is fixed, so $\mathfrak{g}(\tau)$ is a constant number. Thus, we have

$$-\epsilon\mathfrak{F}_\iota(1,\tau)-\frac{2G}{\beta^2}\mathfrak{F}_\iota(\psi,\tau)\leqslant \mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)\mathfrak{F}_\iota(1,\tau)\leqslant \epsilon\mathfrak{F}_\iota(1,\tau)+\frac{2G}{\beta^2}\mathfrak{F}_\iota(\psi,\tau).$$

But

$$\mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)=[\mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)\mathfrak{F}_\iota(1,\tau)]+\mathfrak{g}(\tau)[\mathfrak{F}_\iota(1,\tau)-1]$$

gives

$$\mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)<\epsilon\mathfrak{F}_\iota(1,\tau)+\frac{2G}{\beta^2}\mathfrak{F}_\iota(\psi,\tau)+\mathfrak{g}(\tau)[\mathfrak{F}_\iota(1,\tau)-1]. \tag{4.5}$$

Next, we estimate $\mathfrak{F}_\iota(\psi,\tau)$ as

$$\begin{aligned}\mathfrak{F}_\iota(\psi,\tau)&=\mathfrak{F}_\iota((s-\tau)^2,\tau)=\mathfrak{F}_\iota(s^2-2\tau s+\tau^2,\tau)\\&=\mathfrak{F}_\iota(s^2,\tau)-2\tau\mathfrak{F}_\iota(s,\tau)+\tau^2\mathfrak{F}_\iota(1,\tau)=[\mathfrak{F}_\iota(s^2,\tau)-\tau^2]-2\tau[\mathfrak{F}_\iota(s,\tau)-\tau]+\tau^2[\mathfrak{F}_\iota(1,\tau)-1].\end{aligned}$$

From (4.5), we have

$$\begin{aligned}\mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)&<\epsilon\mathfrak{F}_\iota(1,\tau)+\frac{2G}{\beta^2}\Big\{[\mathfrak{F}_\iota(s^2,\tau)-\tau^2]-2\tau[\mathfrak{F}_\iota(s,\tau)-\tau]+\tau^2[\mathfrak{F}_\iota(1,\tau)-1]\Big\}+\mathfrak{g}(\tau)[\mathfrak{F}_\iota(1,\tau)-1]\\&=\epsilon[\mathfrak{F}_\iota(1,\tau)-1]+\epsilon+\frac{2G}{\beta^2}\Big\{[\mathfrak{F}_\iota(s^2,\tau)-\tau^2]-2\tau[\mathfrak{F}_\iota(s,\tau)-\tau]+\tau^2[\mathfrak{F}_\iota(1,\tau)-1]\Big\}\\&\quad+\mathfrak{g}(\tau)[\mathfrak{F}_\iota(1,\tau)-1].\end{aligned}$$

Then,

$$\begin{aligned}|\mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)|&\leqslant \epsilon+(\epsilon+\frac{2G}{\beta^2}+G)|\mathfrak{F}_\iota(1,\tau)-1|+\frac{4G}{\beta^2}|\mathfrak{F}_\iota(s,\tau)-\tau|+\frac{2G}{\beta^2}|\mathfrak{F}_\iota(s^2,\tau)-\tau^2|\\&\leqslant \epsilon+D(|\mathfrak{F}_\iota(1,\tau)-1|+|\mathfrak{F}_\iota(s,\tau)-\tau|+|\mathfrak{F}_\iota(s^2,\tau)-\tau^2|),\end{aligned}$$

where

$$D=\max\left\{\epsilon+\frac{2G}{\beta^2}+G,\frac{2G}{\beta^2},\frac{4G}{\beta^2}\right\}.$$

Now, for given $\lambda>0$, $\exists\ \epsilon>0$ s.t. $0<\epsilon<\lambda$. Then, by setting

$$\Phi_w(\tau,\lambda)=\left|\left\{w: w\leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}\nu_n q_n|\mathfrak{F}_\iota(\mathfrak{g},\tau)-\mathfrak{g}(\tau)|\geqslant\lambda\right\}\right|$$

and

$$\Phi_{e,w}(\tau,\lambda)=\left|\left\{w: w\leqslant \mathcal{P}_m\vartheta_m \text{ and } u_{s_m-n}\nu_n q_n|\mathfrak{F}_\iota(\mathfrak{g}_e,\tau)-\mathfrak{g}_e(\tau)|\geqslant\frac{\lambda-\epsilon}{3G}\right\}\right|,$$

for $e=0,1,2$, so we get

$$\Phi_w(\tau,\lambda)\leqslant\sum_{e=0}^{2}\Phi_{e,w}(\tau,\lambda).$$

Thus, we have

$$\frac{\|\Phi_w(\tau,\lambda)\|_{C[0,1]}}{\mathcal{P}_m\vartheta_m} \leqslant \sum_{e=0}^{2} \frac{\|\Phi_{e,w}(\tau,\lambda)\|_{C[0,1]}}{\mathcal{P}_m\vartheta_m}. \tag{4.6}$$

By using Definition 2.1 and the assumption about the implication in (4.2), the right side of (4.6) approaches to zero as $m \to \infty$. Hence,

$$\lim_{m\to\infty} \frac{\|\Phi_w(\tau,\lambda)\|_{C[0,1]}}{\mathcal{P}_m\vartheta_m} = 0, \quad (\lambda > 0).$$

Therefore, (4.1) holds true. □

Theorem 4.2. *Suppose $\mathfrak{F}_l$ $(l \in \mathbb{N})$ be a sequence of PLOs from $C[0,1]$ into itself. Then, $\forall\, \mathfrak{g} \in C[0,1]$,*

$$S_{D(NR)\mathfrak{R}} \lim_{l\to\infty} \|\mathfrak{F}_l(\mathfrak{g};\tau) - \mathfrak{g}(\tau)\|_\infty = 0$$

iff

$$S_{D(NR)\mathfrak{R}} \lim_{l\to\infty} \|\mathfrak{F}_l(\mathfrak{g}_e;\tau) - \mathfrak{g}_e(\tau)\|_\infty = 0,$$

where

$$\mathfrak{g}_0(\tau) = 1, \quad \mathfrak{g}_1(\tau) = \tau, \quad \textit{and} \quad \mathfrak{g}_2(\tau) = \tau^2.$$

Proof. This theorem can be proved in the similar manner as done in Theorem 4.1. Therefore, we prefer to skip the details. □

Next, we present an example in support of above theorems.

Example 4.3. Consider the Meyer-König and Zeller operators [1] $M_\sigma(\mathfrak{g};\tau)$ on $C[0,1]$ is defined as

$$M_\sigma(\mathfrak{g};\tau) = \sum_{j=0}^{\infty} \mathfrak{g}\left(\frac{j}{j+\sigma+1}\right)\binom{\sigma+j}{j}\tau^j(1-\tau)^{\sigma-j} \quad (\tau \in [0,1]).$$

Now, we define an operator $\mathfrak{F}_\sigma : C[0,1] \to C[0,1]$ by

$$\mathfrak{F}_\sigma(\mathfrak{g};\tau) = [1+\mathfrak{g}_\sigma(\tau)]\tau(1+\tau D)M_\sigma(\mathfrak{g};\tau), \quad \mathfrak{g} \in C[0,1], \tag{4.7}$$

where the sequence $(\mathfrak{g}_n(\tau))$ of functions are described as in Example 2.4. Now,

$$\mathfrak{F}_\sigma(\mathfrak{g}_0;\tau) = [1+\mathfrak{g}_\sigma(\tau)]\tau(1+\tau D)\mathfrak{g}_0(\tau) = [1+\mathfrak{g}_\sigma(\tau)]\tau,$$
$$\mathfrak{F}_\sigma(\mathfrak{g}_1;\tau) = [1+\mathfrak{g}_\sigma(\tau)]\tau(1+\tau D)\mathfrak{g}_1(\tau) = [1+\mathfrak{g}_\sigma(u)]\tau(1+\tau),$$

and

$$\begin{aligned}\mathfrak{F}_\sigma(\mathfrak{g}_2;\tau) &= [1+\mathfrak{g}_\sigma(\tau)]\tau(1+\tau D)\left\{\mathfrak{g}_2(\tau)\left(\frac{\sigma+2}{\sigma+1}\right) + \frac{\tau}{\sigma+1}\right\}\\ &= [1+\mathfrak{g}_\sigma(\tau)]\left\{\tau^2\left[\left(\frac{\sigma+2}{\sigma+1}\right)\tau + 2\left(\frac{1}{\sigma+1}\right) + 2\tau\left(\frac{\sigma+2}{\sigma+1}\right)\right]\right\}.\end{aligned}$$

Consequently, we have

$$S_{D(NR)\mathfrak{R}} \lim_{\sigma\to\infty} \|\mathfrak{F}_\sigma(\mathfrak{g}_e;\tau) - \mathfrak{g}_e(\tau)\|_\infty = 0,$$

$\forall\, e = 0,1,2$. Hence by Theorem 4.1, we have

$$S_{D(NR)\mathfrak{R}} \lim_{\sigma\to\infty} \|\mathfrak{F}_\sigma(\mathfrak{g};\tau) - \mathfrak{g}(\tau)\|_\infty = 0,$$

$\forall\, \mathfrak{g} \in C[0,1]$. The sequence $(\mathfrak{g}_m)$ of functions described in Example 2.4 is deferred Nörlund and deferred Riesz statistical Riemann summable, but not deferred Nörlund and deferred Riesz statistical Riemann integrable. Thus, our proposed operators given by (4.7) satisfy Theorem 4.2. However, it does not work for Theorem 4.1.

Theorem 4.4. *Suppose* $\mathfrak{F}_{\mathfrak{l}}$ $(\mathfrak{l} \in \mathbb{N})$ *be a sequence of PLOs from* $C[0,1]$ *into itself. Then for all* $\mathfrak{g} \in C[0,1]$,

$$D(NR)L_S \lim_{\mathfrak{l}\to\infty} \|\mathfrak{F}_{\mathfrak{l}}(\mathfrak{g};\tau) - \mathfrak{g}(\tau)\|_\infty = 0$$

iff

$$D(NR)L_S \lim_{\mathfrak{l}\to\infty} \|\mathfrak{F}_{\mathfrak{l}}(\mathfrak{g}_e;\tau) - \mathfrak{g}_e(\tau)\|_\infty = 0,$$

where

$$\mathfrak{g}_0(\tau) = 1, \quad \mathfrak{g}_1(\tau) = \tau, \quad and \quad \mathfrak{g}_2(\tau) = \tau^2.$$

Proof. This theorem can be proved in the similar manner as done in Theorem 4.1. Therefore, we prefer to skip the details. □

Theorem 4.5. *Suppose* $\mathfrak{F}_{\mathfrak{l}}$ $(\mathfrak{l} \in \mathbb{N})$ *be a sequence of PLOs from* $C[0,1]$ *into itself. Then,* $\forall\ \mathfrak{g} \in C[0,1]$,

$$S_{D(NR)L} \lim_{\mathfrak{l}\to\infty} \|\mathfrak{F}_{\mathfrak{l}}(\mathfrak{g};\tau) - \mathfrak{g}(\tau)\|_\infty = 0$$

iff

$$S_{D(NR)L} \lim_{\mathfrak{l}\to\infty} \|\mathfrak{F}_{\mathfrak{l}}(\mathfrak{g}_e;\tau) - \mathfrak{g}_e(\tau)\|_\infty = 0,$$

where

$$\mathfrak{g}_0(\tau) = 1, \quad \mathfrak{g}_1(\tau) = \tau, \quad and \quad \mathfrak{g}_2(\tau) = \tau^2.$$

Proof. This theorem can be proved in the similar manner as done in Theorem 4.1. Therefore, we prefer to skip the details. □

5. Conclusion

In this article, firstly we present the concept of the deferred Nörlund and deferred Riesz statistically Riemann integral, deferred Nörlund and deferred Riesz statistically Lebesgue integral. Further, with the help of certain examples we established some implication relations among them. Finally as an application upon our proposed method, we proved Korovkin-type approximation theorems with algebraic test functions. The results presented in this article not only generalize the earlier works done by several mathematicians but also give a new outlook concerning the evolution of Korovkin-type approximation theorem. As a future work one can study the concept of deferred Cesàro and deferred Euler statistically Riemann integral for sequence of real valued functions.

Acknowledgment

The corresponding author thanks the Council of Scientific and Industrial Research (CSIR), India for partial support under Grant No. 25(0288)/18/EMR-II, dated 24/05/2018.

References

[1] A. Altın, O. Doğru, F. Taşdelen, *The generalization of Meyer-König and Zeller operators by generating functions*, J. Math. Anal. Appl., **312** (2005), 181–194. 4.3

[2] J. Connor, *Two valued measures and summability*, Analysis, **10** (1990), 373–385. 1

[3] K. Demirci, F. Dirik, S. Yıldız, *Deferred Nörlund statistical relative uniform convergence and Korovkin-type approximation theorem*, Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., **70** (2021), 279–289. 1

[4] H. Fast, *Sur La convergence statistique*, Colloq. Math., **2** (1951), 241–244. 1

[5] J. A. Fridy, *On statistical convergent*, Analysis, **5** (1985), 301–314. 1

[6] R. Haloi, M. Sen, B. C. Tripathy, *Statistically lacunary convergence of generalized difference sequences in probabilistic normed spaces*, Appl. Sci., **21** (2019), 107–118. 1

[7] B. B. Jena, S. K. Paikray, H. Dutta, *Statistically Riemann integrable and summable sequence of functions via deferred Cesàro mean*, Bull. Iran. Math. Soc., **48** (2022), 1293–1309. 1

[8] S. A. Mohiuddine, B. A. S. Alamri, *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **113** (2019), 1955–1973. 4

[9] S. A. Mohiuddine, A. Asiri, B. Hazarika, *Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems*, Int. J. Gen. Syst., **48** (2019), 492–506. 4

[10] M. Mursaleen, *λ-statistical convergence*, Math. Slovaca, **50** (2000), 111–115. 1

[11] M. Mursaleen, A. Alotaibi, *Statistical summability and approximation by de la Vallée-Poussin mean*, Appl. Math. Lett., **24** (2011), 320–324. 1

[12] M. Mursaleen, A. Alotaibi, *Korovkin type approximation theorems for functions of two variables through statistical A-summability*, Adv. Differ. Equ., **2012** (2012), 1–10. 4

[13] P. K. Nath, B. C. Tripathy, *Statistical convergence of complex uncertain sequences defined by Orlicz function*, Proyecciones, **39** (2020), 301–315. 1

[14] K. Raj, A. Choudhary, *Relative modular uniform approximation by means of the power series method with applications*, Rev. Un. Mat. Argentina, **60** (2019), 187–208. 4

[15] K. Saini, K. Raj, *Application of statistical convergence in complex uncertain sequence via deferred Reisz mean*, Int. J. Uncertain. Fuzziness Knowlege-Based Syst., **29** (2021), 337-351. 1

[16] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139–150. 1

[17] K. Saini, K. Raj, M. Mursaleen, *Deferred Cesàro and deferred Euler equi-statistical convergence and its applications to Korovkin-type approximation theorem*, Int. J. Gen. Syst., **50** (2021), 567–579. 4

[18] I. J. Schoenberg, *The integrability of certain function and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375. 1

[19] H. M. Srivastava, M. Et, *Lacunary statistical convergence and strongly lacunary summable functions of order α*, Filomat, **31** (2017), 1573–1582. 1

[20] H. M. Srivastava, B. B. Jena, S. K. Paikray, *Statistical Riemann and Lebesgue integrable sequence of functions with Korovkin-type approximation theorems*, Axioms, **10** (2021), 1–16. 1

[21] H. M. Srivastava, B. B. Jena, S. K. Paikray, U. K. Misra, *Generalized equi-statistical convergence of the deferred Nörlund summability and its applications to associated approximation theorems*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **112** (2018), 1487–1501. 1

[22] H. Steinhaus, *Sur la convergence ordinarie et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74. 1