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Insight into degenerate Bell-based Bernoulli polynomials with applications

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Abstract

Recently, the Bell-based Stirling polynomials of the second kind and the Bell-based Bernoulli polynomials [U. Duran, S. Araci, M. Acikgoz, Axioms, **10** (2021), 23 pages] have been considered, and some of their properties and applications in umbral calculus have been derived and analyzed. In this work, a degenerate form of the Bell-based Stirling polynomials of the second kind is defined, and several fundamental properties and formulas for these polynomials are investigated and presented in detail. Then, a degenerate form of the Bell-based Bernoulli polynomials of order α is defined and a plenty of their properties are examined in different aspects. Several correlations with other polynomials and numbers in literature, symmetric identities, implicit summation formulas, derivative properties and addition formulas for the mentioned new polynomials are derived in detail, and some special cases of these results are investigated. Also, the degenerate Bell-based Bernoulli polynomials of order ε are studied in λ-umbral calculus and interesting relations and formulas are developed. Furthermore, the application of λ-umbral calculus to Bell-based degenerate Bernoulli polynomials of order ε shows a correlation with higher-order degenerate derangement polynomials. Finally, a representation of the degenerate differential operator on the degenerate Bell-based Bernoulli polynomials of order ε is provided.

Keywords: Bernoulli polynomials, Bell polynomials, mixed-type polynomials, Stirling numbers of the second kind, degenerate exponential function, umbral calculus.

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1. Introduction

Numerous mathematicians and physicists have examined different extensions of special functions. A new generation of analytical tools for solving large classes of partial differential equations, which are commonly coincided in physical problems, was provided by special functions with more than one variable in particular. Using various families of special polynomials, which provide new mathematical analysis methods, is the only way to treat most of these equations. The use of them is common in computational models of scientific and engineering problems. Therefore, they lead to the derivation of different utility

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identities in a simple manner and researchers are encouraged to consider the potential extensions of new families of special polynomials due to their motivation. For example, see [5, 6, 10, 11, 14, 17, 21, 22, 25]. Let $\lambda \in \mathbb{R} = (-\infty, \infty) \setminus \{0\}$. The λ-extension of falling factorial is provided as (cf. [9–15, 18–21])

$$(\gamma)_{\rho,\lambda} = \gamma(\gamma-\lambda)(\gamma-2\lambda)\cdots(\gamma-(\rho-1)\lambda), \ (\rho \in \mathbb{N}),$$

with $(0)_{\rho,\lambda} = 1$. Upon setting $\lambda = 1$, it is obtained that $(\gamma)_{\rho,1} := (\gamma)_\rho = \gamma(\gamma-1)(\gamma-2)\cdots(\gamma-(\rho-1))$, $(\rho \in \mathbb{N})$ (cf. [9–15, 18–21]). The $\Delta_{\lambda,\gamma}$ difference operator is provided as follows (cf. [11, 13])

$$\Delta_{\lambda,\gamma} f(\gamma) = \frac{f(\gamma+\lambda) - f(\gamma)}{\lambda}, \quad \lambda \neq 0.$$

Let $\lambda \in \mathbb{R}\setminus\{0\}$. The degenerate form of the exponential function is considered as (cf. [9–15, 18–21])

$$e_\lambda^\gamma(z) = (1+\lambda z)^{\frac{\gamma}{\lambda}} = \sum_{\rho=0}^{\infty} (\gamma)_{\rho,\lambda} \frac{z^\rho}{\rho!} \ \text{ and } \ e_\lambda^1(z) := e_\lambda(z),$$

which satisfies the following difference rule:

$$\Delta_{\lambda,\gamma}\left(e_\lambda^\gamma(z)\right) = z e_\lambda^\gamma(z).$$

It is worthy to note that that $\lim_{\lambda \to 0} e_\lambda^\gamma(z) = e^{\gamma z}$. The Stirling polynomials $S_2(\rho, \sigma : \gamma)$ of the second kind (abbreviated with *SPSK*) are provided as follows (*cf.* [2, 4, 5, 7, 17, 24]):

$$\sum_{\rho=0}^{\infty} S_2(\rho, \sigma : \gamma) \frac{z^\rho}{\rho!} = \frac{(e^z - 1)^\sigma}{\sigma!} e^{z\gamma}, \tag{1.1}$$

when $\gamma = 0$ in Eq. (1.1), we have $S_2(\rho, \sigma : 0) := S_2(\rho, \sigma)$ that is the Stirling numbers of the second kind (*SNSK*). These numbers possess important role in combinatorics, as they enumerate the number of ways in which ρ distinguishable objects can be partitioned into σ indistinguishable subsets when each subset has to contain at least one object. The following relation is valid (cf. [2, 4, 5, 7, 17, 24]):

$$\gamma^\rho = \sum_{\sigma=0}^{\rho} S_2(\rho, \sigma)(\gamma)_\sigma, \quad (\gamma \in \mathbb{N}_0). \tag{1.2}$$

The degenerate form of $S_2(\rho, \sigma)$ is considered (*DSPSK*) as (cf. [3, 9–11, 13, 16, 17, 19])

$$\sum_{\rho=0}^{\infty} S_{2,\lambda}(\rho, \sigma : \gamma) \frac{z^\rho}{\rho!} = \frac{(e_\lambda(z) - 1)^\sigma}{\sigma!} e_\lambda^\gamma(z), \tag{1.3}$$

taking $\gamma = 0$ in Eq. (1.3) yields $S_{2,\lambda}(\rho, \sigma : 0) := S_{2,\lambda}(\rho, \sigma)$ that is degenerate Stirling numbers of the second kind (*DSNSK*). A degenerate extension of (1.2) is given by (cf. [14])

$$(\gamma)_{\rho,\lambda} = \sum_{\sigma=0}^{\rho} S_{2,\lambda}(\rho, \sigma)(\gamma)_\sigma. \tag{1.4}$$

For $\sigma \geqslant 0$, the degenerate version of Stirling numbers $S_{1,\lambda}(\rho, \sigma)$ of the first kind (*DSNFK*) are considered as follows (cf. [10, 13, 20]):

$$\sum_{\rho=0}^{\infty} S_{1,\lambda}(\rho, \sigma) \frac{z^\rho}{\rho!} = \frac{(\log_\lambda(1+z))^\sigma}{\sigma!} e_\lambda^\gamma(z) \ \text{ and } \ (\gamma)_\rho = \sum_{\sigma=0}^{\rho} S_{1,\lambda}(\rho, \sigma)(\gamma)_{\sigma,\lambda}. \tag{1.5}$$

From (1.4) and (1.5), it is readily derived that

$$(\gamma)_{\rho,\lambda} = \sum_{\sigma=0}^{\rho} \sum_{\sigma=0}^{\sigma} S_{2,\lambda}(\rho,\sigma) S_{1,\lambda}(\sigma,\sigma)(\gamma)_{\sigma,\lambda}. \tag{1.6}$$

Let $\mathrm{Bel}_\rho(\gamma;\delta)$ be considered as the bivariate Bell polynomials or Bell polynomials of two variable (*BBellP*), which are provided by

$$\sum_{\rho=0}^{\infty} \mathrm{Bel}_\rho(\gamma;\delta)\frac{z^\rho}{\rho!} = e^{\delta(e^z-1)}e^{\gamma z}, \tag{1.7}$$

see [1, 6–8, 17, 23, 24] for applications of *BBellP* in combinatorics and analytic number theory. The Bell polynomials (*BellP*) are derived by $\mathrm{Bel}_\rho(0;\delta) := \mathrm{Bel}_\rho(\delta)$ and also provided by (*cf.* [1, 7, 8, 17, 24]):

$$\sum_{\rho=0}^{\infty} \mathrm{Bel}_\rho(\delta)\frac{z^\rho}{\rho!} = e^{\delta(e^z-1)}. \tag{1.8}$$

When $\delta = 1$ in (1.8), $\mathrm{Bel}_\rho(1) := \mathrm{Bel}_\rho$ denotes the Bell numbers (*BellN*) (cf. [1, 2, 7, 8, 17, 24]):

$$\sum_{\rho=0}^{\infty} \mathrm{Bel}_\rho\frac{z^\rho}{\rho!} = e^{(e^z-1)}. \tag{1.9}$$

The *BellP* [1] emerge in combinatorial analysis as a standard mathematical tool. These polynomials with several extensions have been extensively investigated since the first consideration of them. For example, Carlitz [2] gave differential formulas and relations for *BellP*. Duran et al. [7] introduced Bell-based Stirling polynomials of the second kind and considered Bell-based Bernoulli polynomials, then provided several properties, relations and applications. Then, Khan et al. [8] considered Bell-based Euler polynomials and examined some of their relations and identities. Kim et al. [19] introduced a novel degenerate kind of Bell polynomials and, by utilizing the λ-umbral calculus, gave several interesting formulas. Kim et al. [10] represented degenerate Bell polynomials by the other degenerate Sheffer polynomials such as the degenerate Euler and Bernoulli polynomials, arising from λ-umbral calculus,. Wang et al. [17] studied partial Bell polynomials and then, established several general identities involving Bell polynomials and Sheffer sequences. By (1.2) and (1.8), it is easily observed that (cf. [7])

$$\mathrm{Bel}_\rho(\gamma) = \sum_{\tau=0}^{\rho} S_2(\rho,\tau)\gamma^\tau. \tag{1.10}$$

A degenerate version of (1.7) (*DBBellP*) is considered as follows (cf. [11, 19, 20]):

$$\sum_{\rho=0}^{\infty} \mathrm{Bel}_{\rho,\lambda}(\gamma;\delta)\frac{z^\rho}{\rho!} = e_\lambda^\gamma(z)\, e_\lambda^\delta(e_\lambda(z)-1). \tag{1.11}$$

A degenerate form of (1.8) (*DBellP*) is defined as follows (cf. [11, 19, 20]):

$$\sum_{\rho=0}^{\infty} \mathrm{Bel}_{\rho,\lambda}(\delta)\frac{z^\rho}{\rho!} = e_\lambda^\delta(e_\lambda(z)-1). \tag{1.12}$$

A degenerate version of (1.9) (*DBellN*) is provided as follows

$$\sum_{\rho=0}^{\infty} \mathrm{Bel}_{\rho,\lambda}\frac{z^\rho}{\rho!} = e_\lambda(e_\lambda(z)-1). \tag{1.13}$$

It is noticed that

$$\lim_{\lambda\to 0} \mathrm{Bel}_{\rho,\lambda}(\gamma) = \mathrm{Bel}_{\rho}(\gamma).$$

We state degenerate extension of (1.10) as (cf. [19])

$$\mathrm{Bel}_{\rho,\lambda}(\gamma) = \sum_{\tau=0}^{\rho} S_{2,\lambda}(\rho,\tau)\,\gamma^{\tau}. \tag{1.14}$$

Let $B_{\rho}^{(\varepsilon)}(\gamma)$ be denoted the Bernoulli polynomials of order ε (*BPO*), which are provided by (cf. [4, 5, 7]):

$$\sum_{\rho=0}^{\infty} B_{\rho}^{(\varepsilon)}(\gamma)\frac{z^{\rho}}{\rho!} = \left(\frac{z}{e^{z}-1}\right)^{\varepsilon} e^{\gamma z}, \quad \left(|z|<2\pi;\ \varepsilon\in\mathbb{C};\ 1^{\varepsilon}:=1\right). \tag{1.15}$$

Setting $\gamma=0$, $\varepsilon=1$, and $\gamma+1=\varepsilon=1$ in (1.15), the polynomials $B_{\rho}^{(\varepsilon)}(\gamma)$ reduce to the Bernoulli numbers $B_{\rho}^{(\varepsilon)}$ of order ε (*BNO*), the classical Bernoulli polynomials $B_{\rho}(\gamma)$ (*BP*), and the classical Bernoulli numbers B_{ρ} (*BN*), respectively. The degenerating version of the (1.15) (*DBPO*), is given by (cf. [11, 13, 18, 20, 21]):

$$\sum_{\rho=0}^{\infty} B_{\rho,\lambda}^{(\varepsilon)}(\gamma)\frac{z^{\rho}}{\rho!} = \left(\frac{z}{e_{\lambda}(z)-1}\right)^{\varepsilon} e_{\lambda}^{\gamma}(z), \quad \left(|z|<2\pi\,|1+\lambda z|\,;\varepsilon\in\mathbb{C};1^{\varepsilon}:=1\right). \tag{1.16}$$

Upon setting $\gamma=0$, $\varepsilon=1$, and $\gamma+1=\varepsilon=1$ in (1.16), the polynomials $B_{\rho,\lambda}^{(\varepsilon)}(\gamma)$ reduce to the degenerate Bernoulli numbers $B_{\rho,\lambda}^{(\varepsilon)}$ of order ε (*DBNO*), the degenerate Bernoulli polynomials $B_{\rho,\lambda}(\gamma)$ (*DBP*), and the degenerate Bernoulli numbers $B_{\rho,\lambda}$ (*DBN*), respectively. Along the paper, we will use some abbreviations such as the degenerate Bernoulli numbers written by *DBN*.

In this work, a degenerate form of the Bell-based Stirling polynomials of the second kind is defined, and several fundamental properties and formulas for these polynomials are investigated and presented in detail. Then, a degenerate form of the Bell-based Bernoulli polynomials of order ε is defined and a plenty of their properties are examined in different aspects. Several correlations with other polynomials and numbers in literature, symmetric identities, implicit summation formulas, derivative properties and addition formulas for the mentioned new polynomials are derived in detail, and some special cases of these results are investigated. Also, the degenerate Bell-based Bernoulli polynomials of order ε are studied in λ-umbral calculus and interesting relations and formulas are developed. Furthermore, the application of λ-umbral calculus to Bell-based degenerate Bernoulli polynomials of order ε shows a correlation with higher-order degenerate derangement polynomials. Finally, a representation of the degenerate differential operator on the degenerate Bell-based Bernoulli polynomials of order ε is provided.

2. Degenerate Bell-based Stirling polynomials of the second kind

Here, we first consider a general degenerate class of *SPSK*. By making use of this newly defined polynomials, we obtain new identities and relations. Also we see that they are reduced to classical ones for the special cases. Here, we have one of our primary definitions.

Definition 2.1. The degenerate Bell-based extension of *SPSK* (abbreviated with *DBSPSK*) are defined as follows:

$$\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma,\delta)\frac{z^{\rho}}{\rho!} = \frac{\left(e_{\lambda}(z)-1\right)^{\sigma}}{\sigma!} e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta}\left(e_{\lambda}(z)-1\right). \tag{2.1}$$

Below are the various special circumstances of *DBSPSK* that are examined.

Remark 2.2. Choosing $\gamma = 0$ in (2.1) gives a new extension of *SPSK*, which we call the degenerate Bell-Stirling polynomials of the second kind, as follows:

$$\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\delta)\frac{z^{\rho}}{\rho!} = \frac{(e_{\lambda}(z)-1)^{\sigma}}{\sigma!}e_{\lambda}^{\delta}(e_{\lambda}(z)-1). \tag{2.2}$$

Remark 2.3. *DSPSK* (1.3) and *SNSK* (1.1) can be attained by letting $\lambda \to 0 = \delta$ and $\lambda \to 0 = \gamma = \delta$ in (2.1).

Several properties of *DBSPSK* are analyzed as follows.

Theorem 2.4. DBSPSK *satisfy the following connection with* DSNSK *and* DBBellP, *for* $\rho \in \mathbb{N}$*:*

$$ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma,\delta) = \sum_{\phi=0}^{\rho}\binom{\rho}{\phi}S_{2,\lambda}(\phi,\sigma)\,\mathrm{Bel}_{\rho-\phi,\lambda}(\gamma;\delta). \tag{2.3}$$

Proof. With the aid of (1.3), (1.13), and (2.1), the calculation indicates that

$$\begin{aligned}\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma,\delta)\frac{z^{\rho}}{\rho!} &= \frac{(e_{\lambda}(z)-1)^{\sigma}}{\sigma!}e_{\lambda}^{\gamma}(z)\,e_{\lambda}^{\delta}(e_{\lambda}(z)-1)\\ &= \sum_{\rho=\sigma}^{\infty}S_{2,\lambda}(\rho,\sigma)\frac{z^{\rho}}{\rho!}\sum_{\rho=0}^{\infty}\mathrm{Bel}_{\rho,\lambda}(\gamma;\delta)\frac{z^{\rho}}{\rho!}\\ &= \sum_{\rho=0}^{\infty}\sum_{\phi=0}^{\rho}\binom{\rho}{\phi}S_{2,\lambda}(\phi,\sigma)\,\mathrm{Bel}_{\rho-\phi,\lambda}(\gamma;\delta)\frac{z^{\rho}}{\rho!},\end{aligned}$$

which achieves the desired outcome (2.3). □

Remark 2.5. Theorem 2.4 means the following correlations covering *DBellP* (1.12), *DSNSK* (1.3), and the degenerate Bell-Stirling polynomials of the second kind (2.2):

$$ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\delta) = \sum_{\phi=0}^{\rho}\binom{\rho}{\phi}S_{2,\lambda}(\phi,\sigma)\,\mathrm{Bel}_{\rho-\phi,\lambda}(\delta).$$

Theorem 2.6. DBSPSK *fulfill the following connections with* DSPSK *and* DBellP, *for* $\rho,\sigma \in \mathbb{Z}_{\geqslant 0}$ *with* $\rho \geqslant \sigma$*:*

$$ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma,\delta) = \sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}\,{}_{\mathrm{Bel}}S_{2,\lambda}(\sigma,\sigma:\delta)\,(\gamma)_{\rho-\sigma,\lambda}$$

and

$$ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma,\delta) = \sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}S_{2,\lambda}(\sigma,\sigma:\gamma)\,\mathrm{Bel}_{\rho-\sigma,\lambda}(\delta).$$

Proof. The proof resembles Theorem 2.4. Thereby, we do not include the particulars. □

Theorem 2.7. DBSPSK *fulfill the following summation formulas associated with* DBellP, *for* $\rho,\sigma \in \mathbb{Z}_{\geqslant 0}$ *with* $\rho \geqslant \sigma$*:*

$$ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma_1+\gamma_2,\delta) = \sum_{\phi=0}^{\rho}\binom{\rho}{\phi}\,{}_{\mathrm{Bel}}S_{2,\lambda}(\phi,\sigma:\gamma_1,\delta)\,(\gamma_2)_{\rho-\phi,\lambda}$$

and

$$ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho,\sigma:\gamma,\delta_1+\delta_2) = \sum_{\phi=0}^{\rho}\binom{\rho}{\phi}\,{}_{\mathrm{Bel}}S_{2,\lambda}(\phi,\sigma:\gamma,\delta_1)\,\mathrm{Bel}_{\rho-\phi,\lambda}(\delta_2).$$

Proof. Through the following arrangements

$$e_{\lambda}^{\gamma_1+\gamma_2}(z)\, e_{\lambda}^{\delta}(e_{\lambda}(z)-1) = e_{\lambda}^{\gamma_1}(z)\, e_{\lambda}^{\delta}(e_{\lambda}(z)-1)\, e_{\lambda}^{\gamma_2}(z)$$

and

$$e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta_1+\delta_2}(e_{\lambda}(z)-1) = e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta_1}(e_{\lambda}(z)-1)\, e_{\lambda}^{\delta_2}(e_{\lambda}(z)-1),$$

the proof resembles to that of Theorem 2.4. Thereby, we do not include the particulars. □

Theorem 2.8. DBSPSK *satisfy the following addition formula related to* DSNSK (1.3)*, for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$_{\mathrm{Bel}}S_{2,\lambda}(\rho, \sigma_1+\sigma_2 : \gamma, \delta) = \frac{\sigma_1!\sigma_2!}{(\sigma_1+\sigma_2)!} \sum_{\phi=0}^{\rho} \binom{\rho}{\phi}\, _{\mathrm{Bel}}S_{2,\lambda}(\phi, \sigma_1 : \gamma, \delta)\, S_{2,\lambda}(\rho-\phi, \sigma_2). \tag{2.4}$$

Proof. With the aid of (1.3) and (2.1), the computations shows that

$$\begin{aligned}
\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}S_{2,\lambda}(\rho, \sigma_1+\sigma_2 : \gamma, \delta) \frac{z^{\rho}}{\rho!} &= \frac{(e_{\lambda}(z)-1)^{\sigma_1+\sigma_2}}{(\sigma_1+\sigma_2)!} e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta}(e_{\lambda}(z)-1) \\
&= \frac{\sigma_1!\sigma_2!}{(\sigma_1+\sigma_2)!} \frac{(e_{\lambda}(z)-1)^{\sigma_2}}{\sigma_1!} e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta}(e_{\lambda}(z)-1) \frac{(e_{\lambda}(z)-1)^{\sigma_2}}{\sigma_2!} \\
&= \frac{\sigma_1!\sigma_2!}{(\sigma_1+\sigma_2)!} \sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}S_{2,\lambda}(\rho, \sigma_1 : \gamma, \delta) \frac{z^{\rho}}{\rho!} \sum_{\rho=0}^{\infty} S_{2,\lambda}(\rho, \sigma_2) \frac{z^{\rho}}{\rho!} \\
&= \frac{\sigma_1!\sigma_2!}{(\sigma_1+\sigma_2)!} \sum_{\rho=0}^{\infty} \sum_{\phi=0}^{\rho} \binom{\rho}{\phi} {}_{\mathrm{Bel}}S_{2,\lambda}(\phi, \sigma_1 : \gamma, \delta)\, S_{2,\lambda}(\rho-\phi, \sigma_2) \frac{z^{\rho}}{\rho!},
\end{aligned}$$

which achieves the claimed outcome (2.4). □

Theorem 2.9. DBSPSK, DBellP*, and* DSNSK *fulfill the following correlation for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$S_{2,\lambda}(\rho, \sigma) = \sum_{\phi=0}^{\rho} \binom{\rho}{\phi}\, _{\mathrm{Bel}}S_{2,\lambda}(\phi, \sigma : \gamma, \delta)\, \mathrm{Bel}_{\rho-\phi,\lambda}(-\gamma; -\delta).$$

Proof. Through the following organizing

$$\frac{(e_{\lambda}(z)-1)^{\sigma}}{\sigma!} = \frac{(e_{\lambda}(z)-1)^{\sigma}}{\sigma!} e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta}(e_{\lambda}(z)-1)\, e_{\lambda}^{-\gamma}(z)\, e_{\lambda}^{-\delta}(e_{\lambda}(z)-1),$$

the proof resembles to that of Theorem 2.4. Hence, we do not include the particulars. □

Remark 2.10. Theorem 2.9 means the following correlations covering *DSNSK* (1.3), the degenerate Bell-Stirling polynomials of the second kind (2.2) and *DBellP* (1.12):

$$S_{2,\lambda}(\rho, \sigma) = \sum_{\phi=0}^{\rho} \binom{\rho}{\phi}\, _{\mathrm{Bel}}S_{2,\lambda}(\phi, \sigma : -\delta)\, \mathrm{Bel}_{\rho-\phi,\lambda}(\delta).$$

3. Degenerate Bell-based Bernoulli polynomials and numbers of order ε

Let $B_{\rho}^{(\varepsilon)}(\gamma,\delta)$ indicate the Bell-based Bernoulli polynomials of order ε (*DBPO*), which are provided by (*cf.* [7]):

$$\sum_{\rho=0}^{\infty} B_{\rho}^{(\varepsilon)}(\gamma,\delta)\frac{z^{\rho}}{\rho!}=\left(\frac{z}{e^{z}-1}\right)^{\varepsilon} e^{\gamma z}e^{\delta(e^{z}-1)},\quad (|z|<2\pi;\ \varepsilon\in\mathbb{C};\ 1^{\varepsilon}:=1). \tag{3.1}$$

These polynomials have been studied and analyzed in detail in [7]. Also, applications in umbral calculus have been considered and many interesting formulas have been investigated.

In this chapter, a degenerate form of *DBPO* (3.1) is introduced, and many formulas and properties are intensely derived. Here, we have one of the other our primary definitions.

Definition 3.1. We define the degenerate Bell-based Bernoulli polynomials of order ε (abbreviated with *DBBPO*) as follows:

$$\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^{\rho}}{\rho!}=\left(\frac{z}{e_{\lambda}(z)-1}\right)^{\varepsilon} e_{\lambda}^{\gamma}(z)\, e_{\lambda}^{\delta}(e_{\lambda}(z)-1),\quad (|z|<2\pi|1+\lambda z|;\ \varepsilon\in\mathbb{C};\ 1^{\varepsilon}:=1). \tag{3.2}$$

Below is an analysis of specific cases of *DBBPO*.

Remark 3.2. In the case $\lambda\to 0$, *DBBPO* (3.2) reduces to *DBPO* (3.1).

Remark 3.3. The degenerate Bell-Bernoulli polynomials ${}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\delta)$ of order ε are considered in the case when $\gamma=0$ in (3.2):

$$\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\delta)\frac{z^{\rho}}{\rho!}=\left(\frac{z}{e_{\lambda}(z)-1}\right)^{\varepsilon} e_{\lambda}^{\delta}(e_{\lambda}(z)-1). \tag{3.3}$$

Remark 3.4. *DBBPO* (3.2) reduces to *DBPO* (1.16) in the case when $\delta=0$.

Remark 3.5. *DBBPO* (3.2) reduces to *BP* (1.15) in the case when $\lambda\to 0$, $\delta=0$, and $\varepsilon=1$.

It is also noted that

$${}_{\mathrm{Bel}}B_{\rho,\lambda}^{(1)}(\gamma;\delta):={}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta),$$

which we name the Bell-based degenerate Bernoulli polynomials (abbreviated with *DBBP*).

Our aim now involves elaborating on certain properties of *DBBPO*.

Theorem 3.6. DBBPO *fulfill the following formulas related to* DBBellP (1.11), DBellP (1.12), DBPO (1.16)*), and degenerate Bell-Bernoulli polynomials of order* ε (3.3) *for* $\rho\in\mathbb{Z}_{\geqslant 0}$*:*

$${}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)=\sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}B_{\sigma,\lambda}^{(\varepsilon)}\mathrm{Bel}_{\rho-\sigma,\lambda}(\gamma;\delta), \tag{3.4}$$

$${}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)=\sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}B_{\sigma,\lambda}^{(\varepsilon)}(\gamma)\,\mathrm{Bel}_{\rho-\sigma,\lambda}(\delta), \tag{3.5}$$

$${}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)=\sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}{}_{\mathrm{Bel}}B_{\sigma,\lambda}^{(\varepsilon)}(\delta)\,(\gamma)_{\rho-\sigma,\lambda}. \tag{3.6}$$

Proof. The proving process of (3.4)-(3.6) resembles to the proof of Theorem 2.4. Thereby, we do not include the particulars. □

An addition formula for *DBBPO* is provided as follows.

Theorem 3.7. DBBPO *fulfill the following formula for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$ {}_{\text{Bel}}B^{(\varepsilon_1+\varepsilon_2)}_{\rho,\lambda}(\gamma_1+\gamma_2;\delta_1+\delta_2) = \sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}\,{}_{\text{Bel}}B^{(\varepsilon_1)}_{\sigma,\lambda}(\gamma_1;\delta_1)\,{}_{\text{Bel}}B^{(\varepsilon_2)}_{\rho-\sigma,\lambda}(\gamma_2;\delta_2). \tag{3.7}$$

Proof. Through the following organizing

$$\begin{aligned}&\left(\frac{z}{e_\lambda(z)-1}\right)^{\varepsilon_1+\varepsilon_2} e_\lambda^{\gamma_1+\gamma_2}(z)\, e_\lambda^{\delta_1+\delta_2}(e_\lambda(z)-1)\\ &= \left(\frac{z}{e_\lambda(z)-1}\right)^{\varepsilon_1} e_\lambda^{\gamma_1}(z)\, e_\lambda^{\delta_1}(e_\lambda(z)-1)\left(\frac{z}{e_\lambda(z)-1}\right)^{\varepsilon_2} e_\lambda^{\gamma_2}(z)\, e_\lambda^{\delta_2}(e_\lambda(z)-1),\end{aligned}$$

the proofing process of (3.7) resembles to that of Theorem 2.4. Hence, we do not include the particulars. □

Below is an example of a special case of Theorem 3.7:

$$ {}_{\text{Bel}}B^{(\varepsilon)}_{\rho,\lambda}(\gamma+1;\delta) = \sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}\,{}_{\text{Bel}}B^{(\varepsilon)}_{\sigma,\lambda}(\gamma;\delta)\,(1)_{\rho-\sigma,\lambda}.$$

A difference operator rule for *DBBPO* is as follows.

Theorem 3.8. DBBPO *fulfill the following difference operator formula for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$\Delta_{\lambda,\gamma}\,{}_{\text{Bel}}B^{(\varepsilon)}_{\rho,\lambda}(\gamma;\delta) = \rho\,{}_{\text{Bel}}B^{(\varepsilon)}_{\rho-1,\lambda}(\gamma;\delta).$$

Proof. Through the following organizing difference property:

$$\Delta_{\lambda,\gamma} e_\lambda^{\gamma}(z)\, e_\lambda^{\delta}(e_\lambda(z)-1) = z e_\lambda^{\gamma}(z)\, e_\lambda^{\delta}(e_\lambda(z)-1),$$

the proof is done. □

A relationship for *DBBPO* is given as follows.

Theorem 3.9. DBBPO *fulfill the following identity associated with* DBBellP (1.11)*, for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$\text{Bel}_{\rho,\lambda}(\gamma;\delta) = \frac{{}_{\text{Bel}}B_{\rho+1,\lambda}(\gamma+1;\delta) - {}_{\text{Bel}}B_{\rho+1,\lambda}(\gamma;\delta)}{\rho+1} = \frac{1}{\rho+1}\sum_{\sigma=0}^{\rho}\binom{\rho+1}{\sigma}\,{}_{\text{Bel}}B^{(\varepsilon)}_{\sigma,\lambda}(\gamma;\delta)\,(1)_{\rho-\sigma,\lambda}.$$

Proof. Through the following arrangement

$$e_\lambda^{\gamma}(z)\, e_\lambda^{\delta}(e_\lambda(z)-1) = \frac{e_\lambda(z)-1}{z}\sum_{\rho=0}^{\infty}{}_{\text{Bel}}B_{\rho,\lambda}(\gamma;\delta)\frac{z^\rho}{\rho!},$$

the proof is completed with the aid of Definition 3.1. □

We now present an explicit formula for *DBBP.*

Theorem 3.10. DBBP *fulfill the following relation for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$ {}_{\text{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma-1}\binom{\sigma-1}{\sigma}\frac{(-1)^{\sigma-\sigma-1}}{\sigma!}\frac{(\delta)_{\sigma,\lambda}(\sigma+\gamma)_{\rho+1,\lambda}}{\rho+1}$$

holds for $\rho \in \mathbb{N}_0$.

Proof. With the aid of Definition 3.1, the calculations indicate that

$$\begin{aligned}\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta)\frac{z^{\rho}}{\rho!} &= \frac{ze_{\lambda}^{\gamma}(z)}{e_{\lambda}(z)-1}e_{\lambda}^{\delta}(e_{\lambda}(z)-1)\\ &= ze_{\lambda}^{\gamma}(z)\sum_{\sigma=0}^{\infty}(\delta)_{\sigma,\lambda}\frac{(e_{\lambda}(z)-1)^{\sigma-1}}{\sigma!}\\ &= z\sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma-1}(\delta)_{\sigma,\lambda}\binom{\sigma-1}{\sigma}\frac{(-1)^{\sigma-\sigma-1}}{\sigma!}e_{\lambda}^{\sigma+\gamma}(z)\\ &= \sum_{\rho=0}^{\infty}\sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma-1}(\delta)_{\sigma,\lambda}\binom{\sigma-1}{\sigma}\frac{(-1)^{\sigma-\sigma-1}}{\sigma!}(\sigma+\gamma)_{\rho,\lambda}\frac{z^{\rho-1}}{\rho!},\end{aligned}$$

which achieves the desired outcome. □

Theorem 3.11. DBBPO *fulfill the following formula related to* DSNSK (1.4) *and* DBBellP (1.11) *for* $\rho \in \mathbb{Z}_{\geqslant 0}$ *and* $\sigma \in \mathbb{Z}_{>0}$*:*

$$\mathrm{Bel}_{\rho,\lambda}(\gamma;\delta) = \frac{\rho!\sigma!}{(\rho+\sigma)!}\sum_{\sigma=0}^{\rho+\sigma}\binom{\rho+\sigma}{\sigma}S_{2,\lambda}(\rho+\sigma-\sigma,\tau)\ {}_{\mathrm{Bel}}\mathrm{B}_{\sigma,\lambda}^{(-\sigma)}(\gamma;\delta).$$

Proof. By the following organizing

$$e_{\lambda}^{\gamma}(z)\,e_{\lambda}^{\delta}(e_{\lambda}(z)-1) = \sigma!z^{-\sigma}\frac{(e_{\lambda}(z)-1)^{\sigma}}{\sigma!}\sum_{\rho=0}^{\infty}{}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}^{(-\sigma)}(\gamma;\delta)\frac{z^{\rho}}{\rho!},$$

the proof is over with the aid of Definition 3.1. □

Now, we provide the following theorem covering *DBBPO* and *DSNSK*.

Theorem 3.12. DBBPO *satisfy the following correlation associated with* DSNSK (1.4) *for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$${}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta) = \sum_{\sigma=0}^{\rho}\sum_{\sigma=0}^{\infty}\binom{\rho}{\sigma}(\gamma)_{\sigma}S_{2,\lambda}(\sigma,\sigma)\ {}_{\mathrm{Bel}}\mathrm{B}_{\rho-\sigma,\lambda}^{(\varepsilon)}(\delta). \tag{3.8}$$

Proof. With the aid of (1.3), (3.2), and (3.3), the calculations show that

$$\begin{aligned}\sum_{\rho=0}^{\infty}{}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^{\rho}}{\rho!} &= \frac{z^{\varepsilon}}{(e_{\lambda}(z)-1)^{\varepsilon}}e_{\lambda}^{\delta}(e_{\lambda}(z)-1)(e_{\lambda}(z)-1+1)^{\gamma}\\ &= \frac{z^{\varepsilon}}{(e_{\lambda}(z)-1)^{\varepsilon}}e_{\lambda}^{\delta}(e_{\lambda}(z)-1)\sum_{\sigma=0}^{\infty}(\gamma)_{\sigma}\frac{(e_{\lambda}(z)-1)^{\sigma}}{\sigma!}\\ &= \sum_{\rho=0}^{\infty}\sum_{\sigma=0}^{\rho}\sum_{\sigma=0}^{\infty}\binom{\rho}{\sigma}(\gamma)_{\sigma}S_{2,\lambda}(\sigma,\sigma)\ {}_{\mathrm{Bel}}\mathrm{B}_{\rho-\sigma,\lambda}^{(\varepsilon)}(\delta)\frac{z^{\rho}}{\rho!},\end{aligned}$$

which achieves the intended consequence (3.8). □

Theorem 3.13. DBBPO *fulfill the following addition formula associated with* DSNSK (1.4) *for* $\rho,\sigma \in \mathbb{Z}_{\geqslant 0}$ *with* $\rho \geqslant \sigma$*:*

$$\mathrm{Bel}_{\rho,\lambda}(\gamma_1+\gamma_2;\delta_1+\delta_2) = \frac{\rho!\sigma!}{(\rho+\sigma)!}\sum_{\sigma=0}^{\rho+\sigma}\binom{\rho+\sigma}{\sigma}\ {}_{\mathrm{Bel}}\mathrm{B}_{\sigma,\lambda}^{(\sigma)}(\gamma_2;\delta_2)\ {}_{\mathrm{Bel}}S_{2,\lambda}(\rho+\sigma-\sigma,\sigma:\gamma_1,\delta_1). \tag{3.9}$$

Proof. With the aid of (1.3), (1.13), and (2.1), the calculation indicates that

$$\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}S_{2,\lambda}\left(\rho,\sigma:\gamma_1,\delta_1\right)\frac{z^\rho}{\rho!}\sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}B_{\rho}^{(\sigma)}\left(\gamma_2;\delta_2\right)\frac{z^\rho}{\rho!}=\frac{z^\sigma}{\sigma!}e_\lambda^{\gamma_1+\gamma_2}(z)\,e_\lambda^{\delta_1+\delta_2}\left(e_\lambda(z)-1\right),$$

which achieves the intended consequence (3.9). □

The following series manipulation formulas hold (cf. [7, 8]):

$$\sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma} A\left(\sigma,\sigma-\sigma\right)=\sum_{\sigma,\sigma=0}^{\infty} A\left(\sigma,\sigma\right) \tag{3.10}$$

and

$$\sum_{\rho,\tau=0}^{\infty} f(\rho+\tau)\frac{\gamma^\rho}{\rho!}\frac{\delta^\tau}{\tau!}=\sum_{N=0}^{\infty} f(N)\frac{(\gamma+\delta)^N}{N!}. \tag{3.11}$$

Then, the following implicit summation formula is presented.

Theorem 3.14. DBBPO *fulfill the following relationship for* $\rho\in\mathbb{Z}_{\geqslant 0}$*:*

$$ {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\gamma;\delta)=\sum_{\rho,\tau=0}^{\sigma,\sigma}\binom{\sigma}{\rho}\binom{\sigma}{\tau}\,{}_{\mathrm{Bel}}B_{\sigma+\sigma-\rho-\tau,\lambda}^{(\varepsilon)}(\omega;\delta)\left(\gamma-\omega\right)_{\rho+\tau,\lambda}. \tag{3.12}$$

Proof. Substituting z by $z+\phi$ in (3.2), and with the help of (3.11), it is obtained that

$$e_\lambda^{-\omega}(z+\phi)\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\omega;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!}=\left(\frac{z+\phi}{e_\lambda(z+\phi)-1}\right)^{\varepsilon}e_\lambda^{\delta}\left(e_\lambda(z+\phi)-1\right).$$

Again substituting ω by γ in the previous equation, it is attained that

$$e_\lambda^{-\gamma}(z+\phi)\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!}=\left(\frac{z+\phi}{e_\lambda(z+\phi)-1}\right)^{\varepsilon}e_\lambda^{\delta}\left(e_\lambda(z+\phi)-1\right).$$

Through the previous two equalities, it is computed that

$$e_\lambda^{\gamma-\omega}(z+\phi)\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\omega;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!}=\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!},$$

which means

$$\sum_{\rho,\tau=0}^{\infty}\left(\gamma-\omega\right)_{\rho+\tau,\lambda}\frac{z^\rho}{\rho!}\frac{\phi^\tau}{\tau!}\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\omega;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!}=\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!}.$$

With the aid of (3.10), it is observed that

$$\sum_{\sigma,\sigma=0}^{\infty}\sum_{\rho,\tau=0}^{\sigma,\sigma}\frac{\left(\gamma-\omega\right)_{\rho+\tau,\lambda}\,{}_{\mathrm{Bel}}B_{\sigma+\sigma-\rho-\tau,\lambda}^{(\varepsilon)}(\omega;\delta)}{\rho!\tau!\,(\sigma-\sigma)!\,(\sigma-\tau)!}z^\sigma\phi^\sigma=\sum_{\sigma,\sigma=0}^{\infty} {}_{\mathrm{Bel}}B_{\sigma+\sigma,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^\sigma}{\sigma!}\frac{\phi^\sigma}{\sigma!},$$

which achieves the intended consequence (3.12). □

Here, we provide some symmetric identities for *DBBPO* below.

Theorem 3.15. DBBPO *fulfill the following identity for* $\rho \in \mathbb{Z}_{\geqslant 0}$ *and* $\varkappa, b \in \mathbb{R}$ *and* $\rho \geqslant 0$*:*

$$\sum_{\sigma=0}^{\rho} \binom{\rho}{\sigma} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma,\rho\lambda}(\rho\gamma;\delta)\ {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\sigma,\varkappa\lambda}(\varkappa\gamma;\delta)\,\varkappa^{\rho-\sigma}b^{\sigma} = \sum_{\sigma=0}^{\rho} \binom{\rho}{\sigma} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\sigma,\rho\lambda}(\rho\gamma;\delta)\ {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma,\varkappa\lambda}(\varkappa\gamma;\delta)\,\varkappa^{\sigma}b^{\rho-\sigma}. \tag{3.13}$$

Proof. We choose that

$$\Upsilon = \left(\frac{z^2}{\left(e_{\rho\lambda}(\varkappa z)-1\right)\left(e_{\varkappa\lambda}(\rho z)-1\right)}\right)^{\varepsilon} e_{\lambda}^{2\gamma}(\varkappa b z)\, e_{\rho\lambda}^{\delta}\left(e_{\rho\lambda}(\varkappa z)-1\right) e_{\varkappa\lambda}^{\delta}\left(e_{\varkappa\lambda}(\rho z)-1\right),$$

which is symmetric in $\varkappa$ and ρ. We compute that

$$\begin{aligned}\Upsilon &= \sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho,\rho\lambda}(\rho\gamma;\delta)\frac{(\varkappa z)^{\rho}}{\rho!} \sum_{\rho=0}^{\infty} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho,\varkappa\lambda}(\varkappa\gamma;\delta)\frac{(\rho z)^{\rho}}{\rho!} \\ &= \sum_{\rho=0}^{\infty}\left(\sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma,\rho\lambda}(\rho\gamma;\delta)\ {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\sigma,\varkappa\lambda}(\varkappa\gamma;\delta)\,\varkappa^{\rho-\sigma}b^{\sigma}\right)\frac{z^{\rho}}{\rho!}\end{aligned}$$

and in the same way

$$\Upsilon = \sum_{\rho=0}^{\infty}\left(\sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\sigma,\rho\lambda}(\rho\gamma;\delta)\ {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma,\varkappa\lambda}(\varkappa\gamma;\delta)\,\varkappa^{\sigma}b^{\rho-\sigma}\right)\frac{z^{\rho}}{\rho!},$$

which means the assertion (3.13). □

Here is another symmetric identity for ${}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho,\lambda}(\gamma;\delta)$ as follows.

Theorem 3.16. DBBPO *fulfill the following identity for* $\rho \in \mathbb{Z}_{\geqslant 0}$ *and* $\varkappa, b \in \mathbb{R}$ *and* $\rho \geqslant 0$*:*

$$\sum_{\sigma=0}^{\rho}\sum_{i=0}^{\rho-1}\sum_{j=0}^{\varkappa-1}\binom{\rho}{\sigma} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\sigma,\rho\lambda}\left(i+\frac{\rho}{\varkappa}j+\rho\gamma_1;\delta\right) {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma,\varkappa\lambda}(\varkappa\gamma_2;\delta)\,\rho^{\sigma}\varkappa^{\rho-\sigma} = \sum_{\sigma=0}^{\rho}\sum_{i=0}^{\varkappa-1}\sum_{j=0}^{\rho-1}\binom{\rho}{\sigma} {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\sigma,\varkappa\lambda}\left(i+\frac{\varkappa}{\rho}j+\varkappa\gamma_1;\delta\right) {}_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma,\rho\lambda}(\rho\gamma_2;\delta)\,\varkappa^{\sigma}b^{\rho-\sigma}. \tag{3.14}$$

Proof. We compute that

$$\begin{aligned}\Psi &= \frac{(\varkappa z)^{\varepsilon}(\rho z)^{\varepsilon}}{\left(e_{\rho\lambda}(\varkappa z)-1\right)^{\varepsilon+1}\left(e_{\varkappa\lambda}(\rho z)-1\right)^{\varepsilon+1}}\left(e_{\lambda}(\varkappa b z)-1\right)^2 \\ &\quad\times e_{\lambda}^{\gamma_1+\gamma_2}(\varkappa b z)\, e_{\rho\lambda}^{\delta}\left(e_{\rho\lambda}(\varkappa z)-1\right) e_{\varkappa\lambda}^{\delta}\left(e_{\varkappa\lambda}(\rho z)-1\right) \\ &= \left(\frac{\varkappa z}{e_{\rho\lambda}(\varkappa z)-1}\right)^{\varepsilon}\left(\frac{e_{\lambda}(\varkappa b z)-1}{e_{\rho\lambda}(\varkappa z)-1}\right) e_{\lambda}^{\gamma_1}(\varkappa b z)\, e_{\rho\lambda}^{\delta}\left(e_{\rho\lambda}(\varkappa z)-1\right) \\ &\quad\times\left(\frac{\rho z}{e_{\varkappa\lambda}(\rho z)-1}\right)^{\varepsilon}\left(\frac{e_{\lambda}(\varkappa b z)-1}{e_{\varkappa\lambda}(\varkappa z)-1}\right) e_{\lambda}^{\gamma_2}(\varkappa b z)\, e_{\varkappa\lambda}^{\delta}\left(e_{\varkappa\lambda}(\rho z)-1\right).\end{aligned}$$

With the help of (3.2), it is seen that

$$\Psi = \left(\frac{\varkappa z}{e_{\rho\lambda}(\varkappa z)-1}\right)^{\varepsilon}\sum_{i=0}^{\rho-1} e_{\rho\lambda}^{i}(\varkappa z)\, e_{\rho\lambda}^{\rho\gamma_1}(\varkappa z)\, e_{\rho\lambda}^{\delta}\left(e_{\rho\lambda}(\varkappa z)-1\right)$$

$$\times\left(\frac{\rho z}{e_{\varkappa\lambda}(\rho z)-1}\right)^{\varepsilon}\sum_{j=0}^{\varkappa-1}e_{\varkappa\lambda}^{j}(\rho z)\,e_{\varkappa\lambda}^{\varkappa\gamma_2}(\rho z)\,e_{\varkappa\lambda}^{\delta}\left(e_{\varkappa\lambda}(\rho z)-1\right)$$

$$=\sum_{i=0}^{\rho-1}\sum_{j=0}^{\varkappa-1}\left(\frac{\varkappa z}{e_{\rho\lambda}(\varkappa z)-1}\right)^{\varepsilon}e_{\rho\lambda}^{i+\rho\gamma_1}(\varkappa z)\,e_{\rho\lambda}^{\delta}\left(e_{\rho\lambda}(\varkappa z)-1\right)$$

$$\times\left(\frac{\rho z}{e_{\varkappa\lambda}(\rho z)-1}\right)^{\varepsilon}e_{\varkappa\lambda}^{j+\varkappa\gamma_2}(\rho z)\,e_{\varkappa\lambda}^{\delta}\left(e_{\varkappa\lambda}(\rho z)-1\right)$$

$$=\sum_{\rho=0}^{\infty}\left(\sum_{\sigma=0}^{\rho}\sum_{i=0}^{\rho-1}\sum_{j=0}^{\varkappa-1}\binom{\rho}{\sigma}{}_{\mathrm{Bel}}B_{\sigma,\rho\lambda}^{(\varepsilon)}\left(i+\frac{\rho}{\varkappa}j+\rho\gamma_1;\delta\right){}_{\mathrm{Bel}}B_{\rho-\sigma,\varkappa\lambda}^{(\varepsilon)}(\varkappa\gamma_2;\delta)\,\rho^{\sigma}\varkappa^{\rho-\sigma}\right)\frac{z^{\rho}}{\rho!}$$

and similarly,

$$\Psi=\sum_{\rho=0}^{\infty}\left(\sum_{\sigma=0}^{\rho}\sum_{i=0}^{\varkappa-1}\sum_{j=0}^{\rho-1}\binom{\rho}{\sigma}{}_{\mathrm{Bel}}B_{\sigma,\varkappa\lambda}^{(\varepsilon)}\left(i+\frac{\varkappa}{\rho}j+\varkappa\gamma_1;\delta\right){}_{\mathrm{Bel}}B_{\rho-\sigma,\rho\lambda}^{(\varepsilon)}(\rho\gamma_2;\delta)\,\varkappa^{\sigma}b^{\rho-\sigma}\right)\frac{z^{\rho}}{\rho!},$$

which yields the claimed result (3.14). □

4. Applications

4.1. Applications in λ-Umbral calculus

The umbral calculus has been founded by Rota and has mathematical tools such as linear functionals, differential operators and Sheffer sequences. Recently, the "λ-umbral calculus" has been founded by Kim et al. [11] with the following question: *What if the familiar exponential function appearing in Sheffer sequences property of umbral calculus is changed to degenerate exponential functions*? This asking gave rise to the new concept which is called λ-umbral calculus or known as degenerate umbral calculus. The concept of λ-umbral calculus is reviewed below.

Let $\mathbb{P}$ be the algebra of polynomials in the single variable γ over $\mathbb{C}$ and

$$\mathcal{F}=\left\{f(z)=\sum_{\sigma=0}^{\infty}\varkappa_{\sigma}\frac{z^{\sigma}}{\sigma!}\mid\varkappa_{\sigma}\in\mathbb{C}\right\}.$$

In this calculus, $\langle L|p(\gamma)\rangle_{\lambda}$ refers the action of a λ-linear functional L on the polynomial $p(\gamma)$. A λ-linear functional on $\mathbb{P}$ is defined by setting

$$\left\langle f(z)|\,(\gamma)_{\rho,\lambda}\right\rangle_{\lambda}=\varkappa_{\rho},\quad(\rho\geqslant 0),\tag{4.1}$$

where

$$f(z)=\sum_{\sigma=0}^{\infty}\varkappa_{\sigma}\frac{z^{\sigma}}{\sigma!}.$$

Choosing $f(z)=z^{\sigma}$ in (4.1) yields

$$\left\langle z^{\sigma}|\,(\gamma)_{\rho,\lambda}\right\rangle_{\lambda}=\rho!\delta_{\rho,\sigma},\quad(\rho,\sigma\geqslant 0),\tag{4.2}$$

where $\delta_{\rho,\sigma}=1$ for $\rho=\sigma$ and $\delta_{\rho,\sigma}=1$ for $\rho\neq\sigma$. The λ-differential operator on $\mathbb{P}$ are introduced as follows

$$(z^{\sigma})_{\lambda}(\gamma)_{\rho,\lambda}=\begin{cases}(\rho)_{\sigma}(\gamma)_{\rho-\sigma,\lambda}, & \text{if }\rho\geqslant\sigma\geqslant 0,\\ 0, & \text{if }\rho<\sigma.\end{cases}\tag{4.3}$$

From (4.3), it is derived that for $f(z)=\sum_{\sigma=0}^{\infty}\varkappa_\sigma \frac{z^\sigma}{\sigma!}\in\mathcal{F}$:

$$(f(z))_\lambda (\gamma)_{\rho,\lambda}=\sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}\varkappa_\sigma(\gamma)_{\rho-\sigma,\lambda},\quad(\rho\geqslant 0),$$

and for $\rho\geqslant 0$,

$$\left(e_\lambda^\delta(z)\right)_\lambda(\gamma)_{\rho,\lambda}=(\gamma+\delta)_{\rho,\lambda},\tag{4.4}$$

which yields $\left(e_\lambda^\delta(z)\right)_\lambda p(\gamma)=p(\gamma+\delta)$. A series $f(z)$ with $o(f(z))=1$ and $o(f(z))=0$ is termed a delta series and an invertible series, respectively, where the order $o(f(z))$ of a power series $f(z)$ is the smallest integer σ for which the coefficient of z^σ does not vanish (cf. [24]). Let $g(z)$ and $f(z)$ be an invertible series and a delta series, respectively. Then there exists a unique sequence $S_{\rho,\lambda}(\gamma)$ (of degree ρ) of polynomials fulfilling the orthogonality circumstances

$$\left\langle g(z)(f(z))^\sigma \,|S_{\rho,\lambda}(\gamma)\right\rangle_\lambda=\rho!\delta_{\rho,\sigma},\tag{4.5}$$

where the sequence $S_{\rho,\lambda}(\gamma)$ is Sheffer for $(g(z),f(z))_\lambda$, and is termed the λ-Sheffer sequence for the pair of $(g(z),f(z))_\lambda$, shown by

$$S_{\rho,\lambda}(\gamma)\sim(g(z),f(z))_\lambda.$$

It is noted that for all $\gamma\in\mathbb{C}$:

$$S_{\rho,\lambda}(\gamma)\sim(g(z),f(z))_\lambda\Leftrightarrow\sum_{\rho=0}^{\infty}S_{\rho,\lambda}(\gamma)\frac{z^\rho}{\rho!}=\frac{1}{g(\bar{f}(z))}e_\lambda^\gamma\left(\bar{f}(z)\right),\tag{4.6}$$

where $f\left(\bar{f}(z)\right)=\bar{f}(f(z))=z$. For instance, $(\gamma)_{\rho,\lambda}$ is λ-Sheffer for $(1,z)$ shown by $(\gamma)_{\rho,\lambda}\sim(1,z)_\lambda$. $B_{\rho,\lambda}^{(1)}(\gamma)$ is λ-Sheffer for $\left(\frac{e_\lambda(z)-1}{z},z\right)$ shown by $B_{\rho,\lambda}^{(1)}(\gamma)\sim\left(\frac{e_\lambda(z)-1}{z},z\right)_\lambda$. We observe that for $\rho\geqslant 1$: $(f(z))_\lambda S_{\rho,\lambda}(\gamma)=\rho S_{\rho,\lambda}(\gamma)$, where the sequence $S_{\rho,\lambda}(\gamma)$ is Sheffer for $(g(z),f(z))_\lambda$. For $h(z)$, $f(z)$, $g(z)$, $\sigma(z)\in\mathcal{F}$, and $S_{\rho,\lambda}(\gamma)\sim(g(z),f(z))_\lambda$, $r_{\rho,\lambda}(\gamma)\sim(h(z),\sigma(z))_\lambda$, it is obtained that

$$S_{\rho,\lambda}(\gamma)=\sum_{\sigma=0}^{\rho}r_{\sigma,\lambda}(\gamma)C_{\rho,\sigma},\quad(\rho\geqslant 0),\tag{4.7}$$

where

$$C_{\rho,\sigma}=\frac{1}{\sigma!}\left\langle\frac{h\left(\bar{f}(z)\right)}{g\left(\bar{f}(z)\right)}\sigma\left(\bar{f}(z)\right)^\sigma\Big|(\gamma)_{\rho,\lambda}\right\rangle_\lambda.$$

For $S_{\rho,\lambda}(\gamma)$ being Sheffer for $(g(z),f(z))_\lambda$, then

$$S_{\rho,\lambda}(\gamma)=\sum_{j=0}^{\rho}\frac{1}{j!}\left\langle\frac{1}{g\left(\bar{f}(z)\right)}\left(\bar{f}(z)\right)^j\Big|(\gamma)_{\rho,\lambda}\right\rangle_\lambda(\gamma)_{\sigma,\lambda}.$$

One can see [10–13, 18–20] and cited references therein for more details of λ-umbral calculus theory. Recently, Kim et al. [11] defined and investigated λ-Sheffer sequences and as examples, the authors worked degenerate Bernoulli polynomials and degenerate Euler polynomials under the theory of λ-umbral calculus. After that, several mathematicians studied properties of degenerate poly-Bernoulli, degenerate Hermite, degenerate derangement, degenerate Bell and degenerate Daehee polynomials arising from λ-umbral calculus, cf. [10, 12, 13, 18, 20]. Recall from (3.2) that

$$\sum_{\rho=0}^{\infty}{}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)\frac{z^\rho}{\rho!}=e_\lambda^\gamma(z)\,e_\lambda^\delta(e_\lambda(z)-1)\left(\frac{z}{e_\lambda(z)-1}\right)^\varepsilon.\tag{4.8}$$

As $z \to 0$ in (4.8), it gives ${}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta) = 1$, which implies the generating function of *DBBPO* is invertible. Here, the properties of *DBBP* arising from λ-umbral calculus are sequentially provided as follows. With the aid of (4.8) and (4.6), it is derived that

$$ {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta) \sim \left(\frac{e_\lambda(z)-1}{z}e_\lambda^{-\delta}(e_\lambda(z)-1), z\right) \tag{4.9}$$

and

$$ (z)_\lambda \; {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta) = \rho \; {}_{\mathrm{Bel}}\mathrm{B}_{\rho-1,\lambda}(\gamma;\delta). \tag{4.10}$$

With the help of (4.2) and (4.8), it is seen that

$$ {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta) = \frac{z}{e_\lambda(z)-1}e_\lambda^{\delta}(e_\lambda(z)-1)(\gamma)_{\rho,\lambda} = e_\lambda^{\delta}(e_\lambda(z)-1)\mathrm{B}_{\rho,\lambda}(\gamma) = \sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}\mathrm{Be}\sigma_{\sigma,\lambda}(\delta)\mathrm{B}_{\rho-\sigma,\lambda}(\gamma),$$

which is the particular situation of (3.5). From (4.4) and (4.8), it is also observed that

$$\begin{aligned} {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta) &= \frac{z}{e_\lambda(z)-1}e_\lambda^{\delta}(e_\lambda(z)-1)(\gamma)_{\rho,\lambda} \\ &= \frac{z}{e_\lambda(z)-1}\mathrm{Bel}_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\infty}\frac{\mathrm{B}_{\sigma,\lambda}}{\sigma!}z^\sigma \mathrm{Bel}_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\rho}\binom{\rho}{\sigma}\mathrm{B}_{\sigma,\lambda}\mathrm{Bel}_{\rho-\sigma,\lambda}(\gamma;\delta). \end{aligned}$$

Now, we present our first result as follows.

Theorem 4.1. *There exist constants* $c_0, c_1, \dots, c_\rho$ *as*

$$ p(\gamma) = \sum_{\sigma=0}^{\rho} c_\sigma \; {}_{\mathrm{Bel}}\mathrm{B}_{\sigma,\lambda}(\gamma;\delta),$$

where

$$ c_\sigma = \frac{1}{\sigma!}\left\langle \frac{e_\lambda(z)-1}{z}e_\lambda^{-\delta}(e_\lambda(z)-1)z^\sigma \mid p(\gamma)\right\rangle_\lambda, \tag{4.11}$$

for $\rho \in \mathbb{Z}_{\geqslant 0}$ *and* $p(\gamma) \in \mathbb{P}_\rho$.

Proof. With the aid of (3.2), (4.5), and (4.9), the calculation shows that

$$\left\langle \frac{e_\lambda(z)-1}{z}e_\lambda^{-\delta}(e_\lambda(z)-1)z^\sigma \mid {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta)\right\rangle_\lambda = \rho!\delta_{\rho,\sigma}, \qquad (\rho,\sigma \geqslant 0),$$

which means that

$$\begin{aligned} &\left\langle \frac{e_\lambda(z)-1}{z}e_\lambda^{-\delta}(e_\lambda(z)-1)z^\sigma \mid p(\gamma)\right\rangle_\lambda \\ &= \sum_{\sigma=0}^{\rho} c_\sigma \left\langle \frac{e_\lambda(z)-1}{z}e_\lambda^{-\delta}(e_\lambda(z)-1)z^\sigma \mid {}_{\mathrm{Bel}}\mathrm{B}_{\sigma,\lambda}(\gamma;\delta)\right\rangle_\lambda = \sum_{\sigma=0}^{\rho} c_\sigma \sigma!\delta_{\sigma,\sigma} = \sigma! c_\sigma, \end{aligned}$$

which achieves the intended consequence (4.11). □

Here is an explicit formula for *DBBP*.

Theorem 4.2. DBBP *fulfill the following formula for* $\rho \in \mathbb{Z}_{\geqslant 0}$:

$$ {}_{\mathrm{Bel}}\mathrm{B}_{\rho,\lambda}(\gamma;\delta) = \rho\sum_{\sigma=1}^{\infty}\sum_{\sigma=0}^{\sigma-1}\frac{(\delta)_{\sigma,\lambda}}{\sigma!}\binom{\sigma-1}{\sigma}(-1)^{\sigma-1-\sigma}(\gamma+\sigma)_{\rho-1,\lambda}. \tag{4.12}$$

Proof. With the help of (4.4) and (4.8), it is acquired that

$$\begin{aligned}
{}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) &= \frac{z}{e_\lambda(z)-1} e_\lambda^{\delta}\left(e_\lambda(z)-1\right)(\gamma)_{\rho,\lambda} \\
&= \frac{z}{e_\lambda(z)-1}\sum_{\sigma=0}^{\infty}(\delta)_{\sigma,\lambda}\frac{\left(e_\lambda(z)-1\right)^{\sigma}}{\sigma!}(\gamma)_{\rho,\lambda} \\
&= z\sum_{\sigma=1}^{\infty}\sum_{\sigma=0}^{\sigma-1}\frac{(\delta)_{\sigma,\lambda}}{\sigma!}\binom{\sigma-1}{\sigma}(-1)^{\sigma-1-\sigma}e_\lambda^{\sigma}(z)(\gamma)_{\rho,\lambda} \\
&= \rho\sum_{\sigma=1}^{\infty}\sum_{\sigma=0}^{\sigma-1}\frac{(\delta)_{\sigma,\lambda}}{\sigma!}\binom{\sigma-1}{\sigma}(-1)^{\sigma-1-\sigma}(\gamma+\sigma)_{\rho-1,\lambda},
\end{aligned}$$

which achieves the claimed consequence (4.12). □

A representation of λ-umbral calculus is given as follows.

Theorem 4.3. *The following formula holds for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$\left\langle \frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \middle| (\gamma)_{\rho,\lambda} \right\rangle_\lambda = \sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma+1}\frac{(-\delta)_{\sigma,\lambda}}{\sigma!}\binom{\sigma+1}{\sigma}(-1)^{\sigma-\sigma+1}\left(\frac{(\sigma)_{\rho+1,\lambda}}{\rho+1}\right).$$

Proof. From (4.1), it is readily seen that

$$\begin{aligned}
\left\langle \frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \middle| (\gamma)_{\rho,\lambda} \right\rangle_\lambda &= \frac{1}{\rho+1}\left\langle \frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \mid z(\gamma)_{\rho+1,\lambda} \right\rangle_\lambda \\
&= \frac{1}{\rho+1}\left\langle \sum_{\sigma=0}^{\infty}\frac{(-\delta)_{\sigma,\lambda}}{\sigma!}\left(e_\lambda(z)-1\right)^{\sigma+1} \mid (\gamma)_{\rho+1,\lambda} \right\rangle_\lambda \\
&= \frac{1}{\rho+1}\sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma+1}\frac{(-\delta)_{\sigma,\lambda}}{\sigma!}\binom{\sigma+1}{\sigma}(-1)^{\sigma-\sigma+1}\left\langle e_\lambda^{\sigma}(z) \mid (\gamma)_{\rho+1,\lambda}\right\rangle_\lambda \\
&= \sum_{\sigma=0}^{\infty}\sum_{\sigma=0}^{\sigma+1}\frac{(-\delta)_{\sigma,\lambda}}{\sigma!}\binom{\sigma+1}{\sigma}(-1)^{\sigma-\sigma+1}\left(\frac{(\sigma)_{\rho+1,\lambda}}{\rho+1}\right),
\end{aligned}$$

which completes the proof. □

Theorem 4.4. DBellP (1.12) *fulfill the following representation of* λ*-umbral calculus for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$\left\langle \frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \middle| (\gamma)_{\rho,\lambda} \right\rangle_\lambda = \frac{\mathrm{Bel}_{\rho+1,\lambda}(1;-\delta)-\mathrm{Bel}_{\rho+1,\lambda}(-\delta)}{\rho+1}.$$

Proof. From (4.9) and (4.10), we write

$$ {}_{\mathrm{Be}\sigma}B_{\rho,\lambda}(\gamma;\delta) = \left(\frac{z}{e_\lambda(z)-1}e_\lambda^{\delta}\left(e_\lambda(z)-1\right)\right)_\lambda (\gamma)_{\rho,\lambda}, \qquad (\rho \geqslant 0). \tag{4.13}$$

With the help of (4.1),

$$\begin{aligned}
\left\langle \frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \middle| (\gamma)_{\rho,\lambda} \right\rangle_\lambda &= \left\langle \frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \mid z(\gamma)_{\rho+1,\lambda} \right\rangle_\lambda \\
&= \frac{1}{\rho+1}\left\langle \left(e_\lambda(z)-1\right)e_\lambda^{-\delta}\left(e_\lambda(z)-1\right) \mid (\gamma)_{\rho+1,\lambda} \right\rangle_\lambda
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\rho+1}\left\langle \sum_{\sigma=0}^{\infty} \frac{\left(\mathrm{Bel}_{\sigma,\lambda}(1;-\delta)-\mathrm{Bel}_{\sigma,\lambda}(-\delta)\right)}{\sigma!} z^{\sigma} \middle| (\gamma)_{\rho+1,\lambda}\right\rangle_{\lambda} \\
&= \frac{1}{\rho+1}\sum_{\sigma=0}^{\infty} \frac{\left(\mathrm{Bel}_{\sigma,\lambda}(1;-\delta)-\mathrm{Bel}_{\sigma,\lambda}(-\delta)\right)}{\sigma!}(\rho+1)!\delta_{\rho+1,\sigma} \\
&= \frac{\mathrm{Bel}_{\rho+1,\lambda}(1;-\delta)-\mathrm{Bel}_{\rho+1,\lambda}(-\delta)}{\rho+1}.
\end{aligned}$$

Thereby, the proof is completed. □

Theorem 4.5. *The following formula holds for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\rho}\sum_{\tau=0}^{\sigma}\sum_{j=0}^{\sigma-\tau}\binom{j+\tau}{\tau} S_{2,\lambda}(\rho,\sigma)\, S_{1,\lambda}(\sigma,j+\tau)\ {}_{\mathrm{Bel}}B_{\tau,\lambda}(\delta)\,(\gamma)_{j,\lambda}. \tag{4.14}$$

Proof. With the help of (4.9), (4.13), and (1.6), it is seen that

$$\left(\frac{e_\lambda(z)-1}{z} e_\lambda^{-\delta}\left(e_\lambda(z)-1\right)\right)_{\lambda}\ {}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = (\gamma)_{\rho,\lambda},$$

then the calculations gives that

$$\begin{aligned}
{}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) &= \left(\frac{z}{e_\lambda(z)-1} e_\lambda^{\delta}\left(e_\lambda(z)-1\right)\right)_{\lambda} (\gamma)_{\rho,\lambda} \\
&= \sum_{\sigma=0}^{\rho}\sum_{j=0}^{\sigma} S_{2,\lambda}(\rho,\sigma)\, S_{1,\lambda}(\sigma,j)\left(\frac{z}{e_\lambda(z)-1} e_\lambda^{\delta}\left(e_\lambda(z)-1\right)\right)_{\lambda} (\gamma)_{j,\lambda} \\
&= \sum_{\sigma=0}^{\rho}\sum_{j=0}^{\sigma} S_{2,\lambda}(\rho,\sigma)\, S_{1,\lambda}(\sigma,j)\sum_{\tau=0}^{j}\frac{{}_{\mathrm{Bel}}B_{\tau,\lambda}(\delta)}{\tau!}(z^{\tau})_{\lambda}(\gamma)_{j,\lambda} \\
&= \sum_{\sigma=0}^{\rho}\sum_{j=0}^{\sigma}\sum_{\tau=0}^{j}\binom{j}{\tau} S_{2,\lambda}(\rho,\sigma)\, S_{1,\lambda}(\sigma,j)\ {}_{\mathrm{Bel}}B_{\tau,\lambda}(\delta)\,(\gamma)_{j-\tau,\lambda} \\
&\quad \times \sum_{\sigma=0}^{\rho}\sum_{\tau=0}^{\sigma}\sum_{j=\tau}^{\sigma}\binom{j}{\tau} S_{2,\lambda}(\rho,\sigma)\, S_{1,\lambda}(\sigma,j)\ {}_{\mathrm{Bel}}B_{\tau,\lambda}(\delta)\,(\gamma)_{j-\tau,\lambda} \\
&= \sum_{\sigma=0}^{\rho}\sum_{\tau=0}^{\sigma}\sum_{j=0}^{\sigma-\tau}\binom{j+\tau}{\tau} S_{2,\lambda}(\rho,\sigma)\, S_{1,\lambda}(\sigma,j+\tau)\ {}_{\mathrm{Bel}}B_{\tau,\lambda}(\delta)\,(\gamma)_{j,\lambda},
\end{aligned}$$

which achieves the desired consequence (4.14). □

Theorem 4.6. DBBP *satisfy the following correlation associated with* DBN (1.16) *and* DSNSK (1.4) *for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\rho}\sum_{\sigma=0}^{\sigma}\sum_{\tau=0}^{\sigma}\binom{\rho}{\sigma}\binom{\sigma}{\sigma} B_{\rho-\sigma,\lambda} S_{2,\lambda}(\sigma,\tau)\,(\delta)_{\tau,\lambda}\,(\gamma)_{\sigma-\sigma,\lambda}. \tag{4.15}$$

Proof. Using the aid of (1.14), it is seen that

$$\begin{aligned}
{}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) &= \left\langle \frac{z}{e_\lambda(z)-1} e_\lambda^{\delta}\left(e_\lambda(z)-1\right) \middle| (\gamma)_{\rho,\lambda}\right\rangle_{\lambda} \\
&= \left\langle e_\lambda^{\delta}\left(e_\lambda(z)-1\right) \middle| \left(\frac{z}{e_\lambda(z)-1}\right)_{\lambda} (\gamma)_{\rho,\lambda}\right\rangle_{\lambda}
\end{aligned}$$

$$= \sum_{\sigma=0}^{\rho} \binom{\rho}{\sigma} B_{\rho-\sigma,\lambda} \left\langle \sum_{\sigma=0}^{\infty} \sum_{\tau=0}^{\sigma} (\delta)_{\tau,\lambda} S_{2,\lambda}(\sigma,\tau) \frac{(z^{\sigma})_{\lambda}}{\sigma!} \middle| (\gamma)_{\sigma,\lambda} \right\rangle_{\lambda}$$

$$= \sum_{\sigma=0}^{\rho} \sum_{\sigma=0}^{\sigma} \sum_{\tau=0}^{\sigma} \binom{\rho}{\sigma} \binom{\sigma}{\sigma} B_{\rho-\sigma,\lambda} S_{2,\lambda}(\sigma,\tau)(\delta)_{\tau,\lambda}(\gamma)_{\sigma-\sigma,\lambda},$$

which achieves the asserted consequence (4.15). □

Also it is noted that

$${}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) \sim \left(\frac{e_{\lambda}(z)-1}{z} e_{\lambda}^{-\delta}(e_{\lambda}(z)-1), z \right)_{\lambda} \quad \text{and} \quad B_{\rho,\lambda}^{(s)}(\gamma) \sim \left(\frac{(e_{\lambda}(z)-1)^{s}}{z^{s}}, z \right)_{\lambda}.$$

Theorem 4.7. DBBP *fulfill the following relationship related to* DBN (1.16) *and* DSNSK (1.4) *for* $\rho \in \mathbb{Z}_{\geqslant 0}$:

$${}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\tau=0}^{\rho} \sum_{\sigma=0}^{\rho-\tau} \frac{\binom{\rho}{\tau}\binom{\rho-\tau}{\sigma}}{\binom{\sigma+\varkappa}{s}} {}_{\mathrm{Bel}}B_{\rho-\tau-\sigma,\lambda}(\delta) S_{2,\lambda}(\sigma+s,s) B_{\rho,\lambda}^{(s)}(\gamma). \tag{4.16}$$

Proof. From (4.7), it is seen that

$${}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\tau=0}^{\rho} B_{\rho,\lambda}^{(s)}(\gamma) C_{\rho,\tau}, \quad (\rho \geqslant 0),$$

where

$$C_{\rho,\tau} = \frac{1}{\tau!} \left\langle \frac{z e_{\lambda}^{\delta}(e_{\lambda}(z)-1)}{e_{\lambda}(z)-1} \frac{(e_{\lambda}(z)-1)^{s}}{z^{s}} z^{\tau} \middle| (\gamma)_{\rho,\lambda} \right\rangle_{\lambda}$$

$$= \binom{\rho}{\tau} \left\langle \frac{z e_{\lambda}^{\delta}(e_{\lambda}(z)-1)}{e_{\lambda}(z)-1} \frac{(e_{\lambda}(z)-1)^{s}}{z^{s}} \middle| (\gamma)_{\rho-\tau,\lambda} \right\rangle_{\lambda}$$

$$= \binom{\rho}{\tau} \sum_{\sigma=0}^{\rho-\tau} \frac{S_{2,\lambda}(\sigma+s,s)}{\binom{\sigma+\varkappa}{s}\sigma!} \left\langle \frac{z e_{\lambda}^{\delta}(e_{\lambda}(z)-1)}{e_{\lambda}(z)-1} z^{\sigma} \middle| (\gamma)_{\rho-\tau,\lambda} \right\rangle_{\lambda}$$

$$= \binom{\rho}{\tau} \sum_{\sigma=0}^{\rho-\tau} \frac{S_{2,\lambda}(\sigma+s,s)}{\binom{\sigma+\varkappa}{s}} \binom{\rho-\tau}{\sigma} \left\langle \frac{z e_{\lambda}^{\delta}(e_{\lambda}(z)-1)}{e_{\lambda}(z)-1} \middle| (\gamma)_{\rho-\tau-\sigma,\lambda} \right\rangle_{\lambda}$$

$$= \binom{\rho}{\tau} \sum_{\sigma=0}^{\rho-\tau} \frac{S_{2,\lambda}(\sigma+s,s)}{\binom{\sigma+\varkappa}{s}} \binom{\rho-\tau}{\sigma} {}_{\mathrm{Bel}}B_{\rho-\tau-\sigma,\lambda}(\delta),$$

which achieves the assertion (4.16). □

The degenerate form of derangement polynomials of order $r \in \mathbb{N}$ is introduced as follows (cf. [12])

$$\sum_{\rho=0}^{\infty} d_{\rho,\lambda}^{(r)}(\gamma) \frac{z^{\rho}}{\rho!} = \frac{1}{(1-z)^{r}} e_{\lambda}^{-1}(z) e_{\lambda}^{\gamma}(z).$$

Upon setting $r = 1$, the polynomials $d_{\rho,\lambda}^{(r)}(\gamma)$ reduce to $d_{\rho,\lambda}(\gamma)$ termed as the degenerate derangement polynomials. For $0 \leqslant \rho$, it can be observed from (4.6) that

$$d_{\rho,\lambda}(\gamma) \sim ((1-z) e_{\lambda}(z), z)_{\lambda} \quad \text{and} \quad d_{\rho,\lambda}^{(r)}(\gamma) \sim ((1-z)^{r} e_{\lambda}(z), z)_{\lambda}.$$

For $p(\gamma) \in \mathbb{P}_{\rho}$ being a $(\rho+1)$-dimensional vector space over $\mathbb{C}$ (cf. [11]), it is shown that

$$p(\gamma) = \sum_{\sigma=0}^{\rho} C_{\sigma} d_{\sigma,\lambda}(\gamma), \tag{4.17}$$

where

$$C_\sigma = \frac{1}{\sigma!}\left\langle (1-z)\, e_\lambda(z)\, z^\sigma | p(\gamma)\right\rangle_\lambda \tag{4.18}$$

and

$$p(\gamma) = \sum_{\tau=0}^{\rho} C_\tau^{(r)} d_{\tau,\lambda}^{(r)}(\gamma),$$

where

$$C_\tau^{(r)} = \frac{1}{\tau!}\left\langle (1-z)^r e_\lambda(z)\, z^\sigma | p(\gamma)\right\rangle_\lambda .$$

Let $p(\gamma) = {}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) \in \mathbb{P}_\rho$ in (4.17) and (4.18). Then we get

$$ {}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\rho} C_\sigma d_{\sigma,\lambda}(\gamma),$$

where

$$\begin{aligned}
C_\sigma &= \frac{1}{\sigma!}\left\langle (1-z)\, e_\lambda(z)\, z^\sigma | {}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\left\langle (1-z)\, e_\lambda(z) \,|\, {}_{\mathrm{Bel}}B_{\rho-\sigma,\lambda}(\gamma;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\left\langle (1-z) \,|\, {}_{\mathrm{Bel}}B_{\rho-\sigma,\lambda}(\gamma+1;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\left\langle 1 | {}_{\mathrm{Bel}}B_{\rho-\sigma,\lambda}(\gamma+1;\delta)\right\rangle_\lambda - \binom{\rho}{\sigma}(\rho-\sigma)\left\langle 1 | {}_{\mathrm{Bel}}B_{\rho-\sigma-1,\lambda}(\gamma+1;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\, {}_{\mathrm{Bel}}B_{\rho-\sigma,\lambda}(1;\delta) - \rho\binom{\rho-1}{\sigma}\, {}_{\mathrm{Bel}}B_{\rho-\sigma-1,\lambda}(1;\delta).
\end{aligned}$$

Thus, the following corollary is obtained.

Theorem 4.8. DBBP *satisfy the following identity for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$ {}_{\mathrm{Bel}}B_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\rho}\left(\binom{\rho}{\sigma}\, {}_{\mathrm{Bel}}B_{\rho-\sigma,\lambda}(1;\delta) - \rho\binom{\rho-1}{\sigma}\, {}_{\mathrm{Bel}}B_{\rho-\sigma-1,\lambda}(1;\delta)\right) d_{\sigma,\lambda}(\gamma).$$

Choosing $p(\gamma) = {}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta) \in \mathbb{P}_\rho$ in (4.17) and (4.18) gives the following formula

$$ {}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta) = \sum_{\sigma=0}^{\rho} C_\sigma^{(r)} d_{\sigma,\lambda}^{(r)}(\gamma),$$

where

$$\begin{aligned}
C_\sigma^{(r)} &= \frac{1}{\sigma!}\left\langle (1-z)^r e_\lambda(z)\, z^\sigma | {}_{\mathrm{Bel}}B_{\rho,\lambda}^{(\varepsilon)}(\gamma;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\left\langle (1-z)^r \,|\, {}_{\mathrm{Bel}}B_{\rho-\sigma,\lambda}^{(\varepsilon)}(\gamma+1;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\sum_{j=0}^{r}\binom{r}{j}(-1)^j\binom{\rho-\sigma}{j} j!\left\langle 1 | {}_{\mathrm{Bel}}B_{\rho-\sigma-j,\lambda}^{(\varepsilon)}(\gamma+1;\delta)\right\rangle_\lambda \\
&= \binom{\rho}{\sigma}\sum_{j=0}^{r}\binom{r}{j}\binom{\rho-\sigma}{j}\, {}_{\mathrm{Bel}}B_{\rho-\sigma-j,\lambda}^{(\varepsilon)}(\gamma+1;\delta)(-1)^j j!.
\end{aligned}$$

Thus, the following result can be given.

Theorem 4.9. DBBP *and degenerate derangement polynomials fulfill the following relation for* $\rho \in \mathbb{Z}_{\geqslant 0}$*:*

$$_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho,\lambda}(\gamma;\delta) = \sum_{\sigma=0}^{\rho}\sum_{j=0}^{r}\binom{\rho}{\sigma}\binom{r}{j}\binom{\rho-\sigma}{j}\,_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho-\sigma-j,\lambda}(\gamma+1;\delta)\,d_{\sigma,\lambda}(\gamma)\,(-1)^{j}\,j!.$$

4.2. Applications in degenerate differential operator

The degenerate differential operator is considered by Kim et al. [14, 15] as follows:

$$\left(\gamma\frac{d}{d\gamma}\right)_{\sigma,\lambda} = \left(\gamma\frac{d}{d\gamma}\right)\left(\gamma\frac{d}{d\gamma}-\lambda\right)\left(\gamma\frac{d}{d\gamma}-2\lambda\right)\cdots\left(\gamma\frac{d}{d\gamma}-(\sigma-1)\lambda\right). \tag{4.19}$$

By (4.19), we have

$$\left(\gamma\frac{d}{d\gamma}\right)_{\sigma,\lambda}\gamma^{\rho} = (\rho)_{\sigma,\lambda}\,\gamma^{\rho}.$$

Let f be a formal power series written as $f(\gamma) = \sum_{\rho=0}^{\infty}\varkappa_{\rho}\gamma^{\rho}$ and $\sigma \geqslant 0$. Then the degenerate differential operator of this series is given by

$$\left(\gamma\frac{d}{d\gamma}\right)_{\sigma,\lambda} f(\gamma) = \sum_{\rho=0}^{\infty}\varkappa_{\rho}\,(\rho)_{\sigma,\lambda}\,\gamma^{\rho}.$$

Kim et al. [14, 15] showed that degenerate differential operator plays an important role in boson operators. In this part, we focus on the representation of the degenerate differential operator on *DBBPO* (3.2) as follows. By (1.14) and (3.5), we observe

$$\begin{aligned}
\left(\delta\frac{d}{d\delta}\right)_{\sigma,\lambda}\,_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho,\lambda}(\gamma;\delta) &= \left(\delta\frac{d}{d\delta}\right)_{\sigma,\lambda}\left\{\sum_{j=0}^{\rho}\binom{\rho}{j}B^{(\varepsilon)}_{\rho-j,\lambda}(\gamma)\,\mathrm{Bel}_{j,\lambda}(\delta)\right\}\\
&= \sum_{j=0}^{\rho}\binom{\rho}{j}B^{(\varepsilon)}_{\rho-j,\lambda}(\gamma)\sum_{\tau=0}^{j}S_{2,\lambda}(j,\tau)\left\{\left(\delta\frac{d}{d\delta}\right)_{\sigma,\lambda}\delta^{\tau}\right\}\\
&= \sum_{j=0}^{\rho}\sum_{\tau=0}^{j}\binom{\rho}{j}B^{(\varepsilon)}_{\rho-j,\lambda}(\gamma)\,S_{2,\lambda}(j,\tau)\,(\tau)_{\sigma,\lambda}\,\delta^{\tau}\\
&= \sum_{j=0}^{\rho}\sum_{\tau=0}^{j}\sum_{\sigma=0}^{\sigma}\sum_{r=0}^{\sigma}\binom{\rho}{j}B^{(\varepsilon)}_{\rho-j,\lambda}(\gamma)\,S_{2,\lambda}(j,\tau)\,S_{2,\lambda}(\sigma,\sigma)\,S_{1,\lambda}(\sigma,r)\,(\tau)_{\sigma,\lambda}\,\delta^{\tau}.
\end{aligned}$$

Thus, the following result can be presented.

Theorem 4.10. *The following relation holds true:*

$$\left(\delta\frac{d}{d\delta}\right)_{\sigma,\lambda}\,_{\mathrm{Bel}}B^{(\varepsilon)}_{\rho,\lambda}(\gamma;\delta) = \sum_{j=0}^{\rho}\sum_{\tau=0}^{j}\sum_{\sigma=0}^{\sigma}\sum_{r=0}^{\sigma}\binom{\rho}{j}B^{(\varepsilon)}_{\rho-j,\lambda}(\gamma)\,S_{2,\lambda}(j,\tau)\,S_{2,\lambda}(\sigma,\sigma)\,S_{1,\lambda}(\sigma,r)\,(\tau)_{\sigma,\lambda}\,\delta^{\tau}.$$

5. Conclusion

In this work, a degenerate form of the Bell-based Stirling polynomials of the second kind has been defined, and several fundamental properties and formulas for these polynomials have been investigated and presented in detail. Then, a degenerate form of the Bell-based Bernoulli polynomials of order ε has been defined and a plenty of their properties have been examined in different aspects. Several correlations with other polynomials and numbers in literature, symmetric identities, implicit summation formulas, derivative properties and addition formulas for the mentioned new polynomials have been derived in detail, and some special cases of these results have been investigated. Also, the degenerate Bell-based Bernoulli polynomials of order ε have been studied in λ-umbral calculus and interesting relations and formulas have been developed. Moreover, it has been shown that applications in λ-umbral calculus for the Bell-based degenerate Bernoulli polynomials of order ε give a relation with higher order degenerate derangement polynomials. Finally, a representation of the degenerate differential operator on the degenerate Bell-based Bernoulli polynomials of order ε has been provided. The formulas given in this study are a generalization of the formulas from other studies, some of that are covered in the associated references in [1, 2, 4, 5, 7–15, 17–21, 24]. In subsequent studies, the polynomials in (2.1) and (3.2) can be analyzed in the context of monomiality principle.

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