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Bicomplex Mittag-Leffler function and associated properties

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Abstract

With the increasing importance of the Mittag-Leffler function in physical applications, these days many researchers are studying various generalizations and extensions of the Mittag-Leffler function. In this paper, efforts are made to define the bicomplex extension of the Mittag-Leffler function, and also its analyticity and region of convergence are discussed. Various properties of the bicomplex Mittag-Leffler function including integral representation, recurrence relations, duplication formula, and differential relations are established.

Keywords: Bicomplex numbers, exponential function, Gamma function, Mittag-Leffler function.

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1. Introduction

Bicomplex numbers are being studied for quite a long time and a lot of work has been done in this area. Cockle [9, 10] introduced *Tessarines* between 1848 and 1850 following which Segre [42] introduced the bicomplex numbers. Many properties of the bicomplex numbers have been discovered. During the last few years researchers have aimed to study the different algebraic and geometric properties of bicomplex numbers and its applications (see, e.g., [11, 31, 37, 38, 40, 41]). In the recent developments, efforts have been done to extend the integral transforms [2, 3], holomorphic and meromorphic functions [11–13] a number of functions like Polygamma function [17], Hurwitz Zeta function [18], Gamma and Beta functions [19], Riemann Zeta function [37], bicomplex analysis and Hilbert space [24–29] in the bicomplex variable from their complex counterparts. Bicomplex numbers generalize both: the complex numbers and hyperbolic numbers. The theories which are based on both types of domains, complex and hyperbolic, may be unified by developing that theory for bicomplex functions. For example, the theory of relativistic physics and quantum physics [21]. Further, the theory of bicomplex functions can be applied to electromagnetism by treating the electric and magnetic fields together as a bicomplex field [4].

Recently, various generalizations and the extensions of the Mittag-Leffler function are defined by many authors [1, 5–8, 15, 16, 20, 22, 23, 43] which are useful in the study of fractional calculus. Mittag-Leffler

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function appears while studying the fractional form of various differential equations. For studying the bicomplex version of these fractional differential equations in bicomplex space, a bicomplex version of the Mittag-Leffler function would be required. In this paper, efforts are made to define the bicomplex version of the Mittag-Leffler function.

1.1. *Bicomplex numbers*

Segre [42] defined the set of bicomplex numbers as following.

Definition 1.1 (Bicomplex number)**.** In terms of real components, the set of bicomplex numbers is defined as

$$\mathbb{T} = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + jx_3 \mid x_0,\ x_1,\ x_2,\ x_3\ \in \mathbb{R}\},$$

and in terms of complex numbers it can be written as

$$\mathbb{T} = \{\xi : \xi = z_1 + i_2z_2 \mid z_1,\ z_2\ \in \mathbb{C}\}.$$

We shall use the notations: $x_0 = \mathrm{Re}(\xi),\ x_1 = \mathrm{Im}_{i_1}(\xi),\ x_2 = \mathrm{Im}_{i_2}(\xi),\ x_3 = \mathrm{Im}_j(\xi)$.

Segre discussed the presence of zero divisors which he called *Nullifics*. He noticed that the zero-divisors in bicomplex numbers constitute two ideals which he called the infinite set of nullifics. The set of all zero divisors is called null cone [39] defined as follows:

$$\mathbb{NC} = \mathbb{O}_2 = \{z_1 + z_2i_2 \mid z_1^2 + z_2^2 = 0\}.$$

Two non trivial idempotent zero divisors in $\mathbb{T}$, denoted by e_1 and e_2 are defined as follows [34]:

$$e_1 = \frac{1+i_1i_2}{2} = \frac{1+j}{2},\ e_2 = \frac{1-i_1i_2}{2} = \frac{1-j}{2},\ e_1.e_2 = 0,\ e_1+e_2 = 1,\ \text{and } e_1^2 = e_1,\ e_2^2 = e_2.$$

Definition 1.2 (Idempotent representation)**.** Every element $\xi \in \mathbb{T}$ has unique idempotent representation in terms of e_1 and e_2 defined by

$$\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = \xi_1e_1 + \xi_2e_2,$$

where $\xi_1 = (z_1 - i_1z_2)$ and $\xi_2 = (z_1 + i_1z_2)$.

Projection mappings $P_1 : \mathbb{T} \to T_1 \subseteq \mathbb{C}$, $P_2 : \mathbb{T} \to T_2 \subseteq \mathbb{C}$ for a bicomplex number $\xi = z_1 + i_2z_2$ are defined as (see, e.g. [38]):

$$P_1(\xi) = P_1(z_1 + i_2z_2) = P_1[(z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2] = (z_1 - i_1z_2) \in T_1$$

and

$$P_2(\xi) = P_2(z_1 + i_2z_2) = P_2[(z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2] = (z_1 + i_1z_2) \in T_2,$$

where

$$T_1 = \{\xi_1 = z_1 - i_1z_2\ |z_1, z_2 \in \mathbb{C}\} \text{ and } T_2 = \{\xi_2 = z_1 + i_1z_2\ |z_1, z_2 \in \mathbb{C}\}.$$

Remark 1.3. The bicomplex space $\mathbb{T}$ can be written as the product

$$\mathbb{T} = T_1 \times_e T_2 = \{\xi_1e_1 + \xi_2e_2,\ \xi_1 \in T_1,\ \xi_2 \in T_2\}. \tag{1.1}$$

Definition 1.4 (Bicomplex modulii)**.** Let $\xi = z_1 + i_2z_2 = \xi_1e_1 + \xi_2e_2 = x_0 + x_1i_1 + x_2i_2 + x_3j \in \mathbb{T}$ (see, e.g. [32, 35, 38]). The norm or the real modulus of ξ is given by

$$\|\xi\| = \sqrt{|z_1|^2 + |z_2|^2} = \frac{1}{\sqrt{2}}\sqrt{|\xi_1|^2 + |\xi_2|^2} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

The i_1-modulus of ξ is defined as

$$|\xi|_{i_1} = \sqrt{z_1^2 + z_2^2}.$$

The i_2-modulus of ξ is defined as

$$|\xi|_{i_2} = \sqrt{(|z_1|^2 - |z_2|^2) + 2\,\mathrm{Re}(z_1\bar{z_1})i_2}.$$

The j-modulus of ξ is defined as

$$|\xi|_j = |z_1 - i_1 z_2|e_1 + |z_1 + i_1 z_2|e_2. \tag{1.2}$$

The absolute value of ξ is denoted by $|\xi|_{abs}$, and is defined as

$$|\xi|_{abs} = \sqrt{|z_1^2 + z_2^2|} = \sqrt{|(z_1 - i_1 z_2)(z_1 + i_1 z_2)|} = \sqrt{|\xi_1 \xi_2|} = \sqrt{|\xi_1||\xi_2|}.$$

Definition 1.5 (Argument)**.** Let $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2 = x_0 + x_1 i_1 + x_2 i_2 + x_3 j \in \mathbb{T}$, then hyperbolic argument (see, e.g. [30]) of ξ is given by:

$$\arg_j(\xi) = \arg(\xi_1)e_1 + \arg(\xi_1)e_2. \tag{1.3}$$

Let $U \subseteq \mathbb{T}$ be an open set, and $g : U \to \mathbb{T}$ (see, e.g. [37, 41]). Also $g(z_1 + i_2 z_2) = g_1(z_1, z_2) + i_2 g_2(z_1, z_2)$, then g is $\mathbb{T}$-holomorphic iff g_1 and g_2 are holomorphic in U and

$$\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2} \quad \text{and} \quad \frac{\partial g_2}{\partial z_1} = -\frac{\partial g_1}{\partial z_2} \quad \text{on } U.$$

These equations are called the bicomplex Cauchy-Riemann equations (abbr. bicomplex CR-equations),

$$g' = \frac{\partial g_1}{\partial z_1} + i_2 \frac{\partial g_2}{\partial z_1}.$$

In the following theorem, Riley [35] studied the convergence of bicomplex power series.

Theorem 1.6. *Let*

$$N(\xi) = \sqrt{\|\xi\|^2 + \sqrt{\|\xi\|^4 - |\xi|_{abs}^4}} = \max(|\xi_1|, |\xi_2|), \tag{1.4}$$

then $N(\xi)$ is a norm and if $\sum_{n=0}^{\infty} a_n \xi^n$, $a_n = b_n e_1 + c_n e_2$ is a power series with component series $\sum_{n=0}^{\infty} b_n \xi_1^n$ and $\sum_{n=0}^{\infty} c_n \xi_2^n$, both having the same radius of convergence $R > 0$, then $\sum_{n=0}^{\infty} a_n \xi^n$ converges for $N(\xi) < R$ and diverges for $N(\xi) > R$, where $\|\xi\| = \frac{1}{\sqrt{2}}\sqrt{|\xi_1|^2 + |\xi_2|^2}$ and $|\xi|_{abs} = \sqrt{|\xi_1||\xi_2|}$.

In the following theorem, Ringleb [36] (see also, [35]) discussed the analyticity of a bicomplex function w.r.t. its idempotent complex component functions. This theorem plays a vital role while discussing the convergence of the bicomplex functions.

Theorem 1.7 (Decomposition theorem of Ringleb [36])**.** *Let $f(z)$ be analytic in a region $U \subseteq \mathbb{T}$, and let $T_1 \subseteq \mathbb{C}$ and $T_2 \subseteq \mathbb{C}$ be the component regions of $\mathbb{T}$, in the ξ_1 and ξ_2 planes, respectively. Then there exists a unique pair of complex-valued analytic functions, $f_1(\xi_1)$ and $f_2(\xi_2)$, defined in $U_1 \subseteq T_1$ and $U_2 \subseteq T_2$, respectively, such that*

$$f(z) = f_1(\xi_1)e_1 + f_2(\xi_2)e_2, \xi \in U. \tag{1.5}$$

Conversely, if $f_1(\xi_1)$ is any complex-valued analytic function in a region T_1 and $f_2(\xi_2)$ any complex-valued analytic function in a region T_2, then the bicomplex-valued function f(z) defined by the equation (1.5) is an analytic function of the bicomplex variable ξ in the product-region $U = U_1 \times_e U_2$.

In the Theorem 1.8, Price [34] studied the integration in bicomplex domain w.r.to its idempotent representation. This theorem plays a basic role in the study of integrals of the bicomplex function.

Theorem 1.8. *Let* $U \subseteq \mathbb{T}$. *Let* C_1, C_2 *be two curves defined as*

$$C_1 : z_1 - i_2 z_2 = z_1(t) - i_1 z_2(t) = \xi_1 = \xi_1(t),\ a \leqslant t \leqslant b,$$
$$C_2 : z_1 + i_2 z_2 = z_1(t) + i_1 z_2(t) = \xi_2 = \xi_2(t), a \leqslant t \leqslant b,$$

which have continuous derivatives and whose traces are in $U_1 \subseteq T_1$, $U_2 \subseteq T_2$, *respectively and let* C *be the curve with trace in* U *which is defined as*

$$C : \xi(t) = \xi_1(t)e_1 + \xi_2(t)e_2,\ a \leqslant t \leqslant b.$$

Then the integrals of f, f_1, f_2 *on the curves* C, C_1*, and* C_2 *exist and*

$$\int_C f(\xi)d\xi = \int_{C_1} f_1(z_1 - i_1 z_2)d(z_1 - i_1 z_2)e_1 + \int_{C_2} f_1(z_1 + i_1 z_2)d(z_1 + i_1 z_2)e_2$$

or

$$\int_C f(\xi)d\xi = \int_{C_1} f_1(\xi_1)d(\xi_1)e_1 + \int_{C_2} f_2(\xi_2)d(\xi_2)e_2.$$

We would require the definition of the bicomplex gamma function defined by Goyal et al. [19], in the Euler product form as follows:

$$\frac{1}{\Gamma\xi} = \xi e^{\gamma\xi} \prod_{n=1}^{\infty} \left(\left(1 + \frac{\xi}{n}\right) \exp\left(-\frac{\xi}{n}\right) \right),\ \xi \in \mathbb{T}, \tag{1.6}$$

provided that $z_1 \neq \frac{-(m+l)}{2}$, and $z_2 \neq i_1(\frac{l-m}{2})$, where $m,\ l \in \mathbb{N} \cup \{0\}$. The Euler constant $\gamma(0 \leqslant \gamma \leqslant 1)$ is given by

$$\gamma = \lim_{n\to\infty} (H_n - \log n),\ H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Also, in idempotent form

$$\Gamma\xi = \Gamma\xi_1 e_1 + \Gamma\xi_2 e_2,\ \xi \in \mathbb{T},$$

and in the integral form (see, e.g.[19]), for $p = p_1e_1 + p_2e_2,\ p_1, p_2 \in \mathbb{R}^+$,

$$\Gamma\xi = \int_H e^{-p}p^{\xi-1}dp = \left(\int_0^\infty e^{-p_1}p_1^{\xi_1-1}dp_1\right)e_1 + \left(\int_0^\infty e^{-p_2}p_2^{\xi_2-1}dp_2\right)e_2, \tag{1.7}$$

where $H = (\gamma_1, \gamma_2)$ and $\gamma_1 : 0$ to ∞, $\gamma_2 : 0$ to ∞.

1.2. Mittag-Leffler function and its properties

The Mittag-Leffler function (M-L function) comes intrinsically in the study of the fractional calculus. The importance of the M-L function in science and engineering is continuously increasing. It is very useful in the area of fractional modeling of real life problems.

The one parameter M-L function defined by Mittag-Leffler [33] is given by

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},\ \ \mathrm{Re}(\alpha) > 0,\ \ z \in \mathbb{C}. \tag{1.8}$$

It comes from the Cauchy inequality for the Taylor coefficients and properties of the Gamma function (see, e.g.[16, p.18]) that $\exists$ a number $k \geqslant 0$ and a positive number $r(k)$ such that

$$M_{\mathbb{E}_\alpha(r)} = \max_{|z|=r} |\mathbb{E}_\alpha(z)| < e^{r^k},\ \forall r > r(k), \tag{1.9}$$

hence $\mathbb{E}_\alpha(z)$ is an entire function of finite order. For each $\mathrm{Re}(\alpha) > 0$, the order ρ and type σ of M-L function (1.8) is given by

$$\rho = \limsup_{k\to\infty} \frac{k\log k}{\log\frac{1}{|a_k|}} = \frac{1}{\mathrm{Re}(\alpha)} \tag{1.10}$$

and

$$\sigma = \frac{1}{e\rho} \limsup_{k\to\infty} (k|a_k|^{\frac{\rho}{k}}) = 1. \tag{1.11}$$

2. Bicomplex one-parameter Mittag-Leffler function

Here, we introduce the bicomplex one parameter Mittag-Leffler function defined by

$$\mathbb{E}_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k+1)},$$

where $\xi, \alpha \in \mathbb{T}$, $\xi = z_1 + i_2 z_2$ and $|\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$.

The definition of bicomplex M-L function is well justified by the following theorem.

Theorem 2.1. *Let $\xi, \alpha \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\alpha = \alpha_1 e_1 + \alpha_2 e_2 = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3$ with $|\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$. Then*

$$\mathbb{E}_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k+1)}. \tag{2.1}$$

Proof. Consider the function

$$\mathbb{E}_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k+1)}. \tag{2.2}$$

By using the idempotent representation

$$\mathbb{E}_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi_1^k}{\Gamma(\alpha_1 k+1)} e_1 + \sum_{k=0}^{\infty} \frac{\xi_2^k}{\Gamma(\alpha_2 k+1)} e_2 = \mathbb{E}_{\alpha_1}(\xi_1) e_1 + \mathbb{E}_{\alpha_2}(\xi_2) e_2, \tag{2.3}$$

where $\xi = \xi_1 e_1 + \xi_2 e_2$, $\xi_1 \in T_1$, $\xi_2 \in T_2$ and $\alpha = \alpha_1 e_1 + \alpha_2 e_2$. Now,

$$\mathbb{E}_{\alpha_1}(\xi_1) = \sum_{k=0}^{\infty} \frac{\xi_1^k}{\Gamma(\alpha_1 k+1)} \quad \text{and} \quad \mathbb{E}_{\alpha_2}(\xi_2) = \sum_{k=0}^{\infty} \frac{\xi_2^k}{\Gamma(\alpha_2 k+1)}$$

are complex M-L functions convergent for $\mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2) > 0$, $\xi_1 \in T_1 \subseteq \mathbb{C}$, $\xi_2 \in T_2 \subseteq \mathbb{C}$. Since $\mathbb{E}_{\alpha_1}(\xi_1)$ and $\mathbb{E}_{\alpha_2}(\xi_2)$ are convergent in T_1, T_2, respectively, by Ringleb decomposition theorem, (2.2) is also convergent in $\mathbb{T}$. Further, Let

$$\alpha = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 = \alpha_1 e_1 + \alpha_2 e_2,$$

where $\alpha_1 = (a_0 + a_3) + i_1(a_1 - a_2)$ and $\alpha_2 = (a_0 - a_3) + i_1(a_1 + a_2)$.

Since $\mathrm{Re}(\alpha_1) > 0$ and $\mathrm{Re}(\alpha_2) > 0$,

$$\begin{aligned} &\Rightarrow\ a_0 + a_3 > 0 \text{ and } a_0 - a_3 > 0,\\ &\Rightarrow\ |a_3| < a_0,\\ &\Rightarrow\ |\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha). \end{aligned}$$

This completes the proof. □

By substituting the value of the bicomplex gamma function defined by equation (1.6) in the equation (2.3) we get the following representation for Mittag-Leffler function.

Theorem 2.2. *Let* $\xi, \alpha \in \mathbb{T}$ *where* $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\alpha = a_0 + i_1 a_1 + i_2 a_2 + i_1 i_2 a_3 = \alpha_1 e_1 + \alpha_2 e_2$, *with* $|\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$, *then*

$$\mathbb{E}_\alpha(\xi) = \sum_{k=0}^{\infty} \xi^k (\alpha k + 1) e^{\gamma(\alpha k+1)} \prod_{n=1}^{\infty} \left(\left(1 + \frac{(\alpha k+1)}{n}\right) \exp\left(-\frac{(\alpha k+1)}{n}\right)\right).$$

Remark 2.3. In integral form, with the help of (1.7) the bicomplex M-L function can be represented as

$$\mathbb{E}_\alpha(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\int_H e^{-p} p^{\alpha k} dp},$$

where $H = (\gamma_1, \gamma_2)$ as defined in (1.7).

For different values of the α we obtain various bicomplex functions as special cases. To mention, a few are:

1. for $\alpha = 0$, we get bicomplex binomial function $\mathbb{E}_0(\xi) = \frac{1}{1-\xi}$, $\|\xi\| < 1$;
2. for $\alpha = 1$, we get bicomplex exponential function $\mathbb{E}_1(\pm\xi) = e^{\pm\xi}$;
3. for $\alpha = 2$, we get bicomplex cosine function $\mathbb{E}_2(-\xi^2) = \cos\xi$;
4. for $\alpha = 2$, we get bicomplex hyperbolic cosine function $\mathbb{E}_2(\xi^2) = \cosh\xi$;
5. for $\alpha = 3$, we get the following function: $\mathbb{E}_3(\xi) = \frac{1}{2}\left(e^{\xi^{1/3}} + 2e^{-(1/2)\xi^{1/3}} \cos\left(\frac{\sqrt{3}}{2}\xi^{1/3}\right)\right)$;
6. for $\alpha = 4$, we get following bicomplex relation: $\mathbb{E}_4(\xi) = \frac{1}{2}\left(\cos(\xi^{1/4}) + \cosh(\xi^{1/4})\right)$.

Theorem 2.4. *The bicomplex Mittag-Leffler function defined in equation* (2.1) *satisfies bicomplex Cauchy-Riemann equations.*

Proof. By the result (2.3) we have,

$$\begin{aligned}
\mathbb{E}_\alpha(\xi) &= \mathbb{E}_{\alpha_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2}(\xi_2)e_2 \\
&= \mathbb{E}_{\alpha_1}(z_1 - i_1 z_2)e_1 + \mathbb{E}_{\alpha_2}(z_1 + i_1 z_2)e_2 \\
&= \mathbb{E}_{\alpha_1}(z_1 - i_1 z_2)\left(\frac{1+i_1 i_2}{2}\right) + \mathbb{E}_{\alpha_2}(z_1 + i_1 z_2)\left(\frac{1-i_1 i_2}{2}\right) \\
&= \left(\frac{1}{2}\left(\mathbb{E}_{\alpha_1}(z_1 - i_1 z_2) + \mathbb{E}_{\alpha_2}(z_1 + i_1 z_2)\right)\right) + i_2\left(\frac{i_1}{2}\left(\mathbb{E}_{\alpha_1}(z_1 - i_1 z_2) - \mathbb{E}_{\alpha_2}(z_1 + i_1 z_2)\right)\right) \\
&= f_1(z_1, z_2) + i_2 f_2(z_1, z_2),
\end{aligned}$$

where $f_1(z_1, z_2) = \frac{1}{2}\left(\mathbb{E}_{\alpha_1}(z_1 - i_1 z_2) + \mathbb{E}_{\alpha_2}(z_1 + i_1 z_2)\right)$ and $f_2(z_1, z_2) = \frac{i_1}{2}\left(\mathbb{E}_{\alpha_1}(z_1 - i_1 z_2) - \mathbb{E}_{\alpha_2}(z_1 + i_1 z_2)\right)$. $\mathbb{E}_{\alpha_i}(i = 1, 2)$ are complex M-L functions. Now,

$$\begin{aligned}
\frac{\partial f_1}{\partial z_1} &= \frac{1}{2}\left(\mathbb{E}'_{\alpha_1}(z_1 - i_1 z_2) + \mathbb{E}'_{\alpha_2}(z_1 + i_1 z_2)\right), & \frac{\partial f_1}{\partial z_2} &= \frac{-i_1}{2}\left(\mathbb{E}'_{\alpha_1}(z_1 - i_1 z_2) - \mathbb{E}'_{\alpha_2}(z_1 + i_1 z_2)\right), \\
\frac{\partial f_2}{\partial z_1} &= \frac{i_1}{2}\left(\mathbb{E}'_{\alpha_1}(z_1 - i_1 z_2) - \mathbb{E}'_{\alpha_2}(z_1 + i_1 z_2)\right), & \frac{\partial f_2}{\partial z_2} &= \frac{1}{2}\left(\mathbb{E}'_{\alpha_1}(z_1 - i_1 z_2) + \mathbb{E}'_{\alpha_2}(z_1 + i_1 z_2)\right).
\end{aligned}$$

From the above equations it can be observed that

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2} \quad \text{and} \quad \frac{\partial f_2}{\partial z_1} = -\frac{\partial f_1}{\partial z_2}.$$

Hence, bicomplex Cauchy-Riemann equations are satisfied by the bicomplex M-L function. □

Theorem 2.5. *The bicomplex M-L function* $\mathbb{E}_\alpha(\xi), |\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$ *is an entire function in the bicomplex domain.*

Proof. Let $\sum_{n=0}^{\infty} a_n \xi^n$ represents a bicomplex power series, where $a_n,\ \xi, \in \mathbb{T},\ a_n = b_n e_1 + c_n e_2,\ \xi = \xi_1 e_1 + \xi_2 e_2$. Then by Ringleb decomposition theorem 1.7, the series

$$\sum_{n=0}^{\infty} a_n \xi^n = \left(\sum_{n=0}^{\infty} b_n \xi_1^n\right) e_1 + \left(\sum_{n=0}^{\infty} c_n \xi_2^n\right) e_2$$

converges iff $\sum_{n=0}^{\infty} b_n \xi_1^n$ and $\sum_{n=0}^{\infty} c_n \xi_2^n$ converge in the complex domains (see, e.g. [35]). Now from equation (2.3), the Mittag -Leffler function can be decomposed as

$$\mathbb{E}_\alpha(\xi) = \mathbb{E}_{\alpha_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2}(\xi_2)e_2.$$

Since $\mathbb{E}_{\alpha_1}(\xi_1) = \sum_{k=0}^{\infty} \frac{\xi_1^k}{\Gamma(\alpha_1 k+1)}, \mathrm{Re}(\alpha_1) > 0$, and $\mathbb{E}_{\alpha_2}(\xi_2) = \sum_{k=0}^{\infty} \frac{\xi_2^k}{\Gamma(\alpha_2 k+1)},\ \mathrm{Re}(\alpha_2) > 0$ are complex Mittag Leffler functions with infinite radius of covergence (say R) [16, p.18], then

$$|\xi_1| < R,\ |\xi_2| < R.$$

From equation (1.4),

$$N(\xi) = \sqrt{\|\xi\|^2 + \sqrt{\|\xi\|^4 - |\xi|^4_{abs}}} = \max(|\xi_1|, |\xi_2|) < R.$$

As a consequence pf the Theorem 1.6, $\mathbb{E}_\alpha(\xi)$ converges in the bicomplex domain and has infinite radius of convergence [35]. Since complex M-L function is entire function in $\mathbb{C}$ the bicomplex M-L function is an entire function in $\mathbb{T}$ (Riley [35, p.141]). □

Theorem 2.6 (Order and type)**.** *The bicomplex Mittag-Leffler function* $\mathbb{E}_\alpha(\xi),\ \xi,\ \alpha \in \mathbb{T}$ *is an entire function of finite order* $\rho = \frac{a_0 - a_3 j}{(a_0^2 - a_3^2)}$ *and type* $\sigma = 1$.

Proof. From equation (2.3),

$$\mathbb{E}_\alpha(\xi) = \mathbb{E}_{\alpha_1}(\xi_1)e_1 + \mathbb{E}_{\alpha_2}(\xi_2)e_2.$$

Here $\mathbb{E}_{\alpha_1}(\xi_1)$ and $\mathbb{E}_{\alpha_2}(\xi_2)$ are the complex Mittag-Leffler functions for $\mathrm{Re}(\alpha_1) > 0$ and $\mathrm{Re}(\alpha_2) > 0$, respectively. Since $\mathbb{E}_{\alpha_1}(\xi_1),\ \mathbb{E}_{\alpha_2}(\xi_2)$ are entire functions, there exists numbers $k_1,\ k_2 \geqslant 0$ and positive numbers $r_1(k_1),\ r_2(k_2)$, such that, from equation (1.9), we get

$$M_{\mathbb{E}_{\alpha_1}(r_1)} = \max_{|\xi_1|=r_1} |\mathbb{E}_{\alpha_1}(\xi_1)| < e^{r_1^{k_1}},\ \forall r_1 > r_1(k_2)$$

and

$$M_{\mathbb{E}_{\alpha_2}(r_2)} = \max_{|\xi_2|=r_1} |\mathbb{E}_{\alpha_2}(\xi_2)| < e^{r_2^{k_2}},\ \forall r_2 > r_2(k_2).$$

Let $r = \max(r_1, r_2)$ and $k = \max(k_1, k_2)$, then

$$M_{\mathbb{E}_{\alpha_1}(r_1)} = \max_{|\xi_1|=r_1} |\mathbb{E}_{\alpha_1}(\xi_1)| < e^{r_1^{k_1}} \leqslant e^{r^k},$$

and

$$\begin{aligned} M_{\mathbb{E}_{\alpha_2}(r_2)} &= \max_{|\xi_2|=r_1} |\mathbb{E}_{\alpha_2}(\xi_2)| < e^{r_2^{k_2}} \leqslant e^{r^k}, \\ M_{\mathbb{E}_\alpha(r)} &= \max_{|\xi|_j=r} |\mathbb{E}_\alpha(\xi)|_j \quad \text{(J-modulus of bicomplex number)} \\ &= \max_{|\xi_1|=r} |\mathbb{E}_{\alpha_1}(\xi_1)|e_1 + \max_{|\xi_2|=r} |\mathbb{E}_{\alpha_2}(\xi_2)|e_2 \leqslant e^{r^k}e_1 + e^{r^k}e_2, \quad \forall r > r(k_j) = e^{r^k}. \end{aligned}$$

Hence $\mathbb{E}_\alpha(z)$ is an entire function of finite order.

For the bicomplex M-L function ξ, $\alpha \in \mathbb{T}$, $|\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$, the order ρ is given by

$$\rho = \limsup_{k\to\infty} \frac{k\log k}{\log\Gamma(\alpha k+1)} = \left(\limsup_{k\to\infty} \frac{k\log k}{\log\Gamma(\alpha_1 k+1)}\right) e_1 + \left(\limsup_{k\to\infty} \frac{k\log k}{\log\Gamma(\alpha_2 k+1)}\right) e_2.$$

Now, from equation (1.10),

$$\rho = \left(\frac{1}{\mathrm{Re}(\alpha_1)}\right) e_1 + \left(\frac{1}{\mathrm{Re}(\alpha_2)}\right) e_2 = \left(\frac{1}{a_0+a_3}\right) e_1 + \left(\frac{1}{a_0-a_3}\right) e_2 = \frac{a_0 - a_3 j}{(a_0^2 - a_3^2)},$$

where $a_0 > |a_3| \Rightarrow a_0^2 - a_3^2 \neq 0$. The type σ of the bicomplex M-L function $\mathbb{E}_\alpha(\xi)$ is given by

$$\begin{aligned}
\sigma &= \frac{1}{e\rho} \limsup_{k\to\infty} (k|a_k|_j^{\frac{\rho}{k}}) \\
&= \frac{1}{e\rho} \limsup_{k\to\infty} (k\left|\frac{1}{\Gamma(\alpha k+1)}\right|_j^{\frac{\rho}{k}}) \\
&= \left(\frac{1}{e\rho} \limsup_{k\to\infty} \left(k\left|\frac{1}{\Gamma(\alpha_1 k+1)}\right|^{\frac{\rho}{k}}\right)\right) e_1 + \left(\frac{1}{e\rho} \limsup_{k\to\infty} \left(k\left|\frac{1}{\Gamma(\alpha_2 k+1)}\right|^{\frac{\rho}{k}}\right)\right) e_2 \\
&= 1.e_1 + 1.e_2 \quad \text{(using equation (1.11))} \\
&= 1.
\end{aligned}$$

□

Remark 2.7. There are different modulus such as real, i_1, i_2, and j modulus are defined for a bicomplex number (see, e.g. [38]). In this paper, j modulus has been used for the calculation, since it provides expression in terms of idempotent components of the complex modulus.

2.1. *Properties of bicomplex Mittag-Leffler function*

Integral representation for the complex M-L function $\mathbb{E}_\alpha(z)$ is given by (see, e.g.[14, p.209]):

$$\int_0^\infty e^{-t}\mathbb{E}_\alpha(t^\alpha z)dt = \frac{1}{1-z}, \ z\in\mathbb{C}, \ \alpha \geqslant 0. \tag{2.4}$$

The above integral converges in unit circle and is bounded by the line $\mathrm{Re}(z^{1/\alpha}) = 1$.

Theorem 2.8 (Integral representation for bicomplex M-L function)**.** *Let $\xi \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$ and $\alpha \geqslant 0$, $\|\xi\| < 1$, then*

$$\int_0^\infty e^{-t}\mathbb{E}_\alpha(t^\alpha \xi)dt = \frac{1}{1-\xi}.$$

The above integral converges in the unit circle and is bounded by the plane $\mathrm{Re}(\xi^{1/\alpha}) = 1$, $\mathrm{Im}_j(\xi) = 0$.

Proof. By the integral representation (2.4) and the result (2.3), for $\xi \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\alpha \geqslant 0$ and $|\xi_1| < 1$, $|\xi_2| < 1$,

$$\begin{aligned}
\int_0^\infty e^{-t}\mathbb{E}_\alpha(t^\alpha \xi)dt &= \left(\int_0^\infty e^{-t}\mathbb{E}_\alpha(t^\alpha \xi_1)dt\right) e_1 + \left(\int_0^\infty e^{-t}\mathbb{E}_\alpha(t^\alpha \xi_2)dt\right) e_2 \\
&= \left(\frac{1}{1-\xi_1}\right) e_1 + \left(\frac{1}{1-\xi_2}\right) e_2 \\
&= \frac{1}{1-(\xi_1 e_1 + \xi_2 e_2)} \\
&= \frac{1}{1-\xi}.
\end{aligned} \tag{2.5}$$

In terms of the real components, $\xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3 = \xi_1 e_1 + \xi_2 e_2$. Therefore, $\xi_1 = (x_0 + x_3) + i_1(x_1 - x_2)$, $\xi_2 = (x_0 - x_3) + i_1(x_1 + x_2)$. Since,

$$\begin{aligned}
|\xi_1| < 1 \text{ and } |\xi_2| < 1 &\Rightarrow \sqrt{(x_0+x_3)^2 + (x_1-x_2)^2} < 1 \text{ and } \sqrt{(x_0-x_3)^2 + (x_1+x_2)^2} < 1\\
&\Rightarrow \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2 + 2x_0x_3 - 2x_1x_2} < 1 \text{ and } \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2 - 2x_0x_3 + 2x_1x_2} < 1\\
&\Rightarrow \sqrt{x_0^2 + x_1^2 + x_2^2 + x_2^2} < 1\\
&\Rightarrow \|\xi\| < 1.
\end{aligned}$$

Also,

$$\begin{aligned}
\operatorname{Re} \xi_1^{1/\alpha} = 1,\ \operatorname{Re} \xi_2^{1/\alpha} = 1 &\Rightarrow (x_0 + x_3)^{\frac{1}{\alpha}} = 1,\ (x_0 - x_3)^{\frac{1}{\alpha}} = 1\\
&\Rightarrow (x_0 + x_3) = 1,\ (x_0 - x_3) = 1\\
&\Rightarrow x_0 = 1,\ x_3 = 0\\
&\Rightarrow \operatorname{Re} \xi = 1,\ \operatorname{Im}_j \xi = 0\\
&\Rightarrow \operatorname{Re} \xi^{1/\alpha} = 1,\ \operatorname{Im}_j \xi = 0.
\end{aligned}$$

□

The complex Mittag-Leffler function has following integral representation (see, e.g. [14])

$$\mathbb{E}_\alpha(z) = \frac{1}{2\pi i}\int_\Omega \frac{t^{\alpha-1}e^t}{t^\alpha - z}dt,\ \alpha > 0,\ z \in \mathbb{C}, \tag{2.6}$$

where the path of integration Ω is a loop starting and ending at $-\infty$ and encircling the circular disk $|t| \leqslant |z|^{1/\alpha}$ in the positive sense, $|\arg t| < \pi$ on Ω.

Theorem 2.9. *Let ξ, $\omega \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2$, $\omega = \omega_1 e_1 + \omega_2 e_2$, then bicomplex Mittag-Leffler function has following integral representation*

$$\mathbb{E}_\alpha(\xi) = \frac{1}{2\pi i_1}\int_H \frac{\omega^{\alpha-1}e^(\omega)}{\omega^\alpha - \xi}d\omega,\ \alpha > 0,$$

where the path of integration $H = (\Omega_1, \Omega_2)$ and Ω_1, Ω_2 are loops starting and ending at $-\infty$ and encircling the circular disks $|\omega_1| \leqslant |\xi_1|^{1/\alpha}$, $|\omega_2| \leqslant |\xi_2|^{1/\alpha}$, respectively, in the positive sense, $|\arg \omega_1| < \pi$ on Ω_1 and $|\arg \omega_2| < \pi$ on Ω_2 equivalently, $|\arg_j \omega|_j < \pi$.

Proof. By the integral representation (2.6), result (2.3), and the Theorem 1.8, we have for ξ, $\omega \in \mathbb{T}$,

$$\begin{aligned}
E_\alpha(\xi) &= E_\alpha(\xi_1)e_1 + E_\alpha(\xi_2)e_2\\
&= \frac{1}{2\pi i_1}\int_{\Omega_1} \frac{\omega_1^{\alpha-1}e^{\omega_1}}{\omega_1^\alpha - \xi_1}d\omega_1 e_1 + \frac{1}{2\pi i_1}\int_{\Omega_2} \frac{\omega_2^{\alpha-1}e^{\omega_2}}{\omega_2{}^\alpha - \xi_2}d\omega_2 e_2\\
&= \frac{1}{2\pi i_1}\int_{(\Omega_1,\Omega_2)} \frac{(\omega_1e_1 + \omega_2e_2)^{\alpha-1}e^{(\omega_1e_1+\omega_2e_2)}}{(\omega_1e_1 + \omega_2e_2)^\alpha - (\xi_1e_1 + \xi_2e_2)}d(\omega_1e_1 + \omega_2e_2)\\
&= \frac{1}{2\pi i_1}\int_H \frac{\omega^{\alpha-1}e^\omega}{\omega^\alpha - \xi}d\omega.
\end{aligned}$$

The path of integration is $H = (\Omega_1, \Omega_2)$, where Ω_1, Ω_2 are loops starting and ending at $-\infty$ and encircling the circular disks $|\omega_1| \leqslant |\xi_1|^{1/\alpha}$, $|\omega_2| \leqslant |\xi_2|^{1/\alpha}$, respectively, in the positive sense. Further, since

$$|\arg \omega_1| < \pi \text{ and } |\arg \omega_2| < \pi,$$

from the equations (1.2) and (1.3) we have

$$\arg_j \omega = (\arg \omega_1)e_1 + (\arg \omega_2)e_2 \quad \Rightarrow \quad |\arg_j \omega|_j = |\arg \omega_1|e_1 + |\arg \omega_2|e_2 < \pi e_1 + \pi e_2 = \pi.$$

□

Recurrence relation for the complex M-L function $\mathbb{E}_\alpha(z)$ is given by the following relation, where p, q are the relatively prime natural numbers (see, e.g. [16, p.21])

$$\mathbb{E}_{p/q}(z) = \frac{1}{q}\sum_{l=0}^{q-1} \mathbb{E}_{1/p}(z^{1/q} e^{\frac{2\pi l i_1}{q}}). \tag{2.7}$$

Theorem 2.10 (Recurrence relation for bicomplex M-L function)**.** *Let $\xi \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2$ and $p, q \in \mathbb{N}$ are relatively prime. Then the bicomplex Mittag-Leffler function satisfies*

$$\mathbb{E}_{p/q}(\xi) = \frac{1}{q}\sum_{l=0}^{q-1} \mathbb{E}_{1/p}\left(\xi^{1/q} e^{\frac{2\pi l i_1}{q}}\right).$$

Proof. By the recurrence relation (2.7) and the result (2.3) we have for $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$,

$$\begin{aligned}
\mathbb{E}_{p/q}(\xi) &= \mathbb{E}_{p/q}(\xi_1)e_1 + \mathbb{E}_{p/q}(\xi_2)e_2, \ q \in \mathbb{N}\\
&= \mathbb{E}_{p/q}(z_1 - i_1 z_2)e_1 + \mathbb{E}_{p/q}(z_1 + i_1 z_2)e_2\\
&= \left(\frac{1}{q}\sum_{l=0}^{q-1} \mathbb{E}_{1/p}\left((z_1 - i_1 z_2)^{1/q} e^{\frac{2\pi l i_1}{q}}\right)\right) e_1 + \left(\frac{1}{q}\sum_{l=0}^{q-1} \mathbb{E}_{1/p}\left((z_1 + i_1 z_2)^{1/q} e^{\frac{2\pi l i_1}{q}}\right)\right) e_2\\
&= \frac{1}{q}\sum_{l=0}^{q-1} \mathbb{E}_{1/p}\left(\xi^{1/q} e^{\frac{2\pi l i_1}{q}}\right).
\end{aligned}$$

□

Duplication formula for the complex M-L function $\mathbb{E}_\alpha(z)$ is defined as (see, e.g. [16, p.53]):

$$\mathbb{E}_{2\alpha}(z^2) = \frac{1}{2}\left(\mathbb{E}_\alpha(z) + \mathbb{E}_\alpha(-z)\right), \ \ \mathrm{Re}(\alpha) > 0.$$

Theorem 2.11 (Duplication formula for bicomplex M-L function)**.** *Let $\xi, \alpha \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2$, $|\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$, then*

$$\mathbb{E}_{2\alpha}(\xi^2) = \frac{1}{2}\left(\mathbb{E}_\alpha(\xi) + \mathbb{E}_\alpha(-\xi)\right).$$

Proof. We have for $\xi, \alpha \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2 = \xi_1 e_1 + \xi_2 e_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$ and $\alpha = \alpha_1 e_1 + \alpha_2 e_2$, $|\mathrm{Im}_j(\alpha)| < \mathrm{Re}(\alpha)$,

$$\begin{aligned}
\frac{1}{2}\left(\mathbb{E}_\alpha(\xi) + \mathbb{E}_\alpha(-\xi)\right) &= \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(\alpha k + 1)} + \sum_{k=0}^{\infty} \frac{(-\xi)^k}{\Gamma(\alpha k + 1)}\right)\\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(\xi^k + (-\xi)^k)}{\Gamma(\alpha k + 1)}\\
&= \frac{1}{2}\sum_{k=0}^{\infty} \frac{(\xi^k(1 + (-1)^k))}{\Gamma(\alpha k + 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(2 + 2\frac{\xi^2}{\Gamma(2\alpha+1)} + 2\frac{\xi^4}{\Gamma(4\alpha+1)} + 2\frac{\xi^6}{\Gamma(6\alpha+1)} + \cdots\right) \\
&= \left(1 + \frac{(\xi^2)^1}{\Gamma(1(2\alpha)+1)} + \frac{(\xi^2)^2}{\Gamma(2(2\alpha)+1)} + \frac{(\xi^2)^3}{\Gamma(3(2\alpha)+1)} + \cdots\right) \\
&= \sum_{k=0}^{\infty} \frac{(\xi^2)^k}{\Gamma(2\alpha k+1)} \\
&= \mathbb{E}_{2\alpha}(\xi^2).
\end{aligned}
$$

□

Differential relations for the complex M-L function $\mathbb{E}_\alpha(z)$ are defined by the following relations, where $p, q \in \mathbb{N}$ are relatively prime (see, e.g. [16, p.22]):

$$
\left(\frac{d}{dz}\right)^p \mathbb{E}_p(z^p) = \mathbb{E}_p(z^p), \qquad \frac{d^p}{dz^p}\mathbb{E}_{p/q}(z^{p/q}) = \mathbb{E}_{p/q}(z^{p/q}) + \sum_{k=1}^{q-1} \frac{z^{-kp/q}}{\Gamma\,(1-kp/q)}.
$$

Theorem 2.12 (Differential relations for the bicomplex M-L function). *Let $\xi \in \mathbb{T}$, where $\xi = z_1 + i_2 z_2$, then for p, q, relatively prime natural numbers,*

(i) $\left(\frac{d}{d\xi}\right)^p \mathbb{E}_p(\xi^p) = \mathbb{E}_p(\xi^p)$;

(ii) $\frac{d^p}{d\xi^p}\mathbb{E}_{p/q}(\xi^{p/q}) = \mathbb{E}_{p/q}(\xi^{p/q}) + \sum_{k=1}^{q-1} \frac{\xi^{-kp/q}}{\Gamma\,(1-kp/q)}$.

Proof.

(i)

$$
\begin{aligned}
\left(\frac{d}{d\xi}\right)^p \mathbb{E}_p(\xi^p) &= \left(\frac{d}{d\xi}\right)^p \sum_{k=0}^{\infty} \frac{\xi^{pk}}{\Gamma(pk+1)} \quad \text{(From definition (2.1))} \\
&= \sum_{k=1}^{\infty} \frac{\xi^{pk-p}}{\Gamma(pk-p+1)} = \sum_{k=0}^{\infty} \frac{\xi^{pk}}{\Gamma(pk+1)} \quad \text{(Replacing } k \to k+1) = \mathbb{E}_p(\xi^p).
\end{aligned}
$$

(ii) Again,

$$
\frac{d^p}{d\xi^p}\mathbb{E}_{p/q}(\xi^{p/q}) = \frac{d^p}{d\xi^p}\sum_{k=0}^{\infty} \frac{\xi^{\frac{kp}{q}}}{\Gamma(\frac{kp}{q}+1)} = \sum_{k=0}^{\infty} \frac{\xi^{(\frac{k}{q}-1)p}}{\Gamma(\frac{kp}{q}-p+1)} = \sum_{k=0}^{q-1} \frac{\xi^{(\frac{kp}{q}-p)}}{\Gamma(\frac{kp}{q}-p+1)} + \sum_{k=0}^{\infty} \frac{\xi^{(\frac{kp}{q})}}{\Gamma(\frac{kp}{q}+1)}.
$$

The above equation can further be written as

$$
\frac{d^p}{d\xi^p}\mathbb{E}_{p/q}(\xi^{p/q}) = \sum_{k=1}^{q-1} \frac{\xi^{-kp/q}}{\Gamma\,(1-kp/q)} + \mathbb{E}_{p/q}(\xi^{p/q}).
$$

□

Theorem 2.13. *The function $\mathbb{E}_n(\xi^n)$, $\xi \in \mathbb{T}$, $n = 1, 2, 3, \ldots$, satisfies the n^{th} order ordinary differential equation*

$$
\frac{d^n}{d\xi^n}\left(\mathbb{E}_n(\xi^n)\right) = \mathbb{E}_n(\xi^n).
$$

Proof. For $\alpha > 0$, By replacing ξ by ξ^α in the equation (2.1), we get

$$\mathbb{E}_\alpha(\xi^\alpha) = \sum_{k=0}^{\infty} \frac{\xi^{\alpha k}}{\Gamma(\alpha k+1)} = 1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} + \frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\xi^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots. \tag{2.8}$$

By taking derivative of order α on both sides of the equation (2.8), we get,

$$\begin{aligned} D^\alpha\left(\mathbb{E}_\alpha(\xi^\alpha)\right) &= D^\alpha\left(1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} + \frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{\xi^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right) \\ &= \frac{\Gamma(1)}{\Gamma(1-\alpha)}\xi^{-\alpha} + \frac{\Gamma(\alpha+1)}{\Gamma(1)}\frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\frac{\xi^\alpha}{\Gamma(2\alpha+1)} + \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}\frac{\xi^{2\alpha}}{\Gamma(3\alpha+1)} + \cdots \\ &= \frac{\Gamma(1)}{\Gamma(1-\alpha)}\xi^{-\alpha} + 1 + \frac{\xi^\alpha}{\Gamma(\alpha+1)} + \frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots. \end{aligned} \tag{2.9}$$

Since $\frac{1}{\Gamma(1-\alpha)} = 0$ for $\alpha = n \in \mathbb{N}$, we get from equation (2.9),

$$D^n\left(\mathbb{E}_n(\xi^n)\right) = 1 + \frac{\xi^n}{\Gamma(n+1)} + \frac{\xi^{2n}}{\Gamma(2n+1)} + \cdots = \sum_{k=0}^{\infty} \frac{\xi^{nk}}{\Gamma(nk+1)} = \mathbb{E}_n(\xi^n).$$

□

3. Conclusion

In this paper, one parameter M-L function and its properties in bicomplex space have been defined from its complex counterpart. Various properties and special cases along with recurrence relations, duplication formula, integral representation, differential relation are also derived. We intend to extend the concepts of the fractional calculus in bicomplex space using the M-L function. Since bicomplex space provides a more generalized approach towards the large class of functions appearing in signal theory, electromagnetism, and quantum theory.

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