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New Hadamard-type inequality for new class of geodesic convex functions

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Abstract

This paper aims to introduce the concept of (E, μ, κ)-convex function by using special inequality. Hadamard integral inequality for this new class of geodesic convex function in the case of Lebesgue and Sugeno integrals is given.

Keywords: Convex function, geodesic, Riemannian manifolds.

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1. Introduction

It is commonly known that convexity is used in modern analysis, either directly or indirectly [18]. The idea of convexity has been developed and generalized in numerous directions due to its uses and significance, see [7, 8, 15, 16]. E-convexity of sets and functions, which is a broader function than invexity, was introduced in 1999 [28]. However, Young [27] claims that some of the results in [28] are inaccurate. In [4], the E-convexity was expanded to a semi-E-convexity. See [5, 6, 23] for further information on the E-convex or semi-E-convex functions. Furthermore, Youness and Emam in [29] discuss a novel class of functions called as strongly E-convex functions. In particular, semi-strong E-convexity as well as quasi and pseudo semi-strong E-convexity was added to this class of functions [30].

A manifold is not a linear space, and extensions of concepts and techniques from linear spaces to Riemannian manifolds are natural. Many authors, including Udrist [24] and Rapcsak [20], have studied generalized convex functions in Riemannian manifolds Geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds are investigated in 2012 [10]. Moreover, geodesic semi E-convex functions are introduced in [9]. Recently, geodesic strongly E-convex functions have been introduced, and some of their properties [11].

Based on these ideas, a new class of functions, which are called geodesic semi strongly E-convex functions, are defined and some of their properties are presented in [12]. A class of functions on Riemannian manifolds, which are called geodesic semilocal E-preinvex functions, as a generalization of geodesic semilocal E-convex and geodesic semi E-preinvex functions, are given in [13]. In [21], geodesic E-b-vex

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sets and geodesic E-b-vex functions on a Riemannian manifold are extended to geodesic strongly E-b-vex sets and geodesic strongly E-b-vex functions.

Sugeno integrals are a type of nonlinear integral invented by Sugeno [22] to capture and integrate interactions between criteria of various phenomena. The most well-known integral inequalities for Sugeno integral have been proven, see [1, 25].

2. Preliminaries

In this section, we present some definitions and properties that can be found in many books on differential geometry, such as [24].

Suppose that $\aleph$ is a C^∞ n-dimensional Riemannian manifold, and $T_t\aleph$ is the tangent space to $\aleph$ at t. Also, assume that $\mu_t(y_1, y_2)$ is a positive inner product on the tangent space $T_t\aleph$ ($y_1, y_2 \in T_t\aleph$), which is given for each point of $\aleph$. Then, a C^∞ map $\mu\colon t \longrightarrow \mu_t$, which assigns a positive inner product μ_t to $T_t\aleph, \forall t \in \aleph$ is called a Riemannian metric.

The length of a piecewise C^1 curve $\eta\colon [a_1, a_2] \longrightarrow \aleph$ which is defined as follows:

$$L(\eta) = \int_{a_1}^{a_2} \|\acute{\eta}(y)\| dt.$$

We define $d(t_1, t_2) = \inf\{L(\eta)\colon \eta$ is a piecewise C^1 curve joining t_1 to $t_2\}$ for any points $t_1, t_2 \in \aleph$. Furthermore, a smooth path η is a geodesic if and only if its tangent vector is a parallel vector field along the path η, i.e., η satisfies the equation $\bigtriangledown_{\acute{\eta}(t)}\acute{\eta}(t) = 0$. Every path η is joining $t_1, t_2 \in \aleph$, where $L(\eta) = d(t_1, t_2)$ is a minimal geodesic.

Finally, assume that $(\aleph, \mu)$ is a complete n-dimensional Riemannian manifold with Riemannian connection $\bigtriangledown$. Let $y_1, y_2 \in \aleph$ and $\eta\colon [0, 1] \longrightarrow \aleph$ be a geodesic joining the points y_1 and y_2, which means that $\eta_{y_1,y_2}(0) = x_2$ and $\eta_{y_1,y_2}(1) = y_1$.

A set A in a Riemannian manifold $\aleph$ is called t-convex if A contains every geodesic η_{y_1,y_2} of N whose endpoints y_1 and y_2 belong to A.

Note that the whole of the manifold $\aleph$ is t-convex, and conventionally, so is the empty set. The minimal circle in a hyperboloid is t-convex, but a single point is not. Also, any proper subset of a sphere is not necessarily t-convex.

The following theorem was proved in [24].

Theorem 2.1 ([24]). *The intersection of any number of* t*-convex sets is* t*-convex.*

Remark 2.2. In general, the union of a t-convex set is not necessarily t-convex.

Definition 2.3 ([24]). A function $g\colon A \longrightarrow \mathbb{R}$ is called g-convex function on a t-convex set $A \subset \aleph$ if for every geodesic η_{y_1,y_2}, then

$$g(\eta_{y_1,y_2}(\gamma)) \leqslant \gamma g(y_1) + (1-\gamma)g(y_2)$$

holds $\forall y_1, y_2 \in A$ and $\gamma \in [0, 1]$.

Now let M be a non-empty set and ξ be a σ- algebra of subsets of M.

Definition 2.4 ([19]). Let $N : \xi \longrightarrow [0, \infty)$ be a set function, then N is called a Sugeno measure if it satisfies

1. $N(\phi) = 0$;
2. if $A, B \in \xi$ and $A \subset B$, then $\xi(A) \leqslant \xi(B)$;
3. $A_i \in N$, where $i \in \mathbb{N}$, $A_{i-1} \subset A_i$, then $\lim_{i\longrightarrow\infty} \xi(A_i) = \xi(\cup_{i=1}^{\infty} E_i)$;
4. $A_n \in \xi$, where i$\in \mathbb{N}$,$A_{i-1} \supset A_i$, $\xi(A_1) < \infty$, then $\lim_{i\longrightarrow\infty} \xi(A_i) = \xi(\cap_{i=1}^{\infty} E_i)$.

Assume that (M, ξ, N), which is said to be a sugeno measure space, is a fuzzy measure space. By $H_\xi(M)$, then

$$\chi_\xi(M) = \{h : M\xi \longrightarrow [0, \infty) : h \text{ is measurable with respect to } \xi\}.$$

Definition 2.5 ([17, 22]). Assuming (M, ξ, N) is a fuzzy measure space, $h \in \chi_{\xi}(M)$ and $X \in \xi$, then the Sugeno integral of h on A w.r.t. the N is defined by

$$\int_X h dN = \bigvee_{\alpha \geqslant 0} (\alpha \wedge N(X \cap H_\alpha)),$$

where $H_\alpha = u \in M : h(u) \geqslant \alpha$, $\wedge$ is the prototypical t-normal minimum and $\bigvee$ the prototypical t-conorm maximum. If $X = M$, then

$$\int_X h dN = \bigvee_{\alpha \geqslant 0} (\alpha \wedge N(H_\alpha)).$$

Some properties of the Sugeno integral can be found in [17, 26] such as following.

Theorem 2.6. *Assume that (M, ξ, N) is a fuzzy measure space, $X, Y \in \xi$, and $h_1, h_2 \in \chi_N(M)$, then*

1. $\int_X h_1 dN \leqslant N(X)$;
2. $\int_X a dN = a \wedge N(X)$, *where a is non-negative constant;*
3. *if $h_1 \leqslant h_2$ on X, then $\int_X h_1 dN \leqslant \int_X h_2 dN$;*
4. *if $X \subset Y$, then $\int_X h_1 dN \leqslant \int_Y h_1 dN$.*

3. The main results

In this part of the paper, let us take (M, ξ) be a fuzzy measure space for a given $h \in H^N(M)$ and $X \in \xi$, then

$$\Gamma = \left\{\alpha : \alpha \geqslant 0, N(X \cap h_\alpha) > N(X \cap h_\beta) \text{ for any } \beta > \alpha\right\}.$$

Moreover, $\int_X h dN = \bigvee_{\alpha \in \Gamma} (\alpha \wedge N(X \cap h_\alpha))$ [2].

In the next definition, the concept of (E, μ, κ)-convexity is given.

Definition 3.1. Considering Y_1 and Y_2 are two E-convex sets, where $E : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Assume that $\mu_{E(u_1),E(u_2)} : [0, 1] \longrightarrow Y_1$ is a geodesic arc joining the points$u_1, u_2 \in Y_1$ and $\kappa_{E(\nu_1),E(\nu_2)} : [0, 1] \longrightarrow Y_2$ is a geodesic arc joining the points $\nu_1, \nu_2 \in Y_2$. A real calued function $h : Y_1 \longrightarrow Y_2$ is called a (E, μ, κ)-convex if

$$h\left(\mu_{E(u_1),E(u_2)}(\lambda)\right) \leqslant \kappa_{h(E(\nu_1)),h(E(\nu_2))}(\lambda), \forall u_1, u_2 \in Y_1, \lambda \in [0, 1].$$

Remark 3.2.

1. For a (E, μ, κ)-convex function $h : [x_1, x_2] \longrightarrow [y_1, y_2]$, then
$$h(u) = h\left(\mu_{E(x_1),E(x_2)}\left(\mu^{-1}_{E(x_1),E(x_2)}(u)\right)\right) \leqslant \kappa_{h(E(x_1)),h(E(x_2))}\left(\mu^{-1}_{E(x_1),E(x_2)}(u)\right) \tag{3.1}$$
for all $u \in [x_1, x_2]$, then the inquality is sharp for all $u \in [x_1, x_2]$.
2. If $E = I$, where I is the indenty mapping, then the inquality (3.1) becomes the inquality (2) in [2].

Next, some generalizations of Hadamard in inequality for different geodesic convex functions are given.

Theorem 3.3. *Assume that $Y_1$and Y_2 are two E-convex subsets of $\mathbb{R}$, $x_1, x_2 \in Y_1^o$ with $x_1 < x_2$ and $y_1, y_2 \in Y_2^o$ with $y_1 < y_2$. For the particular geodesic arcs $\mu : [0, 1] \longrightarrow Y_1$ and $\kappa : [0, 1] \longrightarrow Y_2$ defined by $\mu_{E(u_1,u_2)}(\lambda) = (1 - \lambda)E(u_1) + \lambda E(u_2)$ and $\kappa_{\nu_1,\nu_2}(\lambda) = E(\nu_1)^{1-\lambda} E(\nu_2)^\lambda$, then the next inqualities hold.*

1. *If $h : Y_1 \longrightarrow Y_2$ is a (E, μ, μ)- convex function, then*

$$\frac{1}{x_2 - E(x_1)} \int_{E(x_1)}^{E(x_2)} h(E(u)) dE(u) \leqslant \frac{h(E(x_1)) + h(E(x_2))}{2}, \forall u \in [x_1, x_2].$$

2. *If* $h : Y_1 \longrightarrow Y_2 \subseteq (0, \infty)$ *is a* (E, μ, κ)*-convex function with* $h(E(x_1)) \neq h(E(x_2))$*, then*

$$\frac{1}{E(x_2) - E(x_1)} \int_{E(x_1)}^{E(x_2)} h(E(u))dE(u) \leqslant \frac{h(E(x_1))}{\ln\left(\frac{h(E(x_2))}{h(E(x_1))}\right)} \left(\frac{h(E(x_2))}{h(E(x_1))} - 1\right).$$

3. *If* $h : Y_2 \subseteq (0, \infty) \longrightarrow Y_1$ *is a* (E, κ, μ)*-convex function, then*

$$\frac{1}{E(y_1) - E(y_2)} \int_{E(y_1)}^{E(y_2)} h(E(u))dE(u) \leqslant h(E(y_1)) + \frac{h(E(y_2)) - h(E(y_1))}{\ln\left(\frac{h(E(y_2))}{h(E(y_1))}\right)} \left(\frac{E(y_2)\ln\left(\frac{E(y_2)}{E(y_1)}\right)}{E(y_2) - E(y_1)} - 1\right).$$

4. *If* $h : Y_2 : (0, \infty) \longrightarrow Y_2 : (0, \infty)$ *is a* (E, κ, κ)*-convex function with* $h(E(y_1)) \neq h(E(y_2))$*, then*

$$\frac{1}{E(y_1) - E(y_2)} \int_{E(y_1)}^{E(y_2)} h(E(u))dE(u) \leqslant \frac{E(y_1)h(E(y_1))}{E(y_2) - E(y_1)} \left(\frac{\left(\frac{E(y_2)}{E(y_1)}\right)^{\log_{\frac{E(y_2)}{E(y_1)}}(\frac{h(E(y_2))}{h(y_1)}+1)} - 1}{\log_{\frac{E(y_2)}{E(y_1)}}\left(\frac{h(E(y_2))}{h(E(y_1))}\right) + 1}\right).$$

Proof. The first inequality is the well-known Hadamard's inequality for E-convex functions. If we use the inequality (3.1), then we have the following inequalities.

1. The function $h : [x_1, x_2] \longrightarrow [x_1, x_2]$ is (E, μ, μ)-convex iff

$$h(u) \leqslant h(E(x_1)) + \frac{u - E(x_1)}{E(x_2) - E(x_1)}(h(E(x_2)) - h(E(x_1))), \ \forall u \in [x_1, x_2]. \tag{3.2}$$

2. The function $h : [x_1, x_2] \longrightarrow [y_1, y_2]$ is (E, μ, κ)-convex iff

$$h(u) \leqslant h(E(x_1)) \left(\frac{h(E(x_2))}{h(E(x_1))}\right)^{\frac{u-E(x_1)}{E(x_2)-E(x_1)}}, \ \forall u \in [x_1, x_2]. \tag{3.3}$$

3. The function $h : [y_1, y_2] \longrightarrow [x_1, x_2]$ is (E, κ, μ)-convex iff

$$h(u) \leqslant h(E(y_1)) + \log_{\frac{E(y_2)}{E(y_1)}} \frac{u}{h(E(y_1))}(h(E(y_2)) - h(E(y_1))), \ \forall u \in [y_1, y_2]. \tag{3.4}$$

4. The function $h : [x_1, x_2] \longrightarrow [y_1, y_2]$ is (E, μ, κ)-convex iff

$$h(u) \leqslant h(E(y_1)) \left(\frac{h(E(y_2))}{h(E(y_1))}\right)^{\log_{\frac{E(y_2)}{E(y_1)}} \frac{u}{h(E(y_1))}}, \ \forall u \in [y_1, y_2]. \tag{3.5}$$

If we integrate the inequalities (3.2), (3.3), (3.4), and (3.5) from both sides over $[x_1, x_2]$ or $[y_1, y_2]$, we obtain the results in the theorem. □

Theorem 3.4. *Let* $(\mathbb{R}, \xi, N)$ *be the fuzzy measurement space . Assume that* $\mu : [0, 1] \longrightarrow [x_1, x_2]$ *and* $\kappa : [0, 1] \longrightarrow [y_1, y_2]$ *are two invertible geodesic arcs. If* $h : [x_1, x_2] \longrightarrow [y_1, y_2]$ *is a* (E, μ, κ)*-convex function, then*

$$\int_{x_1}^{x_2} h dN \leqslant \begin{cases} \bigvee_{\alpha \in [h(E(x_1)), h(E(x_2))]} \left(\alpha \wedge N\left(\left[\mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)), E(x_2)\right]\right)\right), \\ \quad \text{if } \mu, \kappa \text{ are comonotone}, \\ \bigvee_{\alpha \in [h(E(x_2)), h(E(x_1))]} \left(\alpha \wedge N\left(\left[E(x_1), \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha))\right]\right)\right), \\ \quad \text{if } \mu, \kappa \text{ are countermonotone}. \end{cases}$$

Proof. Since h is a (E, μ, κ)-convex function and by using the property (3 in Theorem 2.6) of fuzzy measure, we get

$$\begin{aligned} \int_{x_1}^{x_2} h dN &= \int_{x_1}^{x_2} h(\mu_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(u)))dN \\ &\leqslant \int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(u))dN. \end{aligned} \tag{3.6}$$

If μ and κ are comonotone, then $\kappa \circ \mu^{-1}$ is an increasing function, then by Definition 2.5

$$\begin{aligned} &\int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(u))dN \\ &\quad = \bigvee_{\alpha \geqslant 0} \left(\alpha \wedge N([E(x_1), E(x_2)] \cap \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))} \geqslant \alpha))\right) \\ &\quad = \bigvee_{\alpha \geqslant 0} \left(\alpha \wedge N(u \geqslant \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)))\right) \\ &\quad = \bigvee_{\alpha \geqslant 0} \left(\alpha \wedge N\left(\left[\mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)), E(x_2)\right]\right)\right). \end{aligned} \tag{3.7}$$

Since $\kappa \circ \mu^{-1}$ is increasing, we get

$$\begin{aligned} &E(x_1) \leqslant \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) < E(x_2) \\ &\Longrightarrow \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(E(x_1))) \leqslant \alpha < \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(E(x_2))) \\ &\Longrightarrow \kappa_{E(x_1),E(x_2)}(0) \leqslant \alpha < \kappa_{E(x_1),E(x_2)}(1) \\ &\Longrightarrow h(E(x_1)) \leqslant \alpha < h(E(x_2)). \end{aligned} \tag{3.8}$$

Thus, $\Gamma = [h(E(x_1)), h(E(x_2))]$ and we only need to consider $\alpha \in [h(E(x_1)), h(E(x_2))]$. It follows from (3.6), (3.7), and (3.8), that

$$\begin{aligned} &\int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(u))dN \\ &\quad \leqslant \bigvee_{\alpha \in [h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge N\left(\left[\mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)), E(x_2)\right]\right)\right). \end{aligned}$$

If μ and κ are countermonotone, then $\kappa \circ \mu^{-1}$ is a decreasing function. Then, by Definition 2.5, we get

$$\begin{aligned} &\int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(u))dN \\ &\quad = \bigvee_{\alpha \geqslant 0} \left(\alpha \wedge N([E(x_1), E(x_2)] \cap \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))} \geqslant \alpha))\right) \\ &\quad = \bigvee_{\alpha \geqslant 0} \left(\alpha \wedge N(u \leqslant \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))} \geqslant \alpha))\right) \\ &\quad = \bigvee_{\alpha \geqslant 0} \left(\alpha \wedge N\left(\left[E(x_1), \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha))\right]\right)\right). \end{aligned} \tag{3.9}$$

Since $\kappa \circ \mu^{-1}$ is decreasing, we get

$$E(x_1) \leqslant \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) < E(x_2)$$

$$\begin{aligned}
&\Longrightarrow \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(E(x_1))) \leqslant \alpha < \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(E(x_2))) \qquad (3.10)\\
&\Longrightarrow \kappa_{E(x_1),E(x_2)}(0) \leqslant \alpha < \kappa_{E(x_1),E(x_2)}(1)\\
&\Longrightarrow h(E(x_1)) \leqslant \alpha < h(E(x_2)).
\end{aligned}$$

Thus, $\Gamma = [h(E(x_2)), h(E(x_1))]$ and we only need to consider $\alpha \in [h(E(x_2)), h(E(x_1))]$. It follows from (3.6), (3.9), and (3.10) that

$$\begin{aligned}
&\int_{x_1}^{x_2} \kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(u))dN\\
&\qquad \leqslant \bigvee_{\alpha\in[h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge N\left(\left[\mu_{E(x_1),E(x_2)}(E(x_2), \kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha))\right]\right)\right).
\end{aligned}$$

□

Remark 3.5. Consider $h : [x_1, x_2] \longrightarrow [y_1, y_2]$ is a (E, μ, κ)-convex function, ξ is the Borel field, and N is the Lebesgue measure on $\mathbb{R}$. Then

$$\int_{x_1}^{x_2} h dN \leqslant \begin{cases} \bigvee_{\alpha\in[h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge (E(x_2) - \mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha))\right), \\ \quad \text{if } \mu, \kappa \text{ are comonotone}, \\ \bigvee_{\alpha\in[h(E(x_2)),h(E(x_1))]} \left(\alpha \wedge (\mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) - E(x_1))\right), \\ \quad \text{if } \mu, \kappa \text{ are countermonotone}. \end{cases}$$

In the following corollaries, consider that $((R), \mathbb{R}, \xi, N)$ is the fuzzy measure space.

Corollary 3.6. *Let* $h : [x_1, x_2] \longrightarrow [x_1, x_2]$ *be a* (E, μ, μ)*-convex function, hence*

$$\int_{x_1}^{x_2} h dN \leqslant \begin{cases} \bigvee_{\alpha\in[h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge N\left(\left[E(x_1) + (E(x_2) - E(x_1))\frac{\alpha - h(E(x_1))}{h(E(x_2)) - h(E(x_1))}, E(x_2)\right]\right)\right), \\ \quad \textit{if } h(E(x_1)) < h(E(x_2)), \\ h(E(x_1)) \wedge N(E(x_1), E(x_2)), \ \textit{if } h(E(x_1)) = h(E(x_2)), \\ \bigvee_{\alpha\in[h(E(x_2)),h(E(x_1))]} \left(\alpha \wedge N\left(\left[E(x_1), E(x_1) + (E(x_2) - E(x_1))\frac{\alpha - h(E(x_1))}{h(E(x_2)) - h(E(x_1))}\right]\right)\right), \\ \quad \textit{if } h(E(x_1)) > h(E(x_2)). \end{cases}$$

If we take

$$\mu_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) = E(x_1) + (E(x_2) - E(x_1))\frac{\alpha - h(E(x_1))}{h(E(x_2)) - h(E(x_1))}$$

in Theorem 2.6, then the Corollary 3.6 can be proved.

Corollary 3.7. *Let* $h : [x_1, x_2] \longrightarrow [y_1, y_2]$ *be a* (E, μ, κ)*-convex function. Then*

$$\int_{x_1}^{x_2} h dN \leqslant \begin{cases} \bigvee_{\alpha\in[h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge N\left(\left[E(x_1) + (E(x_2) - E(x_1))\log_{\frac{h(E(x_2))}{h(E(x_1))}} \frac{\alpha}{h(E(x_1))}, E(x_2)\right]\right)\right), \\ \quad \textit{if } h(E(x_1)) < h(E(x_2)), \\ h(E(x_1)) \wedge N(E(x_1), E(x_2)), \ \textit{if } h(E(x_1)) = h(E(x_2)), \\ \bigvee_{\alpha\in[h(E(x_2)),h(E(x_1))]} \left(\alpha \wedge N\left(\left[E(x_1), E(x_1) + (E(x_2) - E(x_1))\log_{\frac{h(E(x_2))}{h(E(x_1))}} \frac{\alpha}{h(E(x_1))}\right]\right)\right), \\ \quad \textit{if } h(E(x_1)) > h(E(x_2)). \end{cases}$$

If we take

$$\mu_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) = E(x_1) + (E(x_2) - E(x_1)) \log_{\frac{h(E(x_2))}{h(E(x_1))}} \frac{\alpha}{h(E(x_1))}$$

in Theorem 2.6, then the Corollary 3.7 can be proved.

Corollary 3.8. *Let* $h : [y_1, y_2] \longrightarrow [x_1, x_2]$ *be a* (E, κ, μ)*-convex function. Then*

$$\int_{x_1}^{x_2} h dN \leqslant \begin{cases} \bigvee_{\alpha\in[h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge N\left(\left[E(x_1) + (\frac{E(x_2)}{E(x_1)})^{\frac{\alpha-h(E(x_1))}{h(E(x_2))-h(E(x_1))}}, E(x_2)\right]\right)\right), \\ \quad \text{if } h(E(x_1)) < h(E(x_2)), \\ h(E(x_1)) \wedge N(E(x_1), E(x_2)), \text{ if } h(E(x_1)) = h(E(x_2)), \\ \bigvee_{\alpha\in[h(E(x_2)),h(E(x_1))]} \left(\alpha \wedge N\left(\left[E(x_1), E(x_1)(\frac{E(x_2)}{E(x_1)})^{\frac{\alpha-h(E(x_1))}{h(E(x_2))-h(E(x_1))}}\right]\right)\right), \\ \quad \text{if } h(E(x_1)) > h(E(x_2)). \end{cases}$$

If we take

$$\kappa_{E(x_1),E(x_2)}(\mu^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) = E(x_1)(\frac{E(x_2)}{E(x_1)})^{\frac{\alpha-h(E(x_1))}{h(E(x_2))-h(E(x_1))}}$$

in Theorem 2.6, then the Corollary 3.8 can be proved.

Corollary 3.9. *Let* $h : [y_1, y_2] \longrightarrow [y_1, y_2]$ *be a* (E, κ, κ)*-convex function. Then*

$$\int_{x_1}^{x_2} h dN \leqslant \begin{cases} \bigvee_{\alpha\in[h(E(x_1)),h(E(x_2))]} \left(\alpha \wedge N\left(\left[E(x_1)(\frac{E(x_2)}{E(x_1)})^{\log_{\frac{h(E(x_2))}{h(E(x_1))}} \frac{\alpha}{h(E(x_1))}}, E(x_2)\right]\right)\right), \\ \quad \text{if } h(E(x_1)) < h(E(x_2)), \\ h(E(x_1)) \wedge N(E(x_1), E(x_2)), \text{ if } h(E(x_1)) = h(E(x_2)), \\ \bigvee_{\alpha\in[h(E(x_2)),h(E(x_1))]} \left(\alpha \wedge N\left(\left[E(x_1), E(x_1)(\frac{E(x_2)}{E(x_1)})^{\log_{\frac{h(E(x_2))}{h(E(x_1))}} \frac{\alpha}{h(E(x_1))}}\right]\right)\right), \\ \quad \text{if } h(E(x_1)) > h(E(x_2)). \end{cases}$$

If we take

$$\kappa_{E(x_1),E(x_2)}(\kappa^{-1}_{h(E(x_1)),h(E(x_2))}(\alpha)) = E(x_1)(\frac{E(x_2)}{E(x_1)})^{\log_{\frac{h(E(x_2))}{h(E(x_1))}} \frac{\alpha}{h(E(x_1))}}$$

in Theorem 2.6, then Corollary 3.9 can be proven.

Example 3.10.

1. If $E : \mathbb{R} \longrightarrow \mathbb{R}$ is $E(x) = x^2$, then the function $h : [1, 3] \longrightarrow [0, +\infty]$, which is defined as $h(x) = \frac{1}{\ln^2(x+1)}$ is (E, μ, μ)-convex function.
2. If $E : \mathbb{R} \longrightarrow \mathbb{R}$ is $E(x) = x^x$, then the function $h : [1, 2] \longrightarrow [0, +\infty]$, which is defined as $h(x) = x$ is (E, μ, κ)-convex function.
3. If $E : \mathbb{R} \longrightarrow \mathbb{R}$ is $E(x) = x$, then the function $h : [\frac{\pi}{4}, \frac{\pi}{2}]$, which is defined as $h(x) = x^{\sin^2 x}$ is (E, κ, μ)-convex function.
4. If $E : \mathbb{R} \longrightarrow \mathbb{R}$ is $E(x) = 2x$, then the function $h : [1, 2] \longrightarrow [0, +\infty]$, which is defined as $h(x) = (\cosh(x))^{\frac{1}{4}}$ is (E, κ, κ)-convex function.

4. Conclusion

This paper introduces a new class of geodesic convex functions, namely (E, μ, κ)-convex function, and considers and generalizes the Hadamard inequality for (E, μ, κ)-convex function.

References

[1] S. Abbaszadeh, A. Ebadian, *Nonlinear integrals and Hadamard-type inequalities*, Soft Comput., **22** (2018), 2843–2849. 1

[2] S. Abbaszadeh, M. Oraki, A. Ebadian, *Hadamard integral inequality for the class of geodesic convex functions*, Appl. Math. Inf. Sci., **15** (2021) 539–546. 3, 2

[3] I. Abou-Tair, W. T. Sulaiman, *Inequalities via convex functions*, Int. J. Math. Math. Sci., **22** (1999), 543–546.

[4] X. S. Chen, *Some properties of semi-E-convex functions*, J. Math. Anal. Appl., **275** (2002), 251–262. 1

[5] D. I. Duca, L. Lupsa, *On the E-epigraph of an E-convex functions*, J. Optim. Theory Appl., **129** (2006), 341–348. 1

[6] C. Fulga, V. Preda, *Nonlinear programming with E-preinvex and local E-preinvex functions*, European J. Oper. Res., **192** (2009), 737–743. 1

[7] G. Giorgi, *A Simple Way to Prove the Characterization of Differentiable Quasiconvex Functions*, Appl. Math., **5** (2014), 12–26. 1

[8] D. Goldfarb, S. Q. Ma, K. Scheinberg, *Fast alternating linearization methods for minimizing the sum of two convex functions*, Math. Program., **141** (2013), 349–382. 1

[9] A. Iqbal, I. Ahmad, S. Ali, *Some properties of geodesic semi-E-convex functions*, Nonlinear Anal., **74** (2011), 6805–6813. 1

[10] A. Iqbal, S. Ali, I. Ahmad, *On geodesic E-convex sets, geodesic E-convex functions and E-epigraphs*, J. Optim. Theory Appl., **155** (2012), 239–251. 1

[11] A. Kiliçman, W. Saleh, *On Geodesic Strongly E-convex Sets and Geodesic Strongly E-convex Functions*, J. Inequal. Appl., **2015** (2015), 10 pages. 1

[12] A. Kiliçman, W. Saleh, *On geodesic semi strongly E-convex functions*, J. Interdiscip. Math., **19** (2016), 1039–1055. 1

[13] A. Kiliçman, W. Saleh, *On properties of geodesic semilocal E-preinvex functions*, J. Inequal. Appl., **2018** (2018), 13 pages. 1

[14] M. A. Noor, *Fuzzy preinvex functions*, Fuzzy sets and systems, **64** (1994), 95–104.

[15] M. A. Noor, M. U. Awan, K. I. Noor, *On some inequalities for relative semi-convex functions*, J. Inequal. Appl., **2013** (2013), 16 pages. 1

[16] M. A. Noor, K. I. Noor , M. U. Awan, *Generalized convexity and integral inequalities*, Appl. Math. Inf. Sci., **9** (2015), 233–243. 1

[17] E. Pap, *Null-additive set functions*, Springer, Berlin, (1995) 2.5, 2

[18] J. E. Peačarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, Boston, (1992). 1

[19] D. Ralescu, G. Adams, *The fuzzy integral*, J. Math. Anal. Appl., **75** (1980), 562–570. 2.4

[20] T. Rapcsak, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, Kluwer Academic Publishers, Dordrecht, (1997). 1

[21] W. Saleh, *Some Properties of Geodesic Strongly Eb-vex Functions*, Int. J. Anal. Appl., **17** (2019), 388–395. 1

[22] M. Sugeno, *Theory of fuzzy integrals and its applications*, Doctoral Thesis, Tokyo Institute of technology, (1974). 1, 2.5

[23] Y.-R. Syau, E. S. Lee, *Some properties of E-convex functions*, Appl. Math. Lett., **18** (2005), 1074–1080. 1

[24] C. Udrist, *Convex Funcions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, (1994). 1, 2, 2.1, 2.3

[25] H. Vosoughian, *Hadamard integral inequality for the class of semi-harmonically convex functions*, Asian-Eur. J. Math., **15** (2022), 21 pages. 1

[26] Z. Y. Wang, G. J. Klir, *Fuzzy measure theory*, Plenum Press, New York, (1992). 2

[27] X. M. Yang, *On E-convex set, E-convex functions and E-convex programming*, J. Optim. Theory Appl., **109** (2001), 699–703. 1

[28] E. A. Youness, *E-convex sets, E-convex functions, and E-convex programming*, J. Optim. Theory Appl., **102** (1999), 439–450. 1

[29] E. A. Youness, T. Emam, *Strongly E-convex sets and strongly E-convex functions*, J. Interdiscip. Math., **8** (2005), 107–117. 1

[30] E. A. Youness, T. Emam, *Semi strongly E-convex functions*, J. Math. Stat., **1** (2005), 51–57. 1

[31] E. A. Youness, T. Emam, *Characterization of efficient solutions for multi-objective optimization problems involving semi-strong and generalized semi-strong E-convexity*, Acta Math. Sci. Ser. B (Engl. Ed.), **28** (2008), 7–16.