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Modified subgradient extragradient method to solve variational inequalities

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Abstract

In this paper, we introduce a new method to solve pseudomonotone variational inequalities with the Lipschitz condition in a real Hilbert space. This problem is a general mathematical problem in the sense that it unifies a number of the mathematical problems as a particular case, such as the optimization problems, the equilibrium problems, the fixed point problems, the saddle point problems and Nash equilibrium point problems. The new method is constructed around two methods: the extragradient method and the inertial method. The proposed method uses a new stepsize rule based on local operator information rather than its Lipschitz constant or any other line search method. The proposed method does not require any knowledge of the Lipschitz constant of an operator. The strong convergence of the proposed method is well-established. Finally, we conduct a number of numerical experiments to determine the performance and superiority of the proposed method.

Keywords: Variational inequality problem, subgradient extragradient-like method, strong convergence result, Lipschitz continuity, pseudomonotone mapping .

2020 MSC: 65Y05, 65K15, 68W10, 47H05, 47H10.

1. Introduction

This paper examines the problem of classic variational inequalities [29] and the *variational inequality problem* (VIP) for mapping $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ is defined as follows:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } \langle \mathcal{S}(u^*), \nu - u^* \rangle \geqslant 0, \ \forall \nu \in \mathcal{K}, \tag{VIP}$$

where $\mathcal{K}$ is a nonempty, convex and closed subset of a certain Hilbert space $\mathcal{E}$ and $\langle .,. \rangle$ and $\|.\|$ serve as an inner product and the induced norm in $\mathcal{E}$, respectively. Moreover, $\mathbb{R}$, and $\mathbb{N}$ are denoted the sets of real numbers and natural numbers, respectively. It is important to note that the problem (VIP) is equivalent to solve the following problem:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } u^* = P_{\mathcal{K}}[u^* - \rho \mathcal{S}(u^*)].$$

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The variational inequality problem was introduced by Stampacchia [29] in 1964. This is an important mathematical model that unifies several key topics of applied mathematics, such as the network equilibrium problems, the necessary optimality conditions, the complementarity problems and the systems of nonlinear equations (for more details [5, 8, 9, 11, 12, 16, 33]) and others in [1, 10, 15, 17, 19–28]. Korpelevich [13] and Antipin [2] established the following extragradient method:

$$\begin{cases} u_0 \in \mathcal{K}, \\ v_n = P_{\mathcal{K}}[u_n - \rho \mathcal{S}(u_n)], \\ u_{n+1} = P_{\mathcal{K}}[u_n - \rho \mathcal{S}(v_n)]. \end{cases}$$

Recently, the subgradient extragradient method was introduced by Censor et al. [4] for solving the problem (VIP) in a real Hilbert space. Their method takes the form

$$\begin{cases} u_0 \in \mathcal{K}, \\ v_n = P_{\mathcal{K}}[u_n - \rho \mathcal{S}(u_n)], \\ u_{n+1} = P_{\mathcal{E}_n}[u_n - \rho \mathcal{S}(v_n)], \end{cases}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \rho \mathcal{S}(u_n) - v_n, z - v_n \rangle \leqslant 0\}.$$

It is important to note that the proposed well-established method has two serious shortcomings, the first being the fixed constant stepsize, which requires the information or approximation of the Lipschitz constant of the associated operator and is only weakly convergent in Hilbert spaces. From a numerical point of view, it could be challenging to use a fixed step size and thus the convergence rate and performance of the method may be affected. The main purpose of this study is to set up a new inertial-type method that is used to enhance the convergence rate of the iterative sequence and independent of the knowledge Lipschitz constants. These methods have been previously established due to the oscillator equation with a damping and conservative force restoration. This second-order dynamical system is called a heavy friction ball, which was formerly designed by Polyak in [18]. So there is important question that arises:

"Is it possible to introduce a new strongly convergent inertial-type strongly convergent extragradient method with a non-monotone variable step sizerule"?

In this paper, we present a positive answer of the above question, that is, the gradient method indeed establishes a strong convergence sequence by using a variable step sizerule for solving the problem (VIP) combined with pseudomonotone mappings. Motivated by the works of Censor et al. [4] and Polyak [18], we introduce a new inertial extragradient-type method to figure out the problem (VIP) in the setting of an infinite-dimensional real Hilbert space.

The rest of the paper is arranged as follows. The Section 2 consists of the necessary definitions and fundamental lemmas used in this paper. Section 3 consists of an inertial-type iterative scheme and convergence analysis theorem. Section 4 provides numerical results to explain the performance of the new method and compare it with other methods. The following conditions are satisfied in order to study the strong convergence.

($\mathcal{S}$1) The solution set of problem (VIP) is denoted by Π is nonempty.

($\mathcal{S}$2) An operator $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ is said to be pseudomonotone if

$$\big\langle \mathcal{S}(v_1), v_2 - v_1 \big\rangle \geqslant 0 \Longrightarrow \big\langle \mathcal{S}(v_2), v_1 - v_2 \big\rangle \leqslant 0, \ \forall v_1, v_2 \in \mathcal{K}.$$

($\mathcal{S}$3) An operator $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ is said to be *Lipschitz continuous* with constant $L > 0$ if there exists $L > 0$ such that

$$\|\mathcal{S}(v_1) - \mathcal{S}(v_2)\| \leqslant L\|v_1 - v_2\|, \ \forall v_1, v_2 \in \mathcal{K}.$$

($\mathcal{S}$4) An operator $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ is said to be *sequentially weakly continuous* if $\{\mathcal{S}(u_n)\}$ converges weakly to $\mathcal{S}(u)$ for every sequence $\{u_n\}$ that converges weakly to u.

2. Preliminaries

In this section of the text, we have written a number of significant identities and related lemmas and definitions. The *metric projection* $P_{\mathcal{K}}(\nu_1)$ of $\nu_1 \in \mathcal{E}$ is defined by

$$P_{\mathcal{K}}(\nu_1) = \arg\min\{\|\nu_1 - \nu_2\| : \nu_2 \in \mathcal{K}\}.$$

Next, we list some of the important properties of the projection mapping.

Lemma 2.1 ([3]). *Suppose that* $P_{\mathcal{K}} : \mathcal{E} \to \mathcal{K}$ *is a metric projection. Then, we have*

(i) $\nu_3 = P_{\mathcal{K}}(\nu_1)$ *if and only if* $\langle \nu_1 - \nu_3, \nu_2 - \nu_3\rangle \leqslant 0, \ \forall \nu_2 \in \mathcal{K}$;

(ii) $\|\nu_1 - P_{\mathcal{K}}(\nu_2)\|^2 + \|P_{\mathcal{K}}(\nu_2) - \nu_2\|^2 \leqslant \|\nu_1 - \nu_2\|^2, \ \nu_1 \in \mathcal{K}, \nu_2 \in \mathcal{E}$;

(iii) $\|\nu_1 - P_{\mathcal{K}}(\nu_1)\| \leqslant \|\nu_1 - \nu_2\|, \ \nu_2 \in \mathcal{K}, \nu_1 \in \mathcal{E}$.

Lemma 2.2 ([31]). *Let* $\{p_n\} \subset [0, +\infty)$ *be a sequence satisfying the following inequality*

$$p_{n+1} \leqslant (1 - q_n)p_n + q_n r_n, \ \forall n \in \mathbb{N}.$$

Furthermore, $\{q_n\} \subset (0, 1)$ *and* $\{r_n\} \subset \mathbb{R}$ *be two sequences such that*

$$\lim_{n\to+\infty} q_n = 0, \ \sum_{n=1}^{+\infty} q_n = +\infty \text{ and } \limsup_{n\to+\infty} r_n \leqslant 0.$$

Then, $\lim_{n\to+\infty} p_n = 0$.

Lemma 2.3 ([14]). *Suppose that* $\{p_n\}$ *is a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that*

$$p_{n_i} < p_{n_{i+1}}, \quad \forall \quad i \in \mathbb{N}.$$

Then, there is a nondecreasing sequence $m_k \subset \mathbb{N}$ *such that* $m_k \to +\infty$ *as* $k \to +\infty$, *and meet the following requirements for numbers* $k \in \mathbb{N}$:

$$p_{m_k} \leqslant p_{m_{k+1}} \quad \text{and} \quad p_k \leqslant p_{m_{k+1}}.$$

Indeed, $m_k = \max\{j \leqslant k : p_j \leqslant p_{j+1}\}$.

Next, we list some of the important identities that were used to prove the convergence analysis.

Lemma 2.4 ([3]). *For any* $\nu_1, \nu_2 \in \mathcal{E}$ *and* $\ell \in \mathbb{R}$, *the following inequalities are holds.*

(i) $\|\ell\nu_1 + (1 - \ell)\nu_2\|^2 = \ell\|\nu_1\|^2 + (1 - \ell)\|\nu_2\|^2 - \ell(1 - \ell)\|\nu_1 - \nu_2\|^2$.

(ii) $\|\nu_1 + \nu_2\|^2 \leqslant \|\nu_1\|^2 + 2\langle \nu_2, \nu_1 + \nu_2\rangle$.

Lemma 2.5 ([30]). *Assume that* $\mathcal{S} : \mathcal{K} \to \mathcal{E}$ *is a pseudomonotone and continuous mapping. Then,* u^* *is a solution of the problem* (VIP) *if and only if* u^* *is a solution of the following problem.*

$$\text{Find } u \in \mathcal{K} \text{ such that } \langle \mathcal{S}(\nu), \nu - u\rangle \geqslant 0, \ \forall \nu \in \mathcal{K}.$$

3. Main Results

In this section, we present a new inertial-type subgradient extragradient method that includes the new stepsize rule and inertial term, as well as the strong convergence theorem. The main result is as follows.

Algorithm 1

Step 0: Let $u_{-1}, u_0 \in \mathcal{K}$, $\alpha > 0$, $\mu \in (0,1)$, $\rho_0 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty} \varphi_n < +\infty$. Moreover, choose $\{\gamma_n\} \subset (0,1)$ satisfying the following conditions:

$$\lim_{n\to+\infty} \gamma_n = 0 \text{ and } \sum_{n=1}^{+\infty} \gamma_n = +\infty.$$

Step 1: Compute

$$\chi_n = \big[u_n + \alpha_n(u_n - u_{n-1})\big] - \gamma_n\big[u_n + \alpha_n(u_n - u_{n-1})\big],$$

where α_n such that

$$0 \leqslant \alpha_n \leqslant \hat{\alpha_n} \quad \text{and} \quad \hat{\alpha_n} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|u_n - u_{n-1}\|}\right\}, & \text{if } u_n \neq u_{n-1}, \\ \alpha, & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\epsilon_n = \circ(\gamma_n)$ is a positive sequence such that $\lim_{n\to+\infty} \frac{\epsilon_n}{\gamma_n} = 0$.
Step 2: Compute

$$v_n = P_{\mathcal{K}}(\chi_n - \rho_n \mathcal{S}(\chi_n)).$$

If $\chi_n = v_n$, then STOP and v_n is a solution. Otherwise, go to **Step 3**.
Step 3: Compute

$$u_{n+1} = P_{\mathcal{E}_n}(\chi_n - \rho_n \mathcal{S}(v_n)),$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle \chi_n - \rho_n \mathcal{S}(\chi_n) - v_n, z - v_n\rangle \leqslant 0\}.$$

Step 4: Compute

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n + \varphi_n, \frac{\mu\|\chi_n - v_n\|}{\|\mathcal{S}(\chi_n) - \mathcal{S}(v_n)\|}\right\}, & \text{if } \mathcal{S}(\chi_n) - \mathcal{S}(v_n) \neq 0, \\ \rho_n + \varphi_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n = n + 1$ and go back to **Step 1**.

Lemma 3.1. *The sequence $\{\rho_n\}$ generated by (3.2) is convergent to ρ and satisfy the following inequality*

$$\min\left\{\frac{\mu}{L}, \rho_0\right\} \leqslant \rho \leqslant \rho_0 + P, \quad \textit{where} \quad P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof. Due to the Lipschitz continuity of a mapping $\mathcal{S}$, there exists a fixed number $L > 0$. Suppose that $\mathcal{S}(\chi_n) - \mathcal{S}(v_n) \neq 0$ such that

$$\frac{\mu\|\chi_n - v_n\|}{\|\mathcal{S}(\chi_n) - \mathcal{S}(v_n)\|} \geqslant \frac{\mu\|\chi_n - v_n\|}{L\|\chi_n - v_n\|} = \frac{\mu}{L}.$$

By using mathematical induction on the definition of ρ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \rho_0\right\} \leqslant \rho_n \leqslant \rho_0 + P.$$

Let $[\rho_{n+1} - \rho_n]^+ = \max\{0, \rho_{n+1} - \rho_n\}$ and $[\rho_{n+1} - \rho_n]^- = \max\{0, -(\rho_{n+1} - \rho_n)\}$. From the definition of $\{\rho_n\}$, we have

$$\sum_{n=1}^{+\infty} (\rho_{n+1} - \rho_n)^+ = \sum_{n=1}^{+\infty} \max\{0, \rho_{n+1} - \rho_n\} \leqslant P < +\infty.$$

That is, the series $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^+$ is convergent. Next, we need to prove the convergence of $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^-$. Let $\sum_{n=1}^{+\infty}(\rho_{n+1}-\rho_n)^- = +\infty$. Due to the reason that $\rho_{n+1}-\rho_n = (\rho_{n+1}-\rho_n)^+ - (\rho_{n+1}-\rho_n)^-$, thus, we have

$$\rho_{k+1}-\rho_0 = \sum_{n=0}^{k}(\rho_{n+1}-\rho_n) = \sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^+ - \sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^-. \tag{3.3}$$

By allowing $k \to +\infty$ in (3.3), we have $\rho_k \to -\infty$ as $k \to \infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^+$ and $\sum_{n=0}^{k}(\rho_{n+1}-\rho_n)^-$ taking $k \to +\infty$ in (3.3), we obtain $\lim_{n\to\infty}\rho_n = \rho$. This completes the proof. □

Lemma 3.2. *Let $\mathcal{S} : \mathcal{E} \to \mathcal{E}$ be an operator satisfies the conditions* ($\mathcal{S}$1)-($\mathcal{S}$4). *Then, for $u^* \in \Pi \neq \emptyset$, we have*

$$\|u_{n+1}-u^*\|^2 \leqslant \|\chi_n - u^*\|^2 - \Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|\chi_n - v_n\|^2 - \Big(1-\frac{\mu\rho_n}{\rho_{n+1}}\Big)\|u_{n+1}-v_n\|^2.$$

Proof. Consider that

$$\begin{aligned}
\big\|u_{n+1}-u^*\big\|^2 &= \big\|P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - u^*\big\|^2 \\
&= \big\|P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] + [\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)] - u^*\big\|^2 \\
&= \big\|[\chi_n - \rho_n\mathcal{S}(v_n)] - u^*\big\|^2 + \big\|P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)]\big\|^2 \\
&\quad + 2\big\langle P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)], [\chi_n - \rho_n\mathcal{S}(v_n)] - u^*\big\rangle.
\end{aligned} \tag{3.4}$$

It is given that $u^* \in \Pi \subset \mathcal{K} \subset \mathcal{E}_n$, we obtain

$$\begin{aligned}
&\big\|P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)]\big\|^2 \\
&\quad + \big\langle P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)], [\chi_n - \rho_n\mathcal{S}(v_n)] - u^*\big\rangle \\
&= \big\langle [\chi_n - \rho_n\mathcal{S}(v_n)] - P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)], u^* - P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)]\big\rangle \leqslant 0,
\end{aligned}$$

that implies that

$$\begin{aligned}
&\big\langle P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)], [\chi_n - \rho_n\mathcal{S}(v_n)] - u^*\big\rangle \\
&\quad \leqslant -\big\|P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)]\big\|^2.
\end{aligned} \tag{3.5}$$

Combining the expressions (3.4) and (3.5), we have

$$\begin{aligned}
\|u_{n+1}-u^*\|^2 &\leqslant \big\|\chi_n - \rho_n\mathcal{S}(v_n) - u^*\big\|^2 - \big\|P_{\mathcal{E}_n}[\chi_n - \rho_n\mathcal{S}(v_n)] - [\chi_n - \rho_n\mathcal{S}(v_n)]\big\|^2 \\
&\leqslant \|\chi_n - u^*\|^2 - \|\chi_n - u_{n+1}\|^2 + 2\rho_n\langle \mathcal{S}(v_n), u^* - u_{n+1}\rangle.
\end{aligned} \tag{3.6}$$

Since u^* is the solution of problem (VIP), we have

$$\langle \mathcal{S}(u^*), v - u^*\rangle \geqslant 0, \text{ for all } v \in \mathcal{K}.$$

Due to the pseudomonotonicity of $\mathcal{S}$ on $\mathcal{K}$, we obtain

$$\langle \mathcal{S}(v), v - u^*\rangle \geqslant 0, \text{ for all } v \in \mathcal{K}.$$

By substituting $\nu = \nu_n \in \mathcal{K}$, we get

$$\langle \mathcal{S}(\nu_n), \nu_n - u^* \rangle \geqslant 0.$$

Thus, we have

$$\langle \mathcal{S}(\nu_n), u^* - u_{n+1} \rangle = \langle \mathcal{S}(\nu_n), u^* - \nu_n \rangle + \langle \mathcal{S}(\nu_n), \nu_n - u_{n+1} \rangle \leqslant \langle \mathcal{S}(\nu_n), \nu_n - u_{n+1} \rangle. \tag{3.7}$$

By use of expressions (3.6) and (3.7), we have

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &\leqslant \|\chi_n - u^*\|^2 - \|\chi_n - u_{n+1}\|^2 + 2\rho_n \langle \mathcal{S}(\nu_n), \nu_n - u_{n+1} \rangle \\ &\leqslant \|\chi_n - u^*\|^2 - \|\chi_n - \nu_n + \nu_n - u_{n+1}\|^2 + 2\rho_n \langle \mathcal{S}(\nu_n), \nu_n - u_{n+1} \rangle \\ &\leqslant \|\chi_n - u^*\|^2 - \|\chi_n - \nu_n\|^2 - \|\nu_n - u_{n+1}\|^2 + 2\langle \chi_n - \rho_n \mathcal{S}(\nu_n) - \nu_n, u_{n+1} - \nu_n \rangle. \end{aligned} \tag{3.8}$$

By use of the definition $u_{n+1} = P_{\mathcal{E}_n}[\chi_n - \rho_n \mathcal{S}(\nu_n)]$ and ρ_{n+1}, we have

$$\begin{aligned} &2\langle \chi_n - \rho_n \mathcal{S}(\nu_n) - \nu_n, u_{n+1} - \nu_n \rangle \\ &\quad = 2\langle \chi_n - \rho_n \mathcal{S}(\chi_n) - \nu_n, u_{n+1} - \nu_n \rangle + 2\rho_n \langle \mathcal{S}(\chi_n) - \mathcal{S}(\nu_n), u_{n+1} - \nu_n \rangle \\ &\quad = 2\frac{\rho_n}{\rho_{n+1}} \rho_{n+1} \|\mathcal{S}(\chi_n) - \mathcal{S}(\nu_n)\| \|u_{n+1} - \nu_n\| \\ &\quad \leqslant \frac{\mu \rho_n}{\rho_{n+1}} \|\chi_n - \nu_n\|^2 + \frac{\mu \rho_n}{\rho_{n+1}} \|u_{n+1} - \nu_n\|^2. \end{aligned} \tag{3.9}$$

Combining expressions (3.8) and (3.9), we get

$$\begin{aligned} &\|u_{n+1} - u^*\|^2 \\ &\leqslant \|\chi_n - u^*\|^2 - \|\chi_n - \nu_n\|^2 - \|\nu_n - u_{n+1}\|^2 + \frac{\rho_n}{\rho_{n+1}} \left[\mu \|\chi_n - \nu_n\|^2 + \mu \|u_{n+1} - \nu_n\|^2\right] \\ &\leqslant \|\chi_n - u^*\|^2 - \left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) \|\chi_n - \nu_n\|^2 - \left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) \|u_{n+1} - \nu_n\|^2. \end{aligned} \tag{3.10}$$

□

Theorem 3.3. *Let $\{u_n\}$ be a sequence generated by Algorithm 1 and satisfies the conditions* ($\mathcal{S}1$)-($\mathcal{S}4$). *Then, $\{u_n\}$ strongly converges to $u^* \in \Pi$ and $P_\Pi(0) = u^*$.*

Proof. It is given that $\rho_n \to \rho$ such that $\epsilon \in (0, 1 - \mu)$ and

$$\lim_{n \to \infty} \left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) = 1 - \mu > \epsilon > 0.$$

Then, there exits a finite number $n_1 \in \mathbb{N}$ such that

$$\left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) > \epsilon > 0, \ \forall n \geqslant n_1. \tag{3.11}$$

Combining expression (3.10) and (3.11), we obtain

$$\|u_{n+1} - u^*\|^2 \leqslant \|\chi_n - u^*\|^2, \ \forall n \geqslant n_1. \tag{3.12}$$

It is given in expression (3.1) that

$$\lim_{n \to +\infty} \frac{\alpha_n}{\gamma_n} \|u_n - u_{n-1}\| \leqslant \lim_{n \to +\infty} \frac{\epsilon_n}{\gamma_n} \|u_n - u_{n-1}\| = 0. \tag{3.13}$$

By the use of definition of $\{\chi_n\}$ and expression (3.13), we obtain

$$\|\chi_n - u^*\| = \|u_n + \alpha_n (u_n - u_{n-1}) - \gamma_n u_n - \alpha_n \gamma_n (u_n - u_{n-1}) - u^*\|$$

$$= \left\|(1-\gamma_n)(u_n-u^*)+(1-\gamma_n)\alpha_n(u_n-u_{n-1})-\gamma_n u^*\right\| \tag{3.14}$$
$$\leqslant (1-\gamma_n)\left\|u_n-u^*\right\|+(1-\gamma_n)\alpha_n\left\|u_n-u_{n-1}\right\|+\gamma_n\left\|u^*\right\|$$
$$\leqslant (1-\gamma_n)\left\|u_n-u^*\right\|+\gamma_n M_1, \tag{3.15}$$

where

$$(1-\gamma_n)\frac{\alpha_n}{\gamma_n}\left\|u_n-u_{n-1}\right\|+\left\|u^*\right\| \leqslant M_1.$$

Combining expressions (3.12) with (3.15), we obtain

$$\begin{aligned}\|u_{n+1}-u^*\| &\leqslant (1-\gamma_n)\|u_n-u^*\|+\gamma_n M_1\\ &\leqslant \max\left\{\|u_n-u^*\|, M_1\right\}\\ &\ \ \vdots\\ &\leqslant \max\left\{\|u_1-u^*\|, M_1\right\}.\end{aligned}$$

Thus, we conclude that the $\{u_n\}$ is a bounded sequence. Indeed, by expression (3.15) we have

$$\begin{aligned}\left\|\chi_n-u^*\right\|^2 &\leqslant (1-\gamma_n)^2\|u_n-u^*\|^2+\gamma_n^2M_1^2+2M_1\gamma_n(1-\gamma_n)\|u_n-u^*\|\\ &\leqslant \|u_n-u^*\|^2+\gamma_n\left[\gamma_n M_1^2+2M_1(1-\gamma_n)\|u_n-u^*\|\right] \qquad (3.16)\\ &\leqslant \|u_n-u^*\|^2+\gamma_n M_2,\end{aligned}$$

where $\gamma_n M_1^2+2M_1(1-\gamma_n)\|u_n-u^*\| \leqslant M_2$ for some $M_2>0$. From (3.10) and (3.16), we have

$$\|u_{n+1}-u^*\|^2 \leqslant \|u_n-u^*\|^2+\gamma_n M_2-\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|\chi_n-v_n\|^2-\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_{n+1}-v_n\|^2. \tag{3.17}$$

The rest of the proof classified into two parts.

Case 1: Suppose that a fixed number $n_2 \in \mathbb{N}$ $(n_2 \geqslant n_1)$ such that

$$\|u_{n+1}-u^*\| \leqslant \|u_n-u^*\|, \ \forall n \geqslant n_2.$$

The above expression implies that $\lim_{n\to+\infty}\|u_n-u^*\|$ exists and let $\lim_{n\to+\infty}\|u_n-u^*\| = l$ for some $l \geqslant 0$. From the expression (3.17), we have

$$\begin{aligned}&\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|\chi_n-v_n\|^2+\left(1-\frac{\mu\rho_n}{\rho_{n+1}}\right)\|u_{n+1}-v_n\|^2\\ &\leqslant \|u_n-u^*\|^2+\gamma_n M_2-\|u_{n+1}-u^*\|^2.\end{aligned} \tag{3.18}$$

Due to existence of a limit of sequence $\|u_n-u^*\|$ and $\gamma_n \to 0$, we deduce that

$$\|\chi_n-v_n\| \to 0 \quad \text{and} \quad \|u_{n+1}-v_n\| \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.19}$$

By the use of expression (3.19), we have

$$\lim_{n\to+\infty}\|\chi_n-u_{n+1}\| \leqslant \lim_{n\to+\infty}\|\chi_n-v_n\|+\lim_{n\to+\infty}\|v_n-u_{n+1}\| = 0. \tag{3.20}$$

Next, we have to compute

$$\begin{aligned}\|\chi_n-u_n\| &= \|u_n+\alpha_n(u_n-u_{n-1})-\gamma_n\left[u_n+\alpha_n(u_n-u_{n-1})\right]-u_n\|\\ &\leqslant \alpha_n\|u_n-u_{n-1}\|+\gamma_n\|u_n\|+\alpha_n\gamma_n\|u_n-u_{n-1}\|\\ &= \gamma_n\frac{\alpha_n}{\gamma_n}\|u_n-u_{n-1}\|+\gamma_n\|u_n\|+\gamma_n^2\frac{\alpha_n}{\gamma_n}\|u_n-u_{n-1}\| \longrightarrow 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

The last two expressions implies that

$$\lim_{n\to+\infty}\|u_n-u_{n+1}\|\leqslant\lim_{n\to+\infty}\|u_n-\chi_n\|+\lim_{n\to+\infty}\|\chi_n-u_{n+1}\|=0.$$

The above explanation implies that the sequences $\{\chi_n\}$ and $\{v_n\}$ are bounded. By the use of reflexivity of $\mathcal{E}$ and the boundedness of sequence $\{u_n\}$ there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\}\rightharpoonup\hat{u}\in\mathcal{E}$ as $k\to+\infty$. Next, we have to prove that $\hat{u}\in\Pi$. It is given that

$$v_{n_k}=P_{\mathcal{K}}[\chi_{n_k}-\rho\mathcal{S}(\chi_{n_k})],$$

that is equivalent to

$$\langle\chi_{n_k}-\rho\mathcal{S}(\chi_{n_k})-v_{n_k},v-v_{n_k}\rangle\leqslant 0,\ \forall v\in\mathcal{K}.$$

The above inequality implies that

$$\langle\chi_{n_k}-v_{n_k},v-v_{n_k}\rangle\leqslant\rho\langle\mathcal{S}(\chi_{n_k}),v-v_{n_k}\rangle,\ \forall v\in\mathcal{K}.$$

Furthermore, we obtain

$$\frac{1}{\rho}\langle\chi_{n_k}-v_{n_k},v-v_{n_k}\rangle+\langle\mathcal{S}(\chi_{n_k}),v_{n_k}-\chi_{n_k}\rangle\leqslant\langle\mathcal{S}(\chi_{n_k}),v-\chi_{n_k}\rangle,\ \forall v\in\mathcal{K}.\tag{3.21}$$

Due to boundedness of the sequence $\{\chi_{n_k}\}$ implies that $\{\mathcal{S}(\chi_{n_k})\}$ is also bounded. By the use of $\lim_{k\to\infty}\|\chi_{n_k}-v_{n_k}\|=0$ and $k\to\infty$ in (3.21), we obtain

$$\liminf_{k\to\infty}\langle\mathcal{S}(\chi_{n_k}),v-\chi_{n_k}\rangle\geqslant 0,\ \forall v\in\mathcal{K}.$$

Furthermore, we have

$$\langle\mathcal{S}(v_{n_k}),v-v_{n_k}\rangle=\langle\mathcal{S}(v_{n_k})-\mathcal{S}(\chi_{n_k}),v-\chi_{n_k}\rangle+\langle\mathcal{S}(\chi_{n_k}),v-\chi_{n_k}\rangle+\langle\mathcal{S}(v_{n_k}),\chi_{n_k}-v_{n_k}\rangle.\tag{3.22}$$

By the use of $\lim_{k\to\infty}\|\chi_{n_k}-v_{n_k}\|=0$ and $\mathcal{S}$ is L-Lipschitz continuous on $\mathcal{E}$ implies that

$$\lim_{k\to\infty}\|\mathcal{S}(\chi_{n_k})-\mathcal{S}(v_{n_k})\|=0,\tag{3.23}$$

which together with (3.22) and (3.23), we obtain

$$\liminf_{k\to\infty}\langle\mathcal{S}(v_{n_k}),v-v_{n_k}\rangle\geqslant 0,\ \forall v\in\mathcal{K}.$$

Let consider a sequence of positive numbers $\{\epsilon_k\}$ that is decreasing and converges to zero. For each k, we denote m_k by the smallest positive integer such that

$$\langle\mathcal{S}(\chi_{n_i}),v-\chi_{n_i}\rangle+\epsilon_k\geqslant 0,\ \forall i\geqslant m_k.\tag{3.24}$$

Due to $\{\epsilon_k\}$ is decreasing, thus $\{m_k\}$ is increasing.

Case A: If there exists a subsequence $\{\chi_{n_{m_{k_j}}}\}$ of $\{\chi_{n_{m_k}}\}$ such that $\mathcal{S}(\chi_{n_{m_{k_j}}})=0$, for all j. Let $j\to\infty$, we obtain

$$\langle\mathcal{S}(\hat{u}),v-\hat{u}\rangle=\lim_{j\to\infty}\langle\mathcal{S}(\chi_{n_{m_{k_j}}}),v-\hat{u}\rangle=0.$$

Hence $\hat{u}\in\mathcal{K}$, therefore we obtain $\hat{u}\in\Pi$.

Case B: If there exists $n_0 \in \mathbb{N}$ such that for all $n_{m_k} \geqslant n_0$, $\mathcal{S}(\chi_{n_{m_k}}) \neq 0$. Suppose that

$$\Sigma_{n_{m_k}} = \frac{\mathcal{S}(\chi_{n_{m_k}})}{\|\mathcal{S}(\chi_{n_{m_k}})\|^2}, \ \forall n_{m_k} \geqslant n_0.$$

On the basis of the above definition, we can obtain

$$\langle \mathcal{S}(\chi_{n_{m_k}}), \Sigma_{n_{m_k}} \rangle = 1, \ \forall n_{m_k} \geqslant n_0. \tag{3.25}$$

By using expressions (3.24) and (3.25) for all $n_{m_k} \geqslant n_0$, we have

$$\langle \mathcal{S}(\chi_{n_{m_k}}), \ \nu + \epsilon_k \Sigma_{n_{m_k}} - \chi_{n_{m_k}} \rangle \geqslant 0.$$

Due to the pseudomonotonicity of $\mathcal{S}$ for $n_{m_k} \geqslant n_0$, we have

$$\langle \mathcal{S}(\nu + \epsilon_k \Sigma_{n_{m_k}}), \ \nu + \epsilon_k \Sigma_{n_{m_k}} - \chi_{n_{m_k}} \rangle \geqslant 0.$$

For all $n_{m_k} \geqslant n_0$, we have

$$\langle \mathcal{S}(\nu), \nu - \chi_{n_{m_k}} \rangle \geqslant \langle \mathcal{S}(\nu) - \mathcal{S}(\nu + \epsilon_k \Sigma_{n_{m_k}}), \ \nu + \epsilon_k \Sigma_{n_{m_k}} - \chi_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{S}(\nu), \Sigma_{n_{m_k}} \rangle. \tag{3.26}$$

Due to $\{\chi_{n_k}\}$ weakly converges to $\hat{u} \in \mathcal{K}$ with $\mathcal{S}$ is sequentially weakly continuous on the set $\mathcal{K}$, we get $\{\mathcal{S}(\chi_{n_k})\}$ weakly converges to $\mathcal{S}(\hat{u})$. Suppose that $\mathcal{S}(\hat{u}) \neq 0$, we have

$$\|\mathcal{S}(\hat{u})\| \leqslant \liminf_{k \to \infty} \|\mathcal{S}(\chi_{n_k})\|.$$

Since $\{\chi_{n_{m_k}}\} \subset \{\chi_{n_k}\}$ and $\lim_{k \to \infty} \epsilon_k = 0$, we have

$$0 \leqslant \lim_{k \to \infty} \|\epsilon_k \Sigma_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|\mathcal{S}(\chi_{n_{m_k}})\|} \leqslant \frac{0}{\|\mathcal{S}(\hat{u})\|} = 0.$$

Consider that $k \to \infty$ in (3.26), we obtain

$$\langle \mathcal{S}(\nu), \nu - \hat{u} \rangle \geqslant 0, \ \forall \nu \in \mathcal{K}.$$

By the use of Minty Lemma 2.5, we infer that $\hat{u} \in \Pi$. By using the Lipschitz continuity and pseudomonotonicity of a mapping $\mathcal{S}$ implies that the solution set Π is a closed and convex set. It is given that $u^* = P_\Pi(0)$ and by using Lemma 2.1 (ii), we have

$$\langle 0 - u^*, \nu - u^* \rangle \leqslant 0, \ \forall \nu \in \Pi.$$

Next, we have to evaluate

$$\limsup_{n \to +\infty} \langle u^*, u^* - u_n \rangle = \lim_{k \to +\infty} \langle u^*, u^* - u_{n_k} \rangle = \langle u^*, u^* - \hat{u} \rangle \leqslant 0. \tag{3.27}$$

By the use of $\lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0$ and expression (3.27) implies that

$$\limsup_{n \to +\infty} \langle u^*, u^* - u_{n+1} \rangle \leqslant \limsup_{n \to +\infty} \langle u^*, u^* - u_n \rangle + \limsup_{n \to +\infty} \langle u^*, u_n - u_{n+1} \rangle \leqslant 0. \tag{3.28}$$

Take into account the expression (3.14), we have

$$\begin{aligned} &\|\chi_n - u^*\|^2 \\ &= \|u_n + \alpha_n(u_n - u_{n-1}) - \gamma_n u_n - \alpha_n \gamma_n (u_n - u_{n-1}) - u^*\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \big\|(1-\gamma_n)(u_n-u^*)+(1-\gamma_n)\alpha_n(u_n-u_{n-1})-\gamma_n u^*\big\|^2\\
&\leqslant \big\|(1-\gamma_n)(u_n-u^*)+(1-\gamma_n)\alpha_n(u_n-u_{n-1})\big\|^2+2\gamma_n\langle -u^*,\chi_n-u^*\rangle\\
&= (1-\gamma_n)^2\big\|u_n-u^*\big\|^2+(1-\gamma_n)^2\alpha_n^2\big\|u_n-u_{n-1}\big\|^2\\
&\quad +2\alpha_n(1-\gamma_n)^2\big\|u_n-u^*\big\|\big\|u_n-u_{n-1}\big\|+2\gamma_n\langle -u^*,\chi_n-u_{n+1}\rangle+2\gamma_n\langle -u^*,u_{n+1}-u^*\rangle\\
&\leqslant (1-\gamma_n)\big\|u_n-u^*\big\|^2+\alpha_n^2\big\|u_n-u_{n-1}\big\|^2+2\alpha_n(1-\gamma_n)\big\|u_n-u^*\big\|\big\|u_n-u_{n-1}\big\|\\
&\quad +2\gamma_n\big\|u^*\big\|\big\|\chi_n-u_{n+1}\big\|+2\gamma_n\langle -u^*,u_{n+1}-u^*\rangle\\
&= (1-\gamma_n)\big\|u_n-u^*\big\|^2+\gamma_n\Big[\alpha_n\big\|u_n-u_{n-1}\big\|\frac{\alpha_n}{\gamma_n}\big\|u_n-u_{n-1}\big\|\\
&\quad +2(1-\gamma_n)\big\|u_n-u^*\big\|\frac{\alpha_n}{\gamma_n}\big\|u_n-u_{n-1}\big\|+2\big\|u^*\big\|\big\|\chi_n-u_{n+1}\big\|+2\langle u^*,u^*-u_{n+1}\rangle\Big].
\end{aligned}
\tag{3.29}
$$

From expressions (3.12) and (3.29), we obtain

$$
\begin{aligned}
\big\|u_{n+1}-u^*\big\|^2 \leqslant (1-\gamma_n)\big\|u_n-u^*\big\|^2+\gamma_n\Big[\alpha_n\big\|u_n-u_{n-1}\big\|\frac{\alpha_n}{\gamma_n}\big\|u_n-u_{n-1}\big\|&\\
+2(1-\gamma_n)\big\|u_n-u^*\big\|\frac{\alpha_n}{\gamma_n}\big\|u_n-u_{n-1}\big\|+2\big\|u^*\big\|\big\|\chi_n-u_{n+1}\big\|+2\langle u^*,u^*-u_{n+1}\rangle\Big]&.
\end{aligned}
\tag{3.30}
$$

By the use of (3.20), (3.28), (3.30), and applying Lemma 2.2 conclude that $\lim_{n\to+\infty}\big\|u_n-u^*\big\|=0$.

Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i}-u^*\|\leqslant\|u_{n_{i+1}}-u^*\|,\ \forall\, i\in\mathbb{N}.$$

By using Lemma 2.3 there exists a sequence $\{m_k\}\subset\mathbb{N}$ as $m_k\to+\infty$ such that

$$\|u_{m_k}-u^*\|\leqslant\|u_{m_{k+1}}-u^*\| \quad\text{and}\quad \|u_k-u^*\|\leqslant\|u_{m_{k+1}}-u^*\|,\ \text{for all } k\in\mathbb{N}. \tag{3.31}$$

As similar to the Case 1, the expression (3.18) implies that

$$\Big(1-\frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\Big)\|\chi_{m_k}-\nu_{m_k}\|^2+\Big(1-\frac{\mu\rho_{m_k}}{\rho_{m_k+1}}\Big)\|u_{m_k+1}-\nu_{m_k}\|^2\leqslant\|u_{m_k}-u^*\|^2+\gamma_{m_k}M_2-\|u_{m_k+1}-u^*\|^2.$$

Due to $\gamma_{m_k}\to 0$, we deduce the followings:

$$\lim_{k\to+\infty}\|\chi_{m_k}-\nu_{m_k}\|=\lim_{k\to+\infty}\|u_{m_k+1}-\nu_{m_k}\|=0.$$

It continues from that

$$\lim_{k\to+\infty}\|u_{m_{k+1}}-\chi_{m_k}\|\leqslant\lim_{k\to+\infty}\|u_{m_{k+1}}-\nu_{m_k}\|+\lim_{k\to+\infty}\|\nu_{m_k}-\chi_{m_k}\|=0.$$

Next, we will evaluate

$$
\begin{aligned}
\|\chi_{m_k}-u_{m_k}\|&=\|u_{m_k}+\alpha_{m_k}(u_{m_k}-u_{m_k-1})-\gamma_{m_k}\big[u_{m_k}+\alpha_{m_k}(u_{m_k}-u_{m_k-1})\big]-u_{m_k}\|\\
&\leqslant\alpha_{m_k}\|u_{m_k}-u_{m_k-1}\|+\gamma_{m_k}\|u_{m_k}\|+\alpha_{m_k}\gamma_{m_k}\|u_{m_k}-u_{m_k-1}\|\\
&=\gamma_{m_k}\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k}-u_{m_k-1}\|+\gamma_{m_k}\|u_{m_k}\|+\gamma_{m_k}^2\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k}-u_{m_k-1}\|\longrightarrow 0.
\end{aligned}
$$

The above implies that

$$\lim_{k\to+\infty}\|u_{m_k}-u_{m_k+1}\|\leqslant\lim_{k\to+\infty}\|u_{m_k}-\chi_{m_k}\|+\lim_{k\to+\infty}\|\chi_{m_k}-u_{m_k+1}\|=0.$$

By using the same argument as in Case 1, we obtain

$$\limsup_{k\to+\infty}\langle u^*, u^* - u_{m_k+1}\rangle \leqslant 0. \tag{3.32}$$

Combining expressions (3.30) and (3.31), we obtain

$$\begin{aligned}
&\|u_{m_k+1} - u^*\|^2 \\
&\leqslant (1-\gamma_{m_k})\|u_{m_k} - u^*\|^2 + \gamma_{m_k}\Big[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| \\
&\quad + 2(1-\gamma_{m_k})\|u_{m_k} - u^*\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|u^*\|\|\chi_{m_k} - u_{m_k+1}\| + 2\langle u^*, u^* - u_{m_k+1}\rangle\Big] \\
&\leqslant (1-\gamma_{m_k})\|u_{m_{k+1}} - u^*\|^2 + \gamma_{m_k}\Big[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| \\
&\quad + 2(1-\gamma_{m_k})\|u_{m_k} - u^*\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|u^*\|\|\chi_{m_k} - u_{m_k+1}\| + 2\langle u^*, u^* - u_{m_k+1}\rangle\Big].
\end{aligned}$$

The above expression implies that

$$\begin{aligned}
&\|u_{m_k+1} - u^*\|^2 \\
&\leqslant \Big[\alpha_{m_k}\|u_{m_k} - u_{m_k-1}\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| \\
&\quad + 2(1-\gamma_{m_k})\|u_{m_k} - u^*\|\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_k-1}\| + 2\|u^*\|\|\chi_{m_k} - u_{m_k+1}\| + 2\langle u^*, u^* - u_{m_k+1}\rangle\Big].
\end{aligned} \tag{3.33}$$

Since $\gamma_{m_k} \to 0$, and $\|u_{m_k} - u^*\|$ is a bounded, expressions (3.32) and (3.33) imply that

$$\|u_{m_k+1} - u^*\|^2 \to 0, \text{ as } k \to +\infty.$$

It implies that

$$\lim_{n\to+\infty}\|u_k - u^*\|^2 \leqslant \lim_{n\to+\infty}\|u_{m_k+1} - u^*\|^2 \leqslant 0.$$

As a consequence $u_n \to u^*$. This completes the proof of theorem. □

4. Numerical illustrations

This section examines three numerical examples to show the efficiency of the proposed methods. Any of these numerical experiments provide a detailed understanding of how better control parameters can be chosen. Some of them show the advantages of the proposed methods compared to existing ones in the literature.

Example 4.1. First consider the HpHard problem that is taken from [6]. Let $\mathcal{S} : \mathbb{R}^N \to \mathbb{R}^N$ be an operator is defined by $\mathcal{S}(u) = Mu + q$, where $q \in \mathbb{R}^N$ and

$$M = AA^T + B + D,$$

where A is an $N \times N$ matrix, B is an $N \times N$ skew-symmetric matrix and D is an $N \times N$ positive definite diagonal matrix. The set $\mathcal{K}$ is taken in the following way:

$$\mathcal{K} = \{u \in \mathbb{R}^N : -100 \leqslant u_i \leqslant 100\}.$$

It is clear that $\mathcal{S}$ is monotone and Lipschitz continuous through $L = \|M\|$. The control condition are taken as follows: (1) Algorithm 3.1 in [32]: $\rho_0 = 0.15, \mu = 0.75, \alpha_n = 0.65$; (2) Algorithm 1: $\rho_0 = 0.15, \mu = 0.75, \alpha = 0.65, \epsilon_n = \frac{1}{(n+1)^2}, \gamma_n = \frac{1}{10(n+2)}, \varphi_n = \frac{100}{(n+1)^2}$. During this experiment, the initial point is $u_{-1} = u_0 = (2, 2, \ldots, 2)$ and $D_n = \|\chi_n - \nu_n\| \leqslant 10^{-4}$. The numerical results of these methods are shown in Table 1.

Table 1: Numerical illustrations for both methods.

	Algorithm 2 in [32]		Algorithm 1	
N	**Number of Iterations**	**Elapsed Time**	**Number of Iterations**	**Elapsed Time**
5	25	0.123786	13	0.086372
20	41	0.283782	20	0.226123
50	123	0.998352	41	0.641284
100	311	2.241831	45	1.263114
200	225	7.462846	53	3.462451
500	119	12.46582	62	5.568421

Example 4.2. Consider the quadratic fractional programming problem in the following form [7]:

$$\begin{cases} \min f(u) = \dfrac{u^T Q u + a^T u + a_0}{b^T u + b_0}, \\ \text{subject to } u \in \mathcal{K} = \{u \in \mathbb{R}^4 : b^T u + b_0 > 0\}, \end{cases}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \; a_0 = -2, \quad \text{and} \quad b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite on $\mathbb{R}^4$ and consequently f is pseudo-convex on $\mathcal{K}$. Hence, ∇f is pseudo-monotone. Using the quotient rule, we obtain

$$\nabla f(u) = \frac{(b^T u + b_0)(2Qu + a) - b(u^T Q + a^T u + a_0)}{(b^T u + b_0)^2}. \tag{4.1}$$

In this point of view, we can set $\mathcal{S} = \nabla f$ in Theorem 3.3. We minimize f over $\mathcal{K} = \{u \in \mathbb{R}^4 : 1 \leqslant u_i \leqslant 10,\ i = 1,2,3,4\}$. This problem has a unique solution $u^* = (1,1,1,1)^T \in \mathcal{K}$. The control condition are taken as follows: (1) Algorithm 2 in [32]: $\rho_0 = 0.25, \mu = 0.70, \alpha_n = 0.65$; (2) Algorithm 1: $\rho_0 = 0.25, \mu = 0.70, \alpha = 0.65, \epsilon_n = \frac{1}{(n+1)^2}, \varphi_n = \frac{100}{(n+1)^2}, \gamma_n = \frac{1}{100(n+2)}$. During this experiment, the initial points are different and $D_n = \|\chi_n - v_n\| \leqslant 10^{-4}$. The numerical results of these methods are shown in Tables 2-5.

Table 2: Example 4.2: numerical study of Algorithm 2 in [32] and $u_0 = u_1 = [2, -5, 5, -2]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	9.73621052092642	0.833094840825805	9.90241332280372	1.04929123705160
2	9.65398088222099	0.999443650015545	9.83074011956143	1.02908763007986
3	9.57174604515717	1.06492890311881	9.75920250955472	1.00888315702824
4	9.48955289901919	1.12923908105825	9.68783224607597	1.00001777899295
5	9.40743194326346	1.19231579167598	9.61664829940396	1.00000002980971
6	9.32539697955624	1.25416309375667	9.54565902708142	1.00000000018713
7	9.24345882537906	1.31478659951554	9.47487029657638	1.00000000011988
8	9.16162656381652	1.37419323423997	9.40428650097328	1.00000020053245
9	9.07990820317225	1.43239065856804	9.33391113866474	1.00000017038491
10	8.99831103926803	1.48938694679281	9.26374713071480	1.00000014610962
⋮	⋮	⋮	⋮	⋮
111	1.00031601484227	0.999707404385613	1.13272909195984	0.999967307570371
112	1.00031215935883	0.999714823249607	1.11242178130774	0.999968045673433
113	1.00030820768938	0.999722106421074	1.09260313929843	0.999968775268394
114	1.00030416718112	0.999729253535189	1.07326478974856	0.999969496171197
115	1.00030004508967	0.999736264410505	1.05439834405683	0.999970208192181
116	1.00029584834516	0.999743139236891	1.03599542429133	0.999970911118532
117	1.00029157777940	0.999749866096390	1.01804760355120	0.999971604705314
118	1.00028704334254	0.999756607981979	1.00055874144751	0.999972257950801
119	1.00002617317537	0.999977974772264	1.00002397019683	0.999992761540142
CPU time is seconds	0.657361			

Table 3: Example 4.2: numerical study of Algorithm 1 and $u_0 = u_1 = [2, -5, 5, -2]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	5.55377533757632	-2.90587269134480	8.61905463436968	1.72038166468790
2	4.41040024375028	1	7.45023669240132	1.19425608076334
3	12.3448991907237	-12.6397501083870	13.0658963027246	-13.2192576516318
4	3.05882604768791	-7.92358662619279	7.05918169098723	-2.21603141462490
5	2.13444665314296	-7.44999847685431	6.31036964537031	1.12023327884896
6	1.87015108945082	1.00000000097286	6.02864753785448	0.955002424010157
7	1.46861344878822	1.15019734000242	5.43281104076976	1.00000000001322
8	2.05261045125816	-0.507408676818891	4.10590329281081	0.236813108691433
9	1.43898856735670	0.490171438084139	3.43815944442871	1.28802829936381
10	1.29986269980453	0.213085356892448	2.58372157493742	0.853687511480199
⋮	⋮	⋮	⋮	⋮
41	1.00004416092722	0.999975499519528	1.00001716683660	0.999986491695085
42	1.00003823749786	0.999978633212697	1.00001506391126	0.999988330027726
43	1.00003579248111	0.999980505234820	1.00001322064060	0.999989034148032
44	1.00003668241231	0.999980324093982	1.00001308096942	0.999988723923595
45	1.00004014710912	0.999978308970547	1.00001461600524	0.999987666312513
46	1.00004007642355	0.999977850272896	1.00001542714126	0.999987735088239
47	1.00003584694207	0.999980050168924	1.00001399157339	0.999989050797591
48	1.00003405336513	0.999981402563276	1.00001267126779	0.999989569857342
49	1.00003483208481	0.999981190052566	1.00001263286680	0.999989305033961
CPU time is seconds	0.218463			

Table 4: Example 4.2: numerical study of Algorithm 2 in [32] and $u_0 = u_1 = [1, -2, 5, -4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	8.77637333455425	-0.0495715784342590	9.29094270258555	0.811840145149117
2	8.69495843974052	0.997670568432512	9.21911519294898	0.993720619633674
3	8.61358693872012	1.06067800577058	9.14744366755259	0.999984362800348
4	8.53228969180900	1.12238972162512	9.07596267735889	0.999999968846574
5	8.45108721657438	1.18280447433853	9.00468628126089	1.00000020030451
6	8.36999399345207	1.24192670810562	8.93362315817494	1.00000016812645
7	8.28902146271176	1.29976325291497	8.86277934051403	1.00000014262042
8	8.20817933122734	1.35632219180389	8.79215937750138	1.00000012236522
9	8.12747622756739	1.41161229504697	8.72176691752843	1.00000010618077
10	8.04692006065515	1.46564271291772	8.65160502630982	1.00000009284194
⋮	⋮	⋮	⋮	⋮
91	1.00031401521149	0.999711293014288	1.12206970860473	0.999967693819351
92	1.00031010862601	0.999718641089626	1.10201854920601	0.999968427501599
93	1.00030610969925	0.999725853261276	1.08245168429949	0.999969152580824
94	1.00030202573472	0.999732929257881	1.06336073020762	0.999969868869467
95	1.00029786389858	0.999739869024494	1.04473729773544	0.999970576173761
96	1.00029362989013	0.999746673790074	1.02657308709375	0.999971274140246
97	.00028932639364	0.999753330048160	1.00885960830576	0.999971962732758
98	1.00014344006196	0.999865399336672	1.00024238576235	0.999967034603878
99	1.00001368001678	0.999988955179276	1.00001158885806	0.999996120420831
CPU time is seconds	0.438657			

Table 5: Example 4.2: numerical study of Algorithm 1 and $u_0 = u_1 = [1, -2, 5, -4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	3.62540316591384	-4.50907358998138	6.32280270247068	-0.00324645702473703
2	2.35551832614327	1.00000000005651	5.05780638110935	-0.698371567191649
3	1.71910447671045	0.953160264334062	4.29177164937510	0.999999999525121
4	1.15470777752108	0.895315463559908	3.59372641421376	0.999999983377171
5	1.13296432437644	0.650962900197076	2.75601344318112	0.902777544700774
6	1.23231081634727	0.698699138712538	2.24361021384599	0.932339788593485
7	1.09676723997533	0.938983693943763	1.99524423852901	1.01664900928582
8	1.06998496490214	0.914461755684416	1.67181167392322	0.988172769212471
9	1.10337085979123	0.897210465543632	1.39018023991409	0.981585822559744
10	1.06931881399759	0.948386311174497	1.22427628816323	0.995433685763384
⋮	⋮	⋮	⋮	⋮
33	1.00003232233466	0.999983095332643	1.00001083828621	0.999990015035678
34	1.00003157023173	0.999983454013759	1.00001064467299	0.999990250646380
35	1.00003082551930	0.999983859831028	1.00001036357615	0.999990479905837
36	1.00003052571655	0.999984045978129	1.00001021378732	0.999990569779100
37	1.00003025202721	0.999984177903946	1.00001014388048	0.999990654787794
38	1.00002964901522	0.999984471374164	1.00000997843332	0.999990843128007
39	1.00002905083912	0.999984794582010	1.00000975810614	0.999991027421348
CPU time is seconds	0.160578			

Example 4.3. Suppose that the non-linear complementarity problem of Kojima–Shindo while the feasible set $\mathcal{K}$ is

$$\mathcal{K} = \{u \in \mathbb{R}^4 : 1 \leqslant u_i \leqslant 5,\ i = 1, 2, 3, 4\}.$$

Let the mapping $\mathcal{S} : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$\mathcal{S}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}.$$

It is easy to see that $\mathcal{S}$ is not monotone on the set $\mathcal{K}$. By using the Monte-Carlo approach [7] it can be shown that $\mathcal{S}$ is pseudomonotone on $\mathcal{K}$. This problem has a unique solution $u^* = (5, 5, 5, 5)^T$. Actually, in general, it is a very difficult task to check the pseudomonotonicity of any mapping $\mathcal{S}$ in practice. We here employ the Monte-Carlo approach according to the definition of pseudomonotonicity: Generate a large number of pairs of points u and v uniformly in $\mathcal{K}$ satisfying $\mathcal{S}(u)^T(v - u) \geqslant 0$ and then check if $\mathcal{S}(v)^T(v - u) \geqslant 0$. The control conditions are taken as follows: (1) Algorithm 2 in [32]: $\rho_0 = 0.35, \mu = 0.33, \alpha_n = 0.53$; (2) Algorithm 1: $\rho_0 = 0.35, \mu = 0.33, \alpha = 0.53, \epsilon_n = \frac{1}{(n+1)^2}, \varphi_n = \frac{100}{(n+1)^2}, \gamma_n = \frac{1}{(n+2)}$. During this experiment, the initial points are different and $D_n = \|\chi_n - v_n\| \leqslant 10^{-4}$. The numerical results of these methods are shown in Tables 6-9.

Table 6: Example 4.3: numerical study of Algorithm 2 in [32] and $u_0 = u_1 = [1, -1, 3, 5]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	-34.4858669275805	51.8664403130701	74.7007514676511	81.6555636007234
2	-34.7655055354728	52.7053561367472	75.5396672913282	82.4944794244005
3	-68.6885783093673	54.9801618868864	49.8539261662733	48.7706022828314
4	17.3186243752486	21.2315776206067	33.0214233757886	35.6941859283492
5	2.14464032966519	3.41342758196463	6.08695506300084	6.57660634608289
6	2.16077834343589	3.42336001727149	5.00079945477229	5.00277316956197
7	2.17693740637592	3.43332429093693	5.00000861166124	4.99999726228770
8	2.19314690959707	3.44336852111214	4.99999946975626	4.99999956233048
9	2.20940684182298	3.45349275161455	4.99999955360422	4.99999955283736
10	2.22571727346680	3.46369703796100	4.99999959252883	4.99999959253525
⋮	⋮	⋮	⋮	⋮
89	4.86520623565335	4.99999982676687	4.99999982676687	4.99999982676687
90	4.88890753296478	4.99999983165645	4.99999983165645	4.99999983165645
91	4.91260767205413	4.99999983609968	4.99999983609968	4.99999983609968
92	4.93630665260868	4.99999984001939	4.99999984001939	4.99999984001939
93	4.96000446723282	4.99999984294752	4.99999984294752	4.99999984294752
94	4.98370113286901	4.99999999990961	4.99999999990961	4.99999999990961
95	4.99997372263069	5.00000260496717	5.00000260496717	5.00000260496717
96	4.99999974352490	4.99999988910880	4.99999988910880	4.99999988910880
CPU time is seconds	0.498705			

Table 7: Example 4.3: numerical study of Algorithm 1 and $u_0 = u_1 = [1, -1, 3, 5]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	-22.4790198515975	45.1962163525125	61.2227868529569	64.3371909274850
2	-22.7250625769091	45.8890698132465	61.9364390512654	65.0528149718569
3	-46.9191743963588	45.3948805004849	49.2573891717001	50.2790369811032
4	4.76470808480016	31.6637577598404	6.57170133017410	0.238787400016093
5	4.82937349335325	4.99939226879006	5.01026963319225	0.366396789829221
6	34.3710564606067	34.4227198682972	34.4353964718562	0.629949296125400
7	45.0594866095192	45.1112124144806	45.1239165658566	1.91644510896386
8	50.4501740621952	50.5022462269367	50.5150352890470	-1.97169151358552
9	50.5314043274895	50.5834595574629	50.5962444601586	-2.00626829664043
10	50.6403242132421	50.6917607820349	50.7043937855899	-5.12057188697462
⋮	⋮	⋮	⋮	⋮
39	5.00534491845521	5.00534300138664	5.00534228567228	4.98393420214921
40	5.00382749328779	5.00382612764508	5.00382561779860	4.98850257752182
41	5.00271869976732	5.00271773349804	5.00271737275274	4.99183705482842
42	5.00190964154940	5.00190896479704	5.00190871213941	4.99426819503386
43	5.00132055408830	5.00132008712633	5.00131991279145	4.99603731806545
44	5.00089307847857	5.00089276321315	5.00089264551241	4.99732055470556
45	5.00058462738082	5.00058442128729	5.00058434434462	4.99824621008654
46	5.00036429098967	5.00036416273020	5.00036411484598	4.99890728946919
47	5.00021838609581	5.00021830930436	5.00021828063514	4.99934498398791
48	5.00014354541464	5.00014349500391	5.00014347618363	4.99956947666068
CPU time is seconds	0.232178			

Table 8: Example 4.3: numerical study of Algorithm 2 in [32] and $u_0 = u_1 = [-1, 2, -3, 4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	-6.20601542924621	40.0843112633838	48.1337650982707	46.6230007187220
2	-6.34498204992797	40.5012111254306	48.5506649603177	47.0399005807689
3	-12.9307982468928	40.0487124082963	47.5003319282679	46.0967916997756
4	-12.8245110997443	40.1793368929725	47.6269383722835	46.2241156086185
5	-33.1769125028857	34.7884817072125	37.8166900961878	37.2122205719181
6	-3.63715262714896	20.5019731319843	-1.76185029487043	2.54576398430683
7	1.04302948168927	5.05452389604395	1.10650198176853	2.54725466388794
8	1.04978285120354	5.00006058170198	1.11279404365063	2.54921548696884
9	1.05658705719584	4.99999076271743	1.11913985722867	2.55121489498328
10	1.06344317268371	4.99998642519517	1.12554051292081	2.55325387465541
⋮	⋮	⋮	⋮	⋮
71	4.59862733756248	5.00000320217875	4.61146934022478	4.99999682258005
72	4.66305529670103	4.99999994825175	4.67571532974269	5.00000005125787
73	4.72840120415464	5.00000000060611	4.74088184600983	4.99999999898082
74	4.79467821219419	4.99999999981385	4.80698200476301	4.99999999983888
75	4.86189966048480	4.99999999985196	4.87402910964953	4.99999999985158
76	4.93007907865535	4.99999999986557	4.94203665454547	4.99999999986557
77	4.99908430943714	4.99994852203121	5.00068438967529	4.99994852203121
78	4.99998796497466	5.00000039194907	5.00001096999677	5.00000039194907
79	4.99999982724761	5.00000000655933	5.00000015919175	5.00000000655933
CPU time is seconds	0.6462847			

Table 9: Example 4.3: numerical study of Algorithm 1 and $u_0 = u_1 = [-1, 2, -3, 4]^T$.

Iter (n)	u_1	u_2	u_3	u_4
1	67.7657143133604	76.7145530819899	119.464591876179	136.782073520191
2	-68.2650687067241	78.1919764133559	120.945690214610	138.263931342072
3	-112.028480196941	102.736419249501	68.7617248229671	59.8025934358457
4	10.0905654923274	7.52762193553885	48.8773488496240	59.7184201585889
5	1.58599071741342	0.271536323838546	5.42220385209038	5.83759718234703
6	1.76489239217402	0.956593352840844	5.01190765954745	4.97568499729916
7	18.9241448354569	-5.26396384315423	18.9282891809255	19.1336306496812
8	19.1267857198680	-5.33218756241530	19.1290503062070	19.3344556330365
9	19.1362728432230	-6.71061680639647	19.1381744023155	19.3158660670749
10	13.8198984213228	-14.2428711989199	13.8195679016292	13.7892140927893
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
29	5.00036606449632	4.99890329629848	5.00036604809619	5.00036452257377
30	5.00029614550570	4.99911277307117	5.00029613224497	5.00029489874547
31	5.00024019269176	4.99928042449310	5.00024018194207	5.00023918201651
32	5.00021568341312	4.99935386554261	5.00021567376480	5.00021477628717
33	5.00022114499712	4.99933751266914	5.00022113510826	5.00022021525583
34	5.00023906491462	4.99928382965716	5.00023905422789	5.00023806015776
35	5.00024676624015	4.99926074961903	5.00024675521250	5.00024572943005
36	5.00023803185504	4.99928690765120	5.00023802122102	5.00023703205387
37	5.00022237747575	4.99933380427636	5.00022236754428	5.00022144372719
38	5.00020993400317	4.99937108540127	5.00020992463042	5.00020905278423
39	5.00020466823032	4.99938686331783	5.00020465909544	5.00020380937576
CPU time is seconds	0.136686			

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