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A different approach for behavior of fractional plant virus model

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Abstract

In the last few decades many authors pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real problems. It has been shown that fractional-order models are more adequate than previously used integer-order models. In this work, we aim to investigate of different features of the plant virus model with its fractional order equivalent. We present an application for reproduction number for these kind of epidemic models with next generation matrix method. Also, existence and uniqueness of solutions have been showed for this fractional order system. Finally we present some figures according to the given numerical scheme.

Keywords: Fractional differential equation, existence and uniqueness, numerical approximation.

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1. Introduction

Without doubt plants are very important in our life for many things. The plants are used in many areas suc as food, medicine, fiber for clothes and etc. However, plants are subject to virus diseases also similar with human disease. So how do plants get the virus? In plants the most common way that viruses are propagated by means of insects. So insects are very big problem for plants. We know that plants cannot protect themselves. But ıf there are many predators such as birds, bats and herbal remedies, plants can live long and grow healthy. In this paper we consider plant virus model which was modified a plant-virus propagation model with delays in paper [2, 6, 7, 10]. Our aim is to understand plant virus model with fractional order and predict the process of the disease.

In many applied problems which are considered in this area, the zero initial condition on the function $y(t)$ and its integer-order derivatives are used. There are three main reasons for this. The reasons are given as physical interpretation of fractional derivatives, difficulties with numerical approximation of initial conditions. For long years, fractional differentiation and integration operators are very preferred operators that have becoming a very popular in mathematical modelling. Researchers from all over

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the world have considered those operators with their appropriate kernels such as power law (Caputo and Riemann Liouville derivative), exponential law (Caputo-Fabrizio derivative) and Mittag-Leffler law (Atangana-Baleanu derivative) [3–5, 8, 9]. These different works put many effective theorems and applications. Liouville and Riemann gave birth to the well-known fractional integral, and then with its fractional derivative. Among these operators Caputo is the most preferred differential operator which is the convolution of the derivative of a function with power-law [9]. An application of the Laplace transform to this version leads to the normal initial condition. Because of this reason mentioned in last sentence, we have considered our plant virus model via Caputo fractional differential equations. But we noted that the model was considered with its classical version in paper [2] before. In this paper we aim at modelling the spread of plant virus disease. Several cases are considered, conditions under which the unique system solution is obtained are presented in detail. Also we present numerical simulation for solution of the considered equation with this method. The numerical results show that this numerical approach is useful and accurate for obtaining numerical solutions of such equations.

1.1. Some important theorems and definitions of fractional calculus

There is a very close relationship between differential operators and integral operators. This relation is also suitable for fractional calculus. Authors who work with fractional order differentiation want to retain this relation in a suitably generalized sense. In this sense, in this section we give fundamental theorem of classical calculus and important definitions of fractional calculus [9].

Theorem 1.1 (Fundamental theorem of classical calculus)**.** *Let* $k : [t_1, t_2] \to R$ *be a continuous function, and let* $K : [t_1, t_2] \to R$ *be defined by*

$$K := \int_{t_1}^{x} k(t)dt.$$

Then, K *is differentiable and*

$$K' = k.$$

Definition 1.2. Let $\xi > 0$ of a function $k : (0, \infty) \to R$, so Riemann-Liouville fractional integral is given by

$$I_t^{\xi} k(t) = \frac{1}{(\xi - 1)!} \int_0^t (t - x)^{\xi - 1} k(x)dx, \quad x > 0.$$

Definition 1.3. Let $\xi > 0$ of a function $k : (0, \infty) \to R$ and the Riemann-Liouville derivative is given as

$$D_t^{\xi} k(t) = \frac{d}{dt} D_t^{\xi - 1} k(t) = \frac{1}{\Gamma(1 - \xi)} \frac{d}{dt} \int_0^t (t - x)^{-\xi} k(x)dx, \quad 0 < \xi \leqslant 1.$$

Here and following definitions $\Gamma(.)$ is Gamma function and we recall the definition as below:

$$\Gamma(\xi) = \int_0^{\infty} t^{\xi - 1} e^{-t} dt, \quad \text{for } \xi > 0.$$

Definition 1.4. Let $\xi > 0$ of a function $k : (0, \infty) \to R$, the Caputo fractional derivative is given as

$$^{C}D_t^{\xi} k(t) = \frac{1}{\Gamma(1 - \xi)} \int_0^t (t - x)^{-\xi} \frac{d}{dx} k(x)dx, \quad 0 < \xi \leqslant 1.$$

Now we give property for Riemann-Liouville derivative.

Property 1.5. If $k(t)$ is defined in the interval $[t_1, t_2]$ and

$$\frac{1}{\Gamma(\xi)} \int_{t_1}^{t} (t-x)^{\xi-1} k(x) dx = 0,$$

for $\xi > 0$ and for all $t \in [t_1, t_2]$, then

$$k(t) \equiv 0.$$

2. Plant virus model

$$\begin{aligned}
\frac{ds(t)}{dt} &= \mu(k-s) + d\imath - \frac{\beta y}{1+\alpha y} s,\\
\frac{d\imath(t)}{dt} &= \frac{\beta y}{1+\alpha y} s - (d+\mu+\gamma)\,\imath,\\
\frac{dx(t)}{dt} &= \Lambda - \frac{\beta_1 \imath}{1+\alpha_1 \imath} x - \frac{c_1 x}{1+\alpha_3 x} p - mx,\\
\frac{dy(t)}{dt} &= \frac{\beta_1 \imath}{1+\alpha_1 \imath} x - \frac{c_2 y}{1+\alpha_3 y} p - my,\\
\frac{dp(t)}{dt} &= \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x} p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y} p - \delta p.
\end{aligned}$$

The initial conditions are taken as follows:

$$s(0) = s_0,\ \imath(0) = \imath_0,\ x(0) = x_0,\ y(0) = y_0,\ p(0) = p_0.$$

The meanings of the parameters of model considered in this paper was given in paper [2].

2.1. *Positivity and boundness of the solutions*

In this part, we study the positivity of solutions of plant virus model. Here we derive necessary and sufficient conditions such that the solutions have positivity. Our aim is show that solutions have qualitative behavior. So we assume that for any initial conditions for model $s(0) = s_0$, $\imath(0) = \imath_0$, $x(0) = x_0$, $y(0) = y_0$, $p(0) = p_0 \in A^+$ so there exists a unique solution and for $t > 0$ and the solutions satisfies $L^2(0, \infty)$. Let us give definition for a positivity criterion below.

Definition 2.1. The positive cone which is defined in $L^2(0, R^m)$, is

$$A^+ := \left\{ x \in L^2(0, R^m) \left| x^i \geqslant 0,\ i = 1, 2, \ldots, m \right. \right\}.$$

If we can show positivity of solutions, we will state the maximal existence interval of the solutions by $[0, T]$.

Now let us start with $s(t)$ class. We define the following norm

$$\|\varphi\|_\infty = \sup_{t\in[0,T]} |\varphi|.$$

$\overset{\prime}{s}(t)$ is given by

$$\overset{\prime}{s}(t) = \mu(k-s) + d\imath - \frac{\beta y}{1+\alpha y} s.$$

So we have for the function $s(t)$,

$$
\begin{aligned}
s'(t) &= \mu(k-s) + d\imath - \frac{\beta y}{1+\alpha y}s, \ \forall t \geqslant 0\ ,\\
&\geqslant -\left(\mu + \frac{\beta y}{1+\alpha y}\right)s, \ \forall t \geqslant 0,\\
&\geqslant -\left(\mu + \frac{\beta \sup\limits_{t\in[0,T]} |y|}{1+\alpha \sup\limits_{t\in[0,T]} |y|}\right)s, \ \forall t \geqslant 0,\\
&\geqslant -\left(\mu + \frac{\beta \|y\|_\infty}{1+\alpha\|y\|_\infty}\right)s, \ \ \forall t \geqslant 0.
\end{aligned}
$$

Then we have

$$
s(t) \geqslant s_0 e^{-\left(\mu + \frac{\beta\|y\|_\infty}{1+\alpha\|y\|_\infty}\right)t}, \ \ \forall t \geqslant 0.
$$

$\imath'(t)$ is given by

$$
\imath'(t) = \frac{\beta y}{1+\alpha y}s - (d+\mu+\gamma)\imath, \ \ \forall t \geqslant 0 \geqslant -(d+\mu+\gamma)\imath, \ \ \forall t \geqslant 0.
$$

Then we have

$$
\imath(t) \geqslant \imath_0 e^{-(d+\mu+\gamma)t}, \ \ \forall t \geqslant 0.
$$

$x'(t)$ is given by

$$
\begin{aligned}
x'(t) &= \Lambda - \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_1 x}{1+\alpha_3 x}p - mx, \ \ \forall t \geqslant 0\ ,\\
&\geqslant -\left(\frac{\beta_1 \imath}{1+\alpha_1 \imath} + \frac{c_1 x}{1+\alpha_3 x} + m\right)x, \ \ \forall t \geqslant 0\ ,\\
&\geqslant -\left(\frac{\beta_1 \imath}{1+\alpha_1 \imath} + \frac{c_1}{1+\alpha_3} + m\right)x, \ \ \forall t \geqslant 0\ ,\\
&\geqslant -\left(\frac{\beta_1 |\imath|}{1+\alpha_1 |\imath|} + \frac{c_1}{1+\alpha_3} + m\right)x, \ \ \forall t \geqslant 0\ ,\\
&\geqslant -\left(\frac{\beta_1 \sup\limits_{t\in[0,T]} |\imath|}{1+\alpha_1 \sup\limits_{t\in[0,T]} |\imath|} + \frac{c_1}{1+\alpha_3} + m\right)x, \ \ \forall t \geqslant 0\ ,\\
&\geqslant -\left(\frac{\beta_1 \|\imath\|_\infty}{1+\alpha_1 \|\imath\|_\infty} + \frac{c_1}{1+\alpha_3} + m\right)x, \ \ \forall t \geqslant 0.
\end{aligned}
$$

Then we have

$$
x(t) \geqslant x_0 e^{-\left(\frac{\beta_1\|\imath\|_\infty}{1+\alpha_1\|\imath\|_\infty} + \frac{c_1}{1+\alpha_3} + m\right)t}, \ \ \forall t \geqslant 0.
$$

$y'(t)$ is given by

$$
\begin{aligned}
y'(t) &= \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_2 y}{1+\alpha_3 y}p - my, \ \ \forall t \geqslant 0,\\
&\geqslant -\left(\frac{c_2}{1+\alpha_3 y}p + m\right)y, \ \ \forall t \geqslant 0,\\
&\geqslant -\left(\frac{c_2}{1+\alpha_3}p + m\right)y, \ \ \forall t \geqslant 0,
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant -\left(\frac{c_2}{1+\alpha_3}|p|+m\right)y, \quad \forall t \geqslant 0,\\
&\geqslant -\left(\frac{c_2}{1+\alpha_3}\sup_{t\in[0,T]}|p|+m\right)y, \quad \forall t \geqslant 0,\\
&\geqslant -\left(\frac{c_2}{1+\alpha_3}\|p\|_\infty+m\right)y, \quad \forall t \geqslant 0.
\end{aligned}
$$

So we have

$$
y(t) \geqslant y_0 e^{-\left(\frac{c_2}{1+\alpha_3}\|p\|_\infty+m\right)t}, \quad \forall t \geqslant 0.
$$

Finally $p'(t)$ is given by

$$
\begin{aligned}
p'(t) &= \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x}p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y}p - \delta p, \quad \forall t \geqslant 0,\\
&\geqslant -\left(-\frac{\alpha_4 c_1 x}{1+\alpha_3 x} - \frac{\alpha_4 c_2 y}{1+\alpha_3 y} + \delta\right)p, \quad \forall t \geqslant 0,\\
&\geqslant -\left(-\frac{\alpha_4 c_1 |x|}{1+\alpha_3 |x|} - \frac{\alpha_4 c_2 |y|}{1+\alpha_3 |y|)|} + \delta\right)p, \quad \forall t \geqslant 0,\\
&\geqslant -\left(-\frac{\alpha_4 c_1 \sup_{t\in[0,T]}|x|}{1+\alpha_3 \sup_{t\in[0,T]}|x|} - \frac{\alpha_4 c_2 \sup_{t\in[0,T]}|y|}{1+\alpha_3 \sup_{t\in[0,T]}|y|} + \delta\right)p, \quad \forall t \geqslant 0,\\
&\geqslant -\left(-\frac{\alpha_4 c_1 \|x\|_\infty}{1+\alpha_3 \|x\|_\infty} - \frac{\alpha_4 c_2 \|y\|_\infty}{1+\alpha_3 \|y\|_\infty} + \delta\right)p, \quad \forall t \geqslant 0,\\
p'(t) &\geqslant e^{-\left(-\frac{\alpha_4 c_1 \|x\|_\infty}{1+\alpha_3 \|x\|_\infty} - \frac{\alpha_4 c_2 \|y\|_\infty}{1+\alpha_3 \|y\|_\infty} + \delta\right)t}, \quad \forall t \geqslant 0.
\end{aligned}
$$

So the virus model has positive solutions $\forall t \geqslant 0$.

3. Model analysis

R_0 is named as reproduction number which can help us to understand the virus finish completely or not. Virus-free equilibrium point is globally asymptotic stable if $R_0 \leqslant 1$ and virus equilibrium point is globally asymptotically stable if $R_0 > 1$. In this section we obtain the reproduction number using the next generation matrix technique [11]. It can be seen that the plant virus model has a virus free equilibrium point $E_0(k, 0, 0, 0, 0)$. Let us consider following system of differential equation consist with $\dot{\imath}(t)$,

$$
\begin{aligned}
\dot{\imath}(t) &= \frac{\beta y}{1+\alpha y}s - (d+\mu+\gamma)\imath,\\
\dot{y}(t) &= \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_2 y}{1+\alpha_3 y}p - my,\\
\dot{p}(t) &= \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x}p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y}p - \delta p.
\end{aligned}
$$

Then we obtain the following $\digamma$ and ϑ matrices:

$$
\digamma = \begin{pmatrix} \frac{\beta y}{1+\alpha y}s \\ \frac{\beta_1 \imath}{1+\alpha_1 \imath}x \\ \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x}p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y}p \end{pmatrix} \quad \text{and} \quad \vartheta = \begin{pmatrix} (d+\mu+\gamma)\imath \\ \frac{c_2 y}{1+\alpha_3 y}p + my \\ \delta p \end{pmatrix}.
$$

Now let us get F and ν of Jacobian of $\mathcal{F}$ and ϑ matrices. Then

$$F = \begin{pmatrix} 0 & \frac{\beta s}{(1+\alpha y)^2} & 0 \\ \frac{\beta_1 x}{(1+\alpha_1 \imath)^2} & 0 & 0 \\ 0 & \frac{\alpha_4 c_2 p}{(1+\alpha_3 y)^2} & \frac{\alpha_4 c_1 x}{1+\alpha_3 x} + \frac{\alpha_4 c_2 y}{1+\alpha_3 y} \end{pmatrix}$$

and

$$\nu = \begin{pmatrix} (d+\mu+\gamma) & 0 & 0 \\ 0 & \frac{c_2 p}{(1+\alpha_3 y)^2} + m & \frac{c_2 y}{1+\alpha_3 y} \\ 0 & 0 & \delta \end{pmatrix}.$$

So, next generation matrix for the model is

$$N_x = F.\nu^{-1} = \begin{pmatrix} 0 & \frac{\beta_1 x}{(1+\alpha_1 \imath)^2 (d+\mu+\gamma)} & 0 \\ 0 & \frac{(1+\alpha_3 y)^2}{c_2 p + m(1+\alpha_3 y)^2} & \frac{-c_2 y (1+\alpha_3 y)}{\delta\left(c_2 p + m (1+\alpha_3 y)^2\right)} \\ 0 & 0 & \frac{1}{\delta} \end{pmatrix}.$$

Finally, the spectral radius R_0 of the matrix N_x is the basic reproduction number of the model given by

$$R_0 = \frac{(1+\alpha_3 y)^2}{c_2 p + m\,(1+\alpha_3 y)^2}.$$

3.1. Global stability of the virus equilibrium point

In this section we investigate the the global stability of the virus equilibrium point of the plant virus model. Now let us consider model again with its virus equilibrium point $E^*(s^*, \imath^*, x^*, y^*, p^*)$.

$$\begin{aligned}
\dot{s(t)} &= \mu(k-s) + d\imath - \frac{\beta y}{1+\alpha y} s, \\
\dot{\imath(t)} &= \frac{\beta y}{1+\alpha y} s - (d+\mu+\gamma)\,\imath, \\
\dot{x(t)} &= \Lambda - \frac{\beta_1 \imath}{1+\alpha_1 \imath} x - \frac{c_1 x}{1+\alpha_3 x} p - m x, \\
\dot{y(t)} &= \frac{\beta_1 \imath}{1+\alpha_1 \imath} x - \frac{c_2 y}{1+\alpha_3 y} p - m y, \\
\dot{p(t)} &= \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x} p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y} p - \delta p.
\end{aligned}$$

Theorem 3.1. *If $R_0 \geqslant 1$, the point $E^*(s^*, \imath^*, x^*, y^*, p^*)$ is globally asymptotically stable.*

Proof. The proof of this theorem is done by using the idea of Lyapunov function. We put the Lyapunov function associated the system as below:

$$\begin{aligned}
V(E^*(s^*, \imath^*, x^*, y^*, p^*)) = &\left(s - s^* + s^* \log \frac{s^*}{s}\right) + \left(\imath - \imath^* + \imath^* \log \frac{\imath^*}{\imath}\right) \\
&+ \left(x - x^* + x^* \log \frac{x^*}{x}\right) + \left(y - y^* + y^* \log \frac{y^*}{y}\right) + \left(p - p^* + p^* \log \frac{p^*}{p}\right).
\end{aligned}$$

Taking the derivative of Lyapunov function for t, we get

$$\dot{V(t)} = \left(\frac{s-s^*}{s}\right)\dot{s} + \left(\frac{\imath-\imath^*}{\imath}\right)\dot{\imath} + \left(\frac{x-x^*}{x}\right)\dot{x} + \left(\frac{y-y^*}{y}\right)\dot{y} + \left(\frac{p-p^*}{p}\right)\dot{p}.$$

Taking into accont the values in above equation for derivatives

$$\begin{aligned}\dot{V(t)} = &\left(\frac{s-s^*}{s}\right)\left(\mu(k-s)+d\imath-\frac{\beta y}{1+\alpha y}s\right)+\left(\frac{\imath-\imath^*}{\imath}\right)\left(\frac{\beta y}{1+\alpha y}s-(d+\mu+\gamma)\imath\right)\\ &+\left(\frac{x-x^*}{x}\right)\left(\Lambda-\frac{\beta_1\imath}{1+\alpha_1\imath}x-\frac{c_1x}{1+\alpha_3x}p-mx\right)\\ &+\left(\frac{y-y^*}{y}\right)\left(\frac{\beta_1\imath}{1+\alpha_1\imath}x-\frac{c_2y}{1+\alpha_3y}p-my\right)+\left(\frac{p-p^*}{p}\right)\left(\Lambda_p+\frac{\alpha_4c_1x}{1+\alpha_3x}p+\frac{\alpha_4c_2y}{1+\alpha_3y}p-\delta p\right).\end{aligned}$$

Then we write

$$\begin{aligned}\dot{V(t)} =&\mu(k-s)+d\imath-\frac{\beta y}{1+\alpha y}s-\frac{s^*}{s}\mu(k-s)-\frac{s^*}{s}d\imath+\frac{s^*}{s}\frac{\beta y}{1+\alpha y}s\\ &+\frac{\beta y}{1+\alpha y}s-(d+\mu+\gamma)\imath-\frac{\imath^*}{\imath}\frac{\beta y}{1+\alpha y}s+\frac{\imath^*}{\imath}(d+\mu+\gamma)\imath+\Lambda-\frac{\beta_1\imath}{1+\alpha_1\imath}x\\ &-\frac{c_1x}{1+\alpha_3x}p-mx-\frac{x^*}{x}\Lambda+\frac{x^*}{x}\frac{\beta_1\imath}{1+\alpha_1\imath}x+\frac{x^*}{x}\frac{c_1x}{1+\alpha_3x}p+\frac{x^*}{x}mx\\ &+\frac{\beta_1\imath}{1+\alpha_1\imath}x-\frac{c_2y}{1+\alpha_3y}p-my-\frac{y^*}{y}\frac{\beta_1\imath}{1+\alpha_1\imath}x+\frac{y^*}{y}\frac{c_2y}{1+\alpha_3y}p\\ &+\frac{y^*}{y}my+\Lambda_p+\frac{\alpha_4c_1x}{1+\alpha_3x}p+\frac{\alpha_4c_2y}{1+\alpha_3y}p-\delta p-\frac{p^*}{p}\Lambda_p\\ &-\frac{p^*}{p}\frac{\alpha_4c_1x}{1+\alpha_3x}p-\frac{p^*}{p}\frac{\alpha_4c_2y}{1+\alpha_3y}p+\frac{p^*}{p}\delta p.\end{aligned}$$

Let us write above also as

$$\dot{V(t)} = V_1 - V_2,$$

here

$$\begin{aligned}V_1 =&\mu k+d\imath+\frac{s^*\beta y}{1+\alpha y}+s^*\mu+\frac{s^*\beta y}{1+\alpha y}+\frac{\beta y}{1+\alpha y}s+\imath^*(d+\mu+\gamma)\\ &+\Lambda+\frac{x^*\beta_1\imath}{1+\alpha_1\imath}+\frac{x^*c_1p}{1+\alpha_3x}+x^*m+\frac{\beta_1\imath}{1+\alpha_1\imath}x+\frac{y^*c_2p}{1+\alpha_3y}+y^*m+\Lambda_p+\frac{\alpha_4c_1x}{1+\alpha_3x}p+\frac{\alpha_4c_2y}{1+\alpha_3y}p+p^*\delta\end{aligned}$$

and

$$\begin{aligned}V_2 =&\mu s+\frac{\beta y}{1+\alpha y}s+\frac{s^*}{s}\mu k+\frac{s^*}{s}d\imath+(d+\mu+\gamma)\imath+\frac{\imath^*}{\imath}\frac{\beta y}{1+\alpha y}s\\ &+\frac{\beta_1\imath}{1+\alpha_1\imath}x+\frac{c_1x}{1+\alpha_3x}p+mx+\frac{x^*\Lambda}{x}+\frac{c_2yp}{1+\alpha_3y}+my+\frac{y^*}{y}\frac{\beta_1\imath x}{1+\alpha_1\imath}+\delta p+\frac{p^*\Lambda_p}{p}+\frac{p^*\alpha_4c_1x}{1+\alpha_3x}+\frac{p^*\alpha_4c_2y}{1+\alpha_3y}.\end{aligned}$$

Therefore if

$$V_1 - V_2 > 0, \text{ then } \dot{V(t)} > 0;$$
$$V_1 - V_2 = 0, \text{ then } \dot{V(t)} = 0;$$
$$V_1 - V_2 < 0, \text{ then } \dot{V(t)} < 0.$$

□

4. Existence and uniqueness theorem as a method of solution

Remembering that our model given by

$$\frac{ds(t)}{dt} = \mu(k-s)+d\imath-\frac{\beta y}{1+\alpha y}s,$$

$$\begin{aligned}
\frac{d\imath(t)}{dt} &= \frac{\beta y}{1+\alpha y}s - (d+\mu+\gamma)\,\imath,\\
\frac{dx(t)}{dt} &= \Lambda - \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_1 x}{1+\alpha_3 x}p - mx,\\
\frac{dy(t)}{dt} &= \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_2 y}{1+\alpha_3 y}p - my,\\
\frac{dp(t)}{dt} &= \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x}p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y}p - \delta p.
\end{aligned}$$

But in this section, we consider model as below

$$\frac{ds(t)}{dt} = P_1(t,s), \qquad \frac{d\imath(t)}{dt} = P_2(t,\imath), \qquad \frac{dx(t)}{dt} = P_3(t,x), \qquad \frac{dy(t)}{dt} = P_4(t,y), \qquad \frac{dp(t)}{dt} = P_5(t,p).$$

Here we take

$$\begin{aligned}
P_1(t,s) &= \mu(k-s) + d\imath - \frac{\beta y}{1+\alpha y}s,\\
P_2(t,\imath) &= \frac{\beta y}{1+\alpha y}s - (d+\mu+\gamma)\,\imath,\\
P_3(t,x) &= \Lambda - \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_1 x}{1+\alpha_3 x}p - mx,\\
P_4(t,y) &= \frac{\beta_1 \imath}{1+\alpha_1 \imath}x - \frac{c_2 y}{1+\alpha_3 y}p - my,\\
P_5(t,p) &= \Lambda_p + \frac{\alpha_4 c_1 x}{1+\alpha_3 x}p + \frac{\alpha_4 c_2 y}{1+\alpha_3 y}p - \delta p.
\end{aligned}$$

Theorem 4.1. *Assume that there are ten positive constants* j_1, j_2, j_3, j_4, j_5 *and* $\bar{j}_1, \bar{j}_2, \bar{j}_3, \bar{j}_4, \bar{j}_5$, *then the following hold*

i)

$$\begin{aligned}
|P_1(t,s) - P_1(t,s_1)|^2 &\leqslant j_1 |s-s_1|^2,\\
|P_2(t,\imath) - P_2(t,\imath_1)|^2 &\leqslant j_2 |\imath-\imath_1|^2,\\
|P_3(t,x) - P_3(t,x_1)|^2 &\leqslant j_3 |x-x_1|^2,\\
|P_4(t,y) - P_4(t,y_1)|^2 &\leqslant j_4 |y-y_1|^2,\\
|P_5(t,p) - P_5(t,p_1)|^2 &\leqslant j_5 |p-p_1|^2.
\end{aligned}$$

ii)

$$\begin{aligned}
|P_1(t,s)|^2 &\leqslant \bar{j}_1(1+|s|^2),\\
|P_2(t,\imath)|^2 &\leqslant \bar{j}_2(1+|\imath|^2),\\
|P_3(t,x)|^2 &\leqslant \bar{j}_3(1+|x|^2),\\
|P_4(t,y)|^2 &\leqslant \bar{j}_4(1+|y|^2),\\
|P_5(t,p)|^2 &\leqslant \bar{j}_5(1+|p|^2).
\end{aligned}$$

If above conditions are verified, then solution of system exists and is unique. We start with first equation of system $P_1(t,s)$. Then we will try to verify first condition for equation like below

$$|P_1(t,s) - P_1(t,s_1)|^2 \leqslant j_1 |s-s_1|^2.$$

Now we define the following norm

$$\|\varphi\|_\infty^2 = \sup_{t\in[0,T]} |\varphi|^2.$$

Let us consider $s, s_1 \in R^2$ and $t \in [0,T]$,

$$\begin{aligned}
|P_1(t,s) - P_1(t,s_1)|^2 &= \left|\left(-\mu - \frac{\beta y}{1+\alpha y}\right)(s-s_1)\right|^2 \\
&\leqslant \left\{2\mu^2 + \frac{2\beta^2 |y|^2}{1+\alpha |y|^2}\right\} |s-s_1|^2 \\
&\leqslant \left\{2\mu^2 + \frac{2\beta^2 \sup\limits_{t\in[0,T]} |y|^2}{1+\alpha \sup\limits_{t\in[0,T]} |y|^2}\right\} |s-s_1|^2 \\
&\leqslant \left\{2\mu^2 + \frac{2\beta^2 \|y\|_\infty^2}{1+\alpha \|y\|_\infty^2}\right\} |s-s_1|^2 \\
&\leqslant j_1 |s-s_1|^2,
\end{aligned}$$

where

$$j_1 = \left\{2\mu^2 + \frac{2\beta^2 \|y\|_\infty^2}{1+\alpha \|y\|_\infty^2}\right\}.$$

For $\imath, \imath_1 \in R^2$ and $t \in [0,T]$

$$|P_2(t,\imath) - P_2(t,\imath_1)|^2 = |-(d+\mu+\gamma)(\imath-\imath_1)|^2 \leqslant \left\{\left(3d^2+3\mu^2+3\gamma^2\right)\right\} |\imath-\imath_1|^2 \leqslant j_2 |\imath-\imath_1|^2,$$

where

$$j_2 = \left\{3d^2+3\mu^2+3\gamma^2\right\}.$$

For $x, x_1 \in R^2$ and $t \in [0,T]$,

$$\begin{aligned}
|P_3(t,x) - P_3(t,x_1)|^2 &= \left|\left(-\frac{\beta_1 \imath}{1+\alpha_1 \imath} - \frac{c_1 p}{1+\alpha_3 x} - m\right)(x-x_1)\right|^2 \\
&\leqslant \left|\left(\frac{\beta_1 \imath}{1+\alpha_1 \imath} + \frac{c_1 p}{1+\alpha_3} + m\right)(x-x_1)\right|^2 \\
&\leqslant \left|\left\{3\frac{\beta_1^2 |\imath|^2}{1+\alpha_1^2 |\imath|^2} + 3\frac{c_1^2 |p|^2}{1+\alpha_3^2} + 3m^2\right\}(x-x_1)\right|^2 \\
&\leqslant \left\{3\frac{\beta_1^2 \sup\limits_{t\in[0,T]} |\imath|^2}{1+\alpha_1^2 \sup\limits_{t\in[0,T]} |\imath|^2} + 3\frac{c_1^2 \sup\limits_{t\in[0,T]} |p|^2}{(1+\alpha_3)^2} + 3m^2\right\} |x-x_1|^2 \\
&\leqslant \left\{\frac{3\beta_1^2 \|\imath\|_\infty^2}{1+\alpha_1^2 \|\imath\|_\infty^2} + \frac{3c_1^2 \|p\|_\infty^2}{(1+\alpha_3)^2} + 3m^2\right\} |x-x_1|^2 \\
&\leqslant j_3 |x-x_1|^2,
\end{aligned}$$

where

$$j_3 = \left\{\frac{3\beta_1^2 \|\imath\|_\infty^2}{1+\alpha_1^2 \|\imath\|_\infty^2} + \frac{3c_1^2 \|p\|_\infty^2}{(1+\alpha_3)^2} + 3m^2\right\}.$$

For $y, y_1 \in R^2$ and $t \in [0,T]$,

$$|P_4(t,y) - P_4(t,y_1)|^2 = \left|\left(-\frac{c_2 p}{1+\alpha_3 y} - m\right)(y-y_1)\right|^2$$

$$\leqslant \left|\left(-\frac{c_2 p}{1+\alpha_3}-m\right)(y-y_1)\right|^2$$
$$\leqslant \left\{2\frac{c_2^2|p|^2}{(1+\alpha_3)^2}+2m^2\right\}|y-y_1|^2 \leqslant \left\{2\frac{c_2^2\|p\|_\infty^2}{(1+\alpha_3)^2}+2m^2\right\}|y-y_1|^2 \leqslant j_4|y-y_1|^2,$$

where

$$j_4=\left\{2\frac{c_2^2\|p\|_\infty^2}{(1+\alpha_3)^2}+2m^2\right\}.$$

For $p, p_1 \in R^2$ and $t \in [0, T]$,

$$|P_5(t,p)-P_5(t,p_1)|^2 = \left|\left(\frac{\alpha_4 c_1 x}{1+\alpha_3 x}+\frac{\alpha_4 c_2 y}{1+\alpha_3 y}-\delta\right)(p-p_1)\right|^2$$
$$\leqslant \left|\left(\frac{\alpha_4 c_1 x}{1+\alpha_3 x}+\frac{\alpha_4 c_2 y}{1+\alpha_3 y}-\delta\right)(p-p_1)\right|^2$$
$$\leqslant \left\{3\frac{\alpha_4^2 c_1^2|x|^2}{1+\alpha_3|x|^2}+3\frac{\alpha_4^2 c_2^2|y|^2}{1+\alpha_3|y|^2}+3\delta^2\right\}|p-p_1|^2$$
$$\leqslant \left\{3\frac{\alpha_4^2 c_1^2\|x\|_\infty^2}{1+\alpha_3\|x\|_\infty^2}+3\frac{\alpha_4^2 c_2^2\|y\|_\infty^2}{1+\alpha_3\|y\|_\infty^2}+3\delta^2\right\}|p-p_1|^2$$
$$\leqslant j_5|p-p_1|^2,$$

where

$$j_5=\left\{3\frac{\alpha_4^2 c_1^2\|x\|_\infty^2}{1+\alpha_3\|x\|_\infty^2}+3\frac{\alpha_4^2 c_2^2\|y\|_\infty^2}{1+\alpha_3\|y\|_\infty^2}+3\delta^2\right\}.$$

Then we were able to provide the necessary condition (i).

Now we will proof the second condition for plant virus system by following: $\forall(t,s) \in R^2 \times [t_0,T]$, then we will show that

$$|P_1(t,s)|^2 = \left|\mu(k-s)+d\imath-\frac{\beta y}{1+\alpha y}s\right|^2$$
$$\leqslant 4(\mu k)^2+4\mu^2|s|^2+4d^2|\imath|^2+4\frac{\beta^2|y|^2}{(1+\alpha|y|)^2}|s|^2$$
$$\leqslant 4(\mu k)^2+4d^2|\imath|^2+\left(4\mu^2+4\frac{\beta^2|y|^2}{(1+\alpha|y|)^2}\right)|s|^2$$
$$\leqslant 4(\mu k)^2+4d^2\sup_{t\in[0,T]}|\imath|^2+\left(4\mu^2+4\frac{\beta^2\sup_{t\in[0,T]}|y|^2}{\left(1+\alpha\sup_{t\in[0,T]}|y|\right)^2}\right)|s|^2$$
$$\leqslant \left(4(\mu k)^2+4d^2\sup_{t\in[0,T]}|\imath|^2\right)\left(1+\left(\frac{4\mu^2+4\frac{\beta^2\sup_{t\in[0,T]}|y|^2}{\left(1+\alpha\sup_{t\in[0,T]}|y|\right)^2}}{4(\mu k)^2+4d^2\sup_{t\in[0,T]}|\imath|^2}\right)|s|^2\right)$$

$$\leqslant \left(4(\mu k)^2 + 4d^2 \|\imath\|_\infty^2\right)\left(1 + \left(\frac{4\mu^2 + 4\frac{\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)^2}}{4(\mu k)^2 + 4d^2\|\imath\|_\infty^2}\right)|s(t)|^2\right)$$

$$\leqslant \bar{j}_1(1+|s|^2),$$

where

$$\bar{j}_1 = 4(\mu k)^2 + 4d^2\|\imath\|_\infty^2$$

and with under condition

$$\frac{4\mu^2 + 4\frac{\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)^2}}{4(\mu k)^2 + 4d^2\|\imath\|_\infty^2} < 1.$$

Now we continue with second equation $\forall(t,\imath) \in R^2 \times [t_0, T]$, then we will show that

$$|P_2(t,\imath)|^2 = \left|\frac{\beta y}{1+\alpha y}s - (d+\mu+\gamma)\imath\right|^2$$

$$\leqslant \left(2\frac{\beta^2|y|^2}{(1+\alpha|y|)^2}|s|^2 + 2(d+\mu+\gamma)^2|\imath|^2\right)$$

$$\leqslant \left(2\frac{\beta^2 \sup\limits_{t\in[0,T]}|y|^2}{\left(1+\alpha\sup\limits_{t\in[0,T]}|y|\right)^2}\sup_{t\in[0,T]}|s|^2 + 2(d+\mu+\gamma)^2|\imath|^2\right)$$

$$\leqslant \left(2\frac{\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)}\|s\|_\infty^2 + 2(d+\mu+\gamma)^2|\imath|^2\right)$$

$$\leqslant \left(\frac{2\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)}\|s\|_\infty^2\right)\left(1 + \frac{(d+\mu+\gamma)^2}{\frac{\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)}\|s\|_\infty^2}|\imath|^2\right)$$

$$\leqslant \bar{j}_2(1+|\imath|^2),$$

where

$$\bar{j}_2 = \frac{2\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)}\|s\|_\infty^2,$$

and with under condition

$$\frac{(d+\mu+\gamma)^2}{\frac{\beta^2\|y\|_\infty^2}{\left(1+\alpha\|y\|_\infty^2\right)}\|s\|_\infty^2} < 1.$$

$\forall(t,x) \in R^2 \times [t_0, T]$, then we will show that

$$|P_3(t,x)|^2 = \left|\Lambda - \frac{\beta_1\imath}{1+\alpha_1\imath}x - \frac{c_1 x}{1+\alpha_3 x}p - mx\right|^2$$

$$\leqslant \left(4\Lambda^2 + \frac{4\beta_1^2|\imath|^2}{1+\alpha_1^2|\imath|^2}|x|^2 + \frac{4c_1^2|x|^2}{1+\alpha_3^2|x|^2}|p|^2 + 4m^2|x|^2\right)$$

$$\leqslant \left(4\Lambda^2 + \left(\frac{4\beta_1^2|\imath|^2}{1+\alpha_1^2|\imath|^2} + \frac{4c_1^2|p|^2}{1+\alpha_3^2} + 4m^2\right)|x|^2\right)$$

$$\leqslant \left(4\Lambda^2\left(1+\frac{\left(\frac{4\beta_1^2\sup\limits_{t\in[0,T]}|\imath|^2}{1+\alpha_1^2\sup\limits_{t\in[0,T]}|\imath|^2}+\frac{4c_1^2\sup\limits_{t\in[0,T]}|p|^2}{1+\alpha_3^2}+4m^2\right)}{4\Lambda^2}|x|^2\right)\right)$$

$$\leqslant \left(4\Lambda^2\left(1+\frac{\left(\frac{4\beta_1^2\|\imath\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}+\frac{4c_1^2\|p\|_\infty^2}{1+\alpha_3^2}+4m^2\right)}{4\Lambda^2}|x|^2\right)\right)$$

$$\leqslant \bar{j}_3(1+|x|^2),$$

where

$$\bar{j}_3 = 4\Lambda^2,$$

and with under condition

$$\frac{\frac{4\beta_1^2\|\imath\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}+\frac{4c_1^2\|p\|_\infty^2}{1+\alpha_3^2}+4m^2}{4\Lambda^2}<1.$$

$\forall (t,y)\in R^2\times[t_0,T]$, then we will show that

$$|P_4(t,y)|^2 = \left|\frac{\beta_1\imath}{1+\alpha_1\imath}x-\frac{c_2y}{1+\alpha_3y}p-my\right|^2$$

$$\leqslant \left(3\frac{\beta_1^2|\imath|^2|x|^2}{1+\alpha_1^2|\imath|^2}+3\frac{c_2^2|y|^2|p|^2}{1+\alpha_3^2|y|^2}+3m^2|y|^2\right)$$

$$\leqslant \left(3\frac{\beta_1^2|\imath|^2|x|^2}{1+\alpha_1^2|\imath|^2}+3\frac{c_2^2|y|^2|p|^2}{1+\alpha_3^2}+3m^2|y|^2\right)$$

$$\leqslant \left(3\frac{\beta_1^2|\imath|^2|x|^2}{1+\alpha_1^2|\imath|^2}+\left(3\frac{c_2^2|p|^2}{1+\alpha_3^2}+3m^2\right)|y|^2\right)$$

$$\leqslant \left(3\frac{\beta_1^2\|\imath\|_\infty^2\|x\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}+\left(3\frac{c_2^2\|p\|_\infty^2}{1+\alpha_3^2}+3m^2\right)|y|^2\right)$$

$$\leqslant \left(3\frac{\beta_1^2\|\imath\|_\infty^2\|x\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}\right)\left(1+\left(\frac{\frac{c_2^2\|p\|_\infty^2}{1+\alpha_3^2}+m^2}{\frac{\beta_1^2\|\imath\|_\infty^2\|x\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}}\right)|y|^2\right)$$

$$\leqslant \bar{j}_4(1+|y|^2),$$

where

$$\bar{j}_4 = \left(3\frac{\beta_1^2\|\imath\|_\infty^2\|x\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}\right),$$

and with under condition

$$\frac{\frac{c_2^2\|p\|_\infty^2}{1+\alpha_3^2}+m^2}{\frac{\beta_1^2\|\imath\|_\infty^2\|x\|_\infty^2}{1+\alpha_1^2\|\imath\|_\infty^2}}<1.$$

$\forall (t,p)\in R^2\times[t_0,T]$, then we will show that

$$|P_4(t,p)|^2 = \left|\Lambda_p+\frac{\alpha_4c_1x}{1+\alpha_3x}p+\frac{\alpha_4c_2y}{1+\alpha_3y}p-\delta p\right|^2$$

$$
\begin{aligned}
&\leqslant \left(4\Lambda_p^2 + 4\frac{\alpha_4^2 c_1^2 |x|^2}{1+\alpha_3^2 |x|^2} |p|^2 + 4\frac{\alpha_4^2 c_2^2 |y|^2}{1+\alpha_3^2 |y|^2} |p|^2 + 4\delta^2 |p|^2\right) \\
&\leqslant \left(4\Lambda_p^2 + 4\frac{\alpha_4^2 c_1^2 \|x\|_\infty^2}{1+\alpha_3^2 \|x\|_\infty^2} |p|^2 + 4\frac{\alpha_4^2 c_2^2 \|y\|_\infty^2}{1+\alpha_3^2 \|y\|_\infty^2} |p|^2 + 4\delta^2 |p|^2\right) \\
&\leqslant \left(4\Lambda_p^2 + \left(4\frac{\alpha_4^2 c_1^2 \|x\|_\infty^2}{1+\alpha_3^2 \|x\|_\infty^2} + 4\frac{\alpha_4^2 c_2^2 \|y\|_\infty^2}{1+\alpha_3^2 \|y\|_\infty^2} + 4\delta^2\right) |p|^2\right) \\
&\leqslant \left(4\Lambda_p^2 \left(1 + \frac{\frac{\alpha_4^2 c_1^2 \|x\|_\infty^2}{1+\alpha_3^2 \|x\|_\infty^2} + \frac{\alpha_4^2 c_2^2 \|y\|_\infty^2}{1+\alpha_3^2 \|y\|_\infty^2} + \delta^2}{\Lambda_p^2} |p|^2\right)\right) \\
&\leqslant \bar{j}_5(1+|p|^2),
\end{aligned}
$$

where

$$\bar{j}_5 = \left(4\Lambda_p^2\right),$$

and with under condition

$$\frac{\frac{\alpha_4^2 c_1^2 \|x\|_\infty^2}{1+\alpha_3^2 \|x\|_\infty^2} + \frac{\alpha_4^2 c_2^2 \|y\|_\infty^2}{1+\alpha_3^2 \|y\|_\infty^2} + \delta^2}{\Lambda_p^2} < 1.$$

4.1. Numerical scheme for plant virus model with Riemann-Liouville derivative

In this section, algorithms for effecting differintegration to order α will be devised and evaluated for Riemann-Liouville derivative. While putting numerical scheme, we will use Adam-Bashforth numerical scheme [1].

Now, let us write model with Riemann-Liouville derivative as following:

$$
\begin{aligned}
{}_0^{RL}D^\alpha s(t) &= \mu(k - s(t)) + d\imath(t) - \frac{\beta y(t)}{1+\alpha y(t)} s(t), \\
{}_0^{RL}D^\alpha \imath(t) &= \frac{\beta y(t)}{1+\alpha y(t)} s(t) - (d+\mu+\gamma)\,\imath(t), \\
{}_0^{RL}D^\alpha x(t) &= \Lambda - \frac{\beta_1 \imath(t)}{1+\alpha_1 \imath(t)} x(t) - \frac{c_1 x(t)}{1+\alpha_3 x(t)} p(t) - m x(t), \\
{}_0^{RL}D^\alpha y(t) &= \frac{\beta_1 \imath(t)}{1+\alpha_1 \imath(t)} x(t) - \frac{c_2 y(t)}{1+\alpha_3 y(t)} p(t) - m y(t), \\
{}_0^{RL}D^\alpha p(t) &= \Lambda_p + \frac{\alpha_4 c_1 x(t)}{1+\alpha_3 x(t)} p(t) + \frac{\alpha_4 c_2 y(t)}{1+\alpha_3 y(t)} p(t) - \delta p(t),
\end{aligned}
$$

with initial conditions are taken as follows:

$$s(0) = s_0,\ \imath(0) = \imath_0,\ x(0) = x_0,\ y(0) = y_0,\ p(0) = p_0.$$

After above let us consider right side of system as

$${}_0^{RL}D^\alpha s(t) = P_1(t,s),\ {}_0^{RL}D^\alpha \imath(t) = P_2(t,\imath),\ {}_0^{RL}D^\alpha x(t) = P_3(t,x),\ {}_0^{RL}D^\alpha y(t) = P_4(t,y),\ {}_0^{RL}D^\alpha p(t) = P_5(t,p).$$

Now by applying the Riemann-İntegral on system we get following

$$s(t) - s(0) = \frac{1}{\Gamma(\alpha)} \int_0^t P_1(\tau, s(\tau))(t-\tau)^{\alpha-1} d\tau,$$

$$\imath(t)-\imath(0)=\frac{1}{\Gamma(\alpha)}\int_0^t P_2(\tau,\imath(\tau))(t-\tau)^{\alpha-1}d\tau,$$

$$x(t)-x(0)=\frac{1}{\Gamma(\alpha)}\int_0^t P_3(\tau,x(\tau))(t-\tau)^{\alpha-1}d\tau,$$

$$y(t)-y(0)=\frac{1}{\Gamma(\alpha)}\int_0^t P_4(\tau,y(\tau))(t-\tau)^{\alpha-1}d\tau,$$

$$p(t)-p(0)=\frac{1}{\Gamma(\alpha)}\int_0^t P_5(\tau,p(\tau))(t-\tau)^{\alpha-1}d\tau,$$

thus at $t=t_{n+1}=(n+1)\Delta t$, we have the following:

$$s(t_{n+1})-s(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} P_1(t,s(t))(t_{n+1}-t)^{\alpha-1}dt,$$

$$\imath(t_{n+1})-\imath(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} P_2(t,\imath(t))(t_{n+1}-t)^{\alpha-1}dt,$$

$$x(t_{n+1})-x(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} P_3(t,x(t))(t_{n+1}-t)^{\alpha-1}dt,$$

$$y(t_{n+1})-y(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} P_4(t,y(t))(t_{n+1}-t)^{\alpha-1}dt,$$

$$p(t_{n+1})-p(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} P_5(t,p(t))(t_{n+1}-t)^{\alpha-1}dt,$$

and

$$s(t_n)-s(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_n} P_1(t,s(t))(t_n-t)^{\alpha-1}dt,$$

$$\imath(t_n)-\imath(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_n} P_2(t,\imath(t))(t_n-t)^{\alpha-1}dt,$$

$$x(t_n)-x(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_n} P_3(t,x(t))(t_n-t)^{\alpha-1}dt,$$

$$y(t_n)-y(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_n} P_4(t,y(t))(t_n-t)^{\alpha-1}dt,$$

$$p(t_n)-p(0)=\frac{1}{\Gamma(\alpha)}\int_0^{t_n} P_5(t,p(t))(t_n-t)^{\alpha-1}dt.$$

If we subtract from each other, we get

$$s(t_{n+1}) = s(t_n) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}P_1(t,s(t))dt - \frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-t)^{\alpha-1}P_1(t,s(t))dt,$$

$$\imath(t_{n+1}) = \imath(t_n) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}P_2(t,\imath(t))dt - \frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-t)^{\alpha-1}P_2(t,\imath(t))dt,$$

$$x(t_{n+1}) = x(t_n) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}P_3(t,x(t))dt - \frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-t)^{\alpha-1}P_3(t,x(t))dt,$$

$$y(t_{n+1}) = y(t_n) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}P_4(t,y(t))dt - \frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-t)^{\alpha-1}P_4(t,y(t))dt,$$

$$p(t_{n+1}) = p(t_n) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}P_5(t,p(t))dt - \frac{1}{\Gamma(\alpha)}\int_0^{t_n}(t_n-t)^{\alpha-1}P_5(t,p(t))dt.$$

After changing the functions $P_1(t,s(t)), P_2(t,\imath(t)), P_3(t,x(t)), P_4(t,y(t)), P_5(t,p(t))$ with their Lagrange polynomial counterpart respectively as below:

$$\begin{aligned}
P_1(t,s(t)) &\simeq \frac{t-t_{n-1}}{t_n-t_{n-1}}P_1(t_n,s(t_n)) + \frac{t-t_n}{t_{n-1}-t_n}P_1(t_{n-1},s(t_{n-1}))\\
&= \frac{P_1(t_n,s(t_n))}{h}(t-t_{n-1}) - \frac{P_1(t_{n-1},s(t_{n-1}))}{h}(t-t_n),\\
P_2(t,\imath(t)) &\simeq \frac{t-t_{n-1}}{t_n-t_{n-1}}P_1(t_n,\imath(t_n)) + \frac{t-t_n}{t_{n-1}-t_n}P_1(t_{n-1},\imath(t_{n-1}))\\
&= \frac{P_1(t_n,\imath(t_n))}{h}(t-t_{n-1}) - \frac{P_1(t_{n-1},\imath(t_{n-1}))}{h}(t-t_n),\\
P_3(t,x(t)) &\simeq \frac{t-t_{n-1}}{t_n-t_{n-1}}P_1(t_n,x(t_n)) + \frac{t-t_n}{t_{n-1}-t_n}P_1(t_{n-1},x(t_{n-1}))\\
&= \frac{P_1(t_n,x(t_n))}{h}(t-t_{n-1}) - \frac{P_1(t_{n-1},x(t_{n-1}))}{h}(t-t_n),\\
P_4(t,y(t)) &\simeq \frac{t-t_{n-1}}{t_n-t_{n-1}}P_1(t_n,y(t_n)) + \frac{t-t_n}{t_{n-1}-t_n}P_1(t_{n-1},y(t_{n-1}))\\
&= \frac{P_1(t_n,y(t_n))}{h}(t-t_{n-1}) - \frac{P_1(t_{n-1},y(t_{n-1}))}{h}(t-t_n),\\
P_5(t,p(t)) &\simeq \frac{t-t_{n-1}}{t_n-t_{n-1}}P_1(t_n,p(t_n)) + \frac{t-t_n}{t_{n-1}-t_n}P_1(t_{n-1},p(t_{n-1}))\\
&= \frac{P_1(t_n,p(t_n))}{h}(t-t_{n-1}) - \frac{P_1(t_{n-1},p(t_{n-1}))}{h}(t-t_n),
\end{aligned}$$

and then replacing polynomials with their values and doing necessary calculations, the above system can be written as

$$\begin{aligned}
s(t_{n+1}) &= s(t_n) + \frac{P_1(t_n,s(t_n))}{h\Gamma(\alpha)}\left\{\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha}t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha}\right\}\\
&\quad + \frac{P_1(t_{n-1},s(t_{n-1}))}{h\Gamma(\alpha)}\left\{\frac{h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1}\right\} + {}^{1}R_n^{\alpha},\\
\imath(t_{n+1}) &= \imath(t_n) + \frac{P_2(t_n,\imath(t_n))}{h\Gamma(\alpha)}\left\{\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{h}{\alpha}t_n^{\alpha} - \frac{t_n^{\alpha+1}}{\alpha}\right\}
\end{aligned}$$

$$+\frac{P_2(t_{n-1},\imath(t_{n-1}))}{h\Gamma(\alpha)}\left\{\frac{h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{t_n^{\alpha}}{\alpha+1}\right\}+{}^{2}R_n^{\alpha},$$

$$x(t_{n+1})=x(t_n)+\frac{P_3(t_n,x(t_n))}{h\Gamma(\alpha)}\left\{\frac{2h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{h}{\alpha}t_n^{\alpha}-\frac{t_n^{\alpha+1}}{\alpha}\right\}$$

$$+\frac{P_3(t_{n-1},x(t_{n-1}))}{h\Gamma(\alpha)}\left\{\frac{h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{t_n^{\alpha}}{\alpha+1}\right\}+{}^{3}R_n^{\alpha},$$

$$y(t_{n+1})=y(t_n)+\frac{P_4(t_n,y(t_n))}{h\Gamma(\alpha)}\left\{\frac{2h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{h}{\alpha}t_n^{\alpha}-\frac{t_n^{\alpha+1}}{\alpha}\right\}$$

$$+\frac{P_4(t_{n-1},y(t_{n-1}))}{h\Gamma(\alpha)}\left\{\frac{h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{t_n^{\alpha}}{\alpha+1}\right\}+{}^{4}R_n^{\alpha},$$

$$p(t_{n+1})=p(t_n)+\frac{P_5(t_n,p(t_n))}{h\Gamma(\alpha)}\left\{\frac{2h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{h}{\alpha}t_n^{\alpha}-\frac{t_n^{\alpha+1}}{\alpha}\right\}$$

$$+\frac{P_5(t_{n-1},p(t_{n-1}))}{h\Gamma(\alpha)}\left\{\frac{h}{\alpha}t_{n+1}^{\alpha}-\frac{t_{n+1}^{\alpha+1}}{\alpha+1}+\frac{t_n^{\alpha}}{\alpha+1}\right\}+{}^{5}R_n^{\alpha},$$

where

$${}^{1}R_n^{\alpha}<\frac{h^{3+\alpha}M_1}{12\Gamma(\alpha+1)}\left\{(n+1)^{\alpha}+n^2\right\},\qquad {}^{2}R_n^{\alpha}<\frac{h^{3+\alpha}M_2}{12\Gamma(\alpha+1)}\left\{(n+1)^{\alpha}+n^2\right\},$$

$${}^{3}R_n^{\alpha}<\frac{h^{3+\alpha}M_3}{12\Gamma(\alpha+1)}\left\{(n+1)^{\alpha}+n^2\right\},\qquad {}^{4}R_n^{\alpha}<\frac{h^{3+\alpha}M_4}{12\Gamma(\alpha+1)}\left\{(n+1)^{\alpha}+n^2\right\},$$

$${}^{5}R_n^{\alpha}<\frac{h^{3+\alpha}M_5}{12\Gamma(\alpha+1)}\left\{(n+1)^{\alpha}+n^2\right\}.$$

At the system above M_1, M_2, M_3, M_4, and M_5 are bound of functions respectively.

4.2. Numerical simulations

In this numerical part, we show numerical simulation of the given of fractional plant virus. We have made use of the model with the Riemann-Liouville derivative and the numerical scheme that was suggested by Adam-Bashforth where the Lagrange polynomial interpolation is used. Figures 1 and 2 give the solution of system with same results.

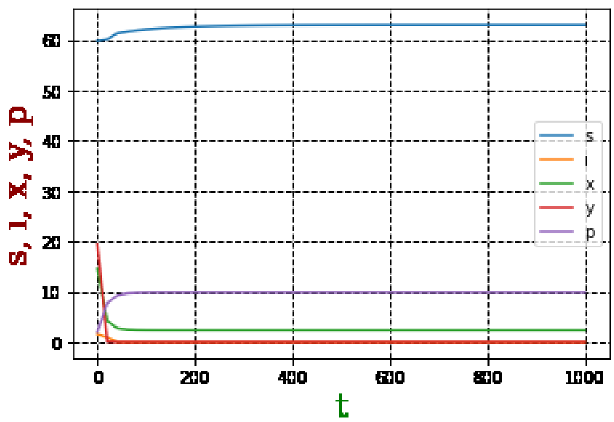

Figure 1: Numerical simulation of system.

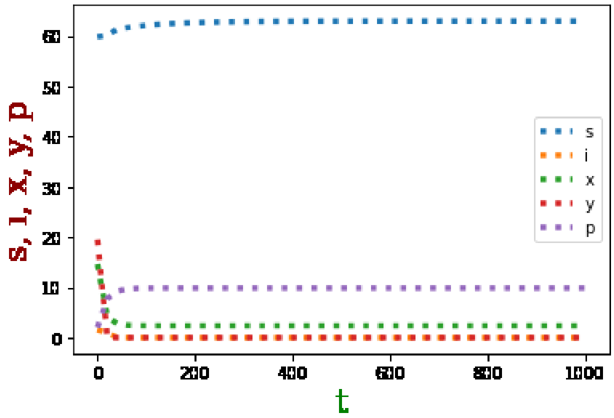

Figure 2: Numerical simulation of system.

5. Conclusion

In this work, we considered plant virus model with its fractional order counterpart. The model was considered to show global stability of equilibrium points. Also, using linear growth and Lipschitz rules, we obtained the conditions for the existence and the uniqueness of the system solutions . Finally, numerical simulations are given to showed effective of method.

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