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Perfect 2-Colorings of Johnson Graph $J(10, 3)$

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Abstract

A perfect 2-coloring of a graph G with a matrix $A = \{a_{ij}\}_{i,j=1,2}$ is a coloring of the vertices of G into the set of colors $\{1, 2\}$ such that the number of vertices of the color j adjacent with the fixed vertex x of the color i does not depend on a choice of the vertex x and equals to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring. We can consider perfect coloring as a generalization of the concept of completely regular codes presented by P. Delsarte for the first time. The parameter matrices of all perfect 2-colorings of the Johnson graph $J(10, 3)$ are listed in this paper.

Keywords: Perfect 2-colorings, Johnson graph $J(10, 3)$, parameter matrices.
2010 MSC: 05C15, 68R05, 51E05.

1. Introduction

We start with the essential definitions and concepts. Denote the collection of binary vectors of length n by E^n. Refer as the *weight* of a vector x in E^n to the number of nonzero coordinates of x. The vertex set of a *Johnson graph* $J(n, \omega)$ is defined as the collection of all vectors in E^n of weight ω; the set of edges of this graph consists of the pairs of vectors differing in exactly two coordinates. Showing that $J(n, \omega)$ is a regular graph of degree $w(n-\omega)$ is fairly straightforward [2].

By a *perfect* 2*-coloring* of vertices of G with a matrix $A = \{a_{ij}\}_{i,j=1,2}$, we mean a map T from the set of vertices V to the set of colors 1 and 2 (white and black) such that the color composition of the neighborhood of any vertex depends only on its color, and the number of vertices of color j adjacent to a fixed vertex of color i is a_{ij}. The matrix A is called the *parameter matrix* of a perfect coloring [2, 8].

A set of k-element subsets (called *blocks*) of an n-element set such that every t-element subset appears in λ blocks, is called a $t-(n, k, \lambda)-design$ [2].

θ is called an eigenvalue of a graph G whenever θ is an eigenvalue of the adjacency matrix of G [2]. In [4] it is shown that the eigenvalues of $J(n, \omega)$ are exhausted by

$$\theta_k = (\omega - k)(n - \omega - k), 0 \leqq k \leqq \omega.$$

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If $A = \{a_{ij}\}_{i,j=1,2}$ is a parameter matrix of perfect 2-colorings of the graph $J(n, \omega)$, then $a_{11} + a_{12} = a_{21} + a_{22} = \theta_0$ (i.e. $a_{11} + a_{12} = a_{21} + a_{22} = \omega(n-\omega)$), and $a_{11} - a_{21} = \theta_k, 1 \leqq k \leqq \omega$. We call $a_{11} - a_{21}$ henceforth the minimal eigenvalue of the perfect 2-coloring with parameter matrix A [1, 2].

In this paper we list all perfect 2-colorings of the Johnson graph $J(10, 3)$. The problem for $J(n, 2)$,$J(6, 3)$, $J(7, 3)$, $J(8, 3)$, $J(8, 4)$ and $J(n, 3)$ (n odd) has already been settled.

2. Preliminaries

As we frequently use two lemmas and two constructions in this paper, we present them here:

Lemma 2.1 ([2, Corollary 1])**.** *Suppose that a graph $J(n, \omega)$ admits a perfect 2-coloring with parameter matrix $A = \{a_{ij}\}_{i,j=1,2}$ whose minimal eigenvalue $a_{11} - a_{21}$ is the kth eigenvalue of $J(n, \omega)$. Then*

$$\frac{a_{21}}{a_{12} + a_{21}} . \begin{pmatrix} n-i \\ \omega - i + j \end{pmatrix}$$

is an integer for every i and j with $0 \leqslant j \leqslant i \leqslant k-1$.

Lemma 2.2 ([2, Proposition 3])**.** *A perfect 2-coloring of $J(n, \omega)$ having the parameter matrix $A = \{a_{ij}\}_{i,j=1,2}$ and eigenvalue $-\omega$ exists if and only if there is some $(\omega-1) - (n, \omega, \frac{a_{21}}{a_{12}+a_{21}}.(n-\omega+1)) - design$.*

Construction 2.3 ([2, Construction 1])**.** Take $j \in \{1, ..., n\}$. Color white all vectors of weight ω and length n whose jth coordinate is equal to 0, and color black all vectors whose jth coordinate is equal to 1.

This is a perfect 2-coloring of the vertices of $J(n, \omega)$ having parameter matrix

$$\begin{bmatrix} \omega(n-\omega-1) & \omega \\ n-\omega & (n-\omega)(\omega-1) \end{bmatrix}.$$

The minimal eigenvalue of this perfect 2-coloring is

$$(\omega-1)(n-\omega-1)-1;$$

thus, it is the first eigenvalue of $J(n, \omega)$. A. Meyerowitz [7] described constructively all completely regular designs of strength 0. In the terminology of his paper, completely regular designs of strength 0 and covering radius 1 are perfect 2-colorings of $J(n, \omega)$ with the minimal eigenvalue $(\omega-1)(n-\omega-1)-1$, and conversely.

Construction 2.4 ([2, Construction 3])**.** Consider the complete bipartite graph each part of which contains m vertices. Remove from it a perfect matching and denote the resulting graph by G. Enumerate the vertices of G in some way from 1 to $2m$. Divide the vertices of $J(2m, 3)$ into the orbits of the automorphism group of G. To this end, consider all three-vertex sets of vertices of G. Divide these sets into the following groups inside which $Aut(G)$ acts transitively:

(1) all three vertices belong to one part of G;

(2) two vertices belong to one part of G, while the third, to the other, and the latter is adjacent to both vertices of the first part;

(3) two vertices belong to one part of G, while the third, to the other, and the latter is adjacent only to one vertex of the two vertices the first part.

Therefore, the action of $Aut(G)$ divides the vertices $J(2m, 3)$ into three orbits. By [2, Theorem 5], every orbit coloring of a graph G is perfect. Considering G, it is not difficult to see that the parameter matrix $B_{3\times 3}$ of the orbit coloring of the vertices of $J(2m, 3)$ corresponding to $Aut(G)$ is

$$\begin{bmatrix} 3(m-3) & 3(m-2) & 6 \\ m-2 & 5(m-3)+2 & 6 \\ m-2 & 3(m-2) & 2m-1 \end{bmatrix}.$$

This matrix enjoys the following interesting property: in every column, the two off-diagonal entries are equal, which enables us to use [2, Lemma 1] to merge the corresponding colors. Therefore, we obtain three series of parameter matrices of perfect 2-colorings of $J(2m, 3)$ as follows:

$$\begin{bmatrix} 3(2m-5) & 6 \\ 4(m-2) & 2m-1 \end{bmatrix}, \begin{bmatrix} 3(m-3) & 3m \\ m-2 & 5m-7 \end{bmatrix}, \begin{bmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-13 \end{bmatrix}.$$

3. Main Result

Now we have the following theorem which is the main result of this paper.

Theorem 3.1. *The list*

$$\begin{bmatrix} 18 & 3 \\ 7 & 14 \end{bmatrix},$$

$$\begin{bmatrix} 6 & 15 \\ 3 & 18 \end{bmatrix}, \begin{bmatrix} 9 & 12 \\ 6 & 15 \end{bmatrix}, \begin{bmatrix} 12 & 9 \\ 9 & 12 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 18 \\ 6 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 12 \\ 12 & 9 \end{bmatrix}$$

exhausts the parameter matrices of perfect 2*-colorings of* $J(10, 3)$.

Proof. As mentioned before, the minimal eigenvalue of a perfect 2-coloring of the Johnson graph $J(10, 3)$ is equal to $\theta_k = (7-k)(3-k)-k, k = 1, 2, 3$. Therefore we have three cases:

Case 1. $k = 1$. In this case, $\theta_1 = 11$. By [7], there is only one perfect 2-coloring of $J(10, 3)$ with $a_{11} - a_{21} = \theta_1$. Using Construction 2.1 gives us the parameter matrix of it. It is

$$\begin{bmatrix} 18 & 3 \\ 7 & 14 \end{bmatrix}.$$

Case 2. $k = 2$. In this case, $\theta_2 = 3$. By [1] and knowing that $a_{11} + a_{12} = a_{21} + a_{22} = \omega(n-\omega)$ and up to renaming the colors, there are 9 potentially perfect 2-colorings with parameter matrices listed below:

$$\begin{bmatrix} 6 & 15 \\ 3 & 18 \end{bmatrix}, \begin{bmatrix} 9 & 12 \\ 6 & 15 \end{bmatrix}, \begin{bmatrix} 12 & 9 \\ 9 & 12 \end{bmatrix}, \tag{3.1}$$

$$\begin{bmatrix} 4 & 17 \\ 1 & 20 \end{bmatrix}, \begin{bmatrix} 5 & 16 \\ 2 & 19 \end{bmatrix}, \begin{bmatrix} 7 & 14 \\ 4 & 17 \end{bmatrix}, \begin{bmatrix} 8 & 13 \\ 5 & 16 \end{bmatrix}, \begin{bmatrix} 10 & 11 \\ 7 & 14 \end{bmatrix}, \begin{bmatrix} 11 & 10 \\ 8 & 13 \end{bmatrix}. \tag{3.2}$$

By taking $k = 2$ and $j = i = 1$ in Lemma 2.1, we conclude that

$$\frac{a_{21}}{a_{12} + a_{21}} \binom{n-i}{\omega - i + j} = \frac{a_{21}}{18} \binom{9}{3} = \frac{14a_{21}}{3}$$

must be integer (in all matrices of the above lists we have $a_{12} + a_{21} = 18$). Therefore, the matrices on the list (3.2) are not acceptable. We use Construction 2.4. By taking $m = 5$ we get the matrices on the list (3.1) are three proper parameter matrices.

Note 1. Fortunately, we found three fitted constructions for three matrices on the list (3.1) (Construction 2.4), see [5].

Note 2. The existence of a perfect 2-coloring with parameter matrix $\begin{bmatrix} 12 & 9 \\ 9 & 12 \end{bmatrix}$ (obtained by

$$\begin{bmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-4 \end{bmatrix}$$

in Construction 2.4, by taking $m = 5$), was left as an open case in [6].

Case 3. $k = 3$. In this case, $\theta_3 = -3$. As well as Case 2 and applying [1] and knowing that $a_{11} + a_{12} = a_{21} + a_{22} = \omega(n-\omega)$, up to renaming the colors, we get 10 potentially perfect 2-colorings with parameter matrices listed below:

$$\begin{bmatrix} 3 & 18 \\ 6 & 15 \end{bmatrix}, \begin{bmatrix} 9 & 12 \\ 12 & 9 \end{bmatrix}, \tag{3.3}$$

$$\begin{bmatrix} 0 & 21 \\ 3 & 18 \end{bmatrix}, \begin{bmatrix} 1 & 20 \\ 4 & 17 \end{bmatrix}, \begin{bmatrix} 2 & 19 \\ 5 & 16 \end{bmatrix}, \begin{bmatrix} 4 & 17 \\ 7 & 14 \end{bmatrix},$$
$$\begin{bmatrix} 5 & 16 \\ 8 & 13 \end{bmatrix}, \begin{bmatrix} 6 & 15 \\ 9 & 12 \end{bmatrix}, \begin{bmatrix} 7 & 14 \\ 10 & 11 \end{bmatrix}, \begin{bmatrix} 8 & 13 \\ 11 & 10 \end{bmatrix}. \tag{3.4}$$

Note 3. We explain why we consider $\begin{bmatrix} 0 & 21 \\ 3 & 18 \end{bmatrix}$ as a potentially parameter matrix in Case 3, whereas we do not similarly suppose $\begin{bmatrix} 3 & 18 \\ 0 & 21 \end{bmatrix}$ in Case 2. Considering $a_{11} = 0$ does not produce any obstacle, where $a_{21} = 0$ causes that the graph be disconnected, which contradicts the fact that Johnson graphs are connected.

Again we use Lemma 2.1. By taking $k = 3$ and $j = 1$, $i = 2$, we deduce that

$$\frac{a_{21}}{a_{12} + a_{21}} \binom{n-i}{\omega - i + j} = \frac{a_{21}}{24} \binom{8}{2} = \frac{7}{6} a_{21}$$

must be integer (in all matrices of the above lists we have $a_{12} + a_{21} = 24$). It means that the matrices on the list (3.4) are not acceptable. For the matrices on the list (3.3) we apply Lemma 2.2.

For the matrix $\begin{bmatrix} 3 & 18 \\ 6 & 15 \end{bmatrix}$ and using Lemma 2.2, we get $\lambda = \frac{a_{21}}{a_{12}+a_{21}}(n - \omega + 1) = \frac{6}{24} \times 8 = 2$, where λ is one of the parameters of a $t-(n, k, \lambda)-design$, mentioned as a (block) design in Introduction. By [3] we conclude that there exist a perfect 2-coloring of $J(10, 3)$ with parameter matrix $\begin{bmatrix} 3 & 18 \\ 6 & 15 \end{bmatrix}$.

For the matrix $\begin{bmatrix} 9 & 12 \\ 12 & 9 \end{bmatrix}$, we again use Lemma 2.2 and get $\lambda = \frac{a_{21}}{a_{12}+a_{21}}(n - \omega + 1) = \frac{12}{24} \times 8 = 4$. Applying again [3], we deduce that there is a perfect 2-coloring of $J(10, 3)$ with parameter matrix $\begin{bmatrix} 9 & 12 \\ 12 & 9 \end{bmatrix}$, and the proof is completed. □

4. Conclusion

Perfect colorings of graphs is a new field in mathematics, related to graph theory, coding theory, and combinatorics, including designs. Perfect 2-colorings of Johnson graphs $J(n, \omega)$ is more attractive for mathematicians. The cases $J(n, 2)$ and $J(n, 3)$ with odd n have already been settled, and the case $J(n, 3)$ in general is gradually being resolved. The next challenge will be perfect 2-colorings of $J(n, 4)$. Any progress in this case will simultaneously solve the counterpart problem in designs.

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