Modified subgradient extragradient method to solve variational inequalities

Kanikar Muangchoo

Faculty of Science and Technology, Rajamangala University of Technology Phra Nakhon (RMUTP), 1381 Pracharat 1 Road, Wongsawang, Bang Sue, Bangkok 10800, Thailand.

Abstract

In this paper, we introduce a new method to solve pseudomonotone variational inequalities with the Lipschitz condition in a real Hilbert space. This problem is a general mathematical problem in the sense that it unifies a number of the mathematical problems as a particular case, such as the optimization problems, the equilibrium problems, the fixed point problems, the saddle point problems and Nash equilibrium point problems. The new method is constructed around two methods: the extragradient method and the inertial method. The proposed method uses a new stepsize rule based on local operator information rather than its Lipschitz constant or any other line search method. The proposed method does not require any knowledge of the Lipschitz constant of an operator. The strong convergence of the proposed method is well-established. Finally, we conduct a number of numerical experiments to determine the performance and superiority of the proposed method.

Keywords: Variational inequality problem, subgradient extragradient-like method, strong convergence result, Lipschitz continuity, pseudomonotone mapping.

2020 MSC: 65Y05, 65K15, 68W10, 47H05, 47H10.

1. Introduction

This paper examines the problem of classic variational inequalities [29] and the variational inequality problem (VIP) for mapping $S : \mathcal{E} \rightarrow \mathcal{E}$ is defined as follows:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } \langle S(u^*), v - u^* \rangle \geq 0, \quad \forall v \in \mathcal{K},$$

(VIP)

where $\mathcal{K}$ is a nonempty, convex and closed subset of a certain Hilbert space $\mathcal{E}$ and $\langle ., . \rangle$ and $\| . \|$ serve as an inner product and the induced norm in $\mathcal{E}$, respectively. Moreover, $\mathbb{R}$, and $\mathbb{N}$ are denoted the sets of real numbers and natural numbers, respectively. It is important to note that the problem (VIP) is equivalent to solve the following problem:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } u^* = \text{P}_\mathcal{K}[u^* - \rho S(u^*)].$$

Email address: kanikar.m@rmutp.ac.th (Kanikar Muangchoo)
doi: 10.22436/jmcs.025.02.03

Received: 2021-01-08  Revised: 2021-03-26  Accepted: 2021-04-15
The variational inequality problem was introduced by Stampacchia [29] in 1964. This is an important mathematical model that unifies several key topics of applied mathematics, such as the network equilibrium problems, the necessary optimality conditions, the complementarity problems and the systems of nonlinear equations (for more details [5, 8, 9, 11, 12, 16, 33]) and others in [1, 10, 15, 17, 19–28]. Korpelevich [13] and Antipin [2] established the following extragradient method:

\[
\begin{align*}
  &u_0 \in \mathcal{K}, \\
  &v_n = P_{\mathcal{K}}[u_n - \rho S(u_n)], \\
  &u_{n+1} = P_{\mathcal{K}}[u_n - \rho S(v_n)].
\end{align*}
\]

Recently, the subgradient extragradient method was introduced by Censor et al. [4] for solving the problem (VIP) in a real Hilbert space. Their method takes the form

\[
\begin{align*}
  &u_0 \in \mathcal{K}, \\
  &v_n = P_{\mathcal{K}}[u_n - \rho S(u_n)], \\
  &u_{n+1} = P_{E_n}[u_n - \rho S(v_n)],
\end{align*}
\]

where \( E_n = \{ z \in \mathcal{E} : \langle u_n - \rho S(u_n) - v_n, z - v_n \rangle \leq 0 \} \).

It is important to note that the proposed well-established method has two serious shortcomings, the first being the fixed constant stepsize, which requires the information or approximation of the Lipschitz constant of the associated operator and is only weakly convergent in Hilbert spaces. From a numerical point of view, it could be challenging to use a fixed step size and thus the convergence rate and performance of the method may be affected. The main purpose of this study is to set up a new inertial-type method that is used to enhance the convergence rate of the iterative sequence and independent of the knowledge Lipschitz constants. These methods have been previously established due to the oscillator equation with a damping and conservative force restoration. This second-order dynamical system is called a heavy friction ball, which was formerly designed by Polyak in [18]. So there is important question that arises:

“Is it possible to introduce a new strongly convergent inertial-type strongly convergent extragradient method with a non-monotone variable step sizerule”?

In this paper, we present a positive answer of the above question, that is, the gradient method indeed establishes a strong convergence sequence by using a variable step sizerule for solving the problem (VIP) combined with pseudomonotone mappings. Motivated by the works of Censor et al. [4] and Polyak [18], we introduce a new inertial extragradient-type method to figure out the problem (VIP) in the setting of an infinite-dimensional real Hilbert space.

The rest of the paper is arranged as follows. The Section 2 consists of the necessary definitions and fundamental lemmas used in this paper. Section 3 consists of an inertial-type iterative scheme and convergence analysis theorem. Section 4 provides numerical results to explain the performance of the new method and compare it with other methods. The following conditions are satisfied in order to study the strong convergence.

(S1) The solution set of problem (VIP) is denoted by \( \Pi \) is nonempty.

(S2) An operator \( S : \mathcal{E} \to \mathcal{E} \) is said to be pseudomonotone if

\[
\langle S(v_1), v_2 - v_1 \rangle \geq 0 \implies \langle S(v_2), v_1 - v_2 \rangle \leq 0, \quad \forall v_1, v_2 \in \mathcal{K}.
\]

(S3) An operator \( S : \mathcal{E} \to \mathcal{E} \) is said to be \emph{Lipschitz continuous} with constant \( L > 0 \) if there exists \( L > 0 \) such that

\[
\|S(v_1) - S(v_2)\| \leq L\|v_1 - v_2\|, \quad \forall v_1, v_2 \in \mathcal{K}.
\]

(S4) An operator \( S : \mathcal{E} \to \mathcal{E} \) is said to be \emph{sequentially weakly continuous} if \( S(u_n) \) converges weakly to \( S(u) \) for every sequence \( \{u_n\} \) that converges weakly to \( u \).
2. Preliminaries

In this section of the text, we have written a number of significant identities and related lemmas and definitions. The metric projection \( P_{\mathcal{K}}(v_1) \) of \( v_1 \in \mathcal{E} \) is defined by

\[
P_{\mathcal{K}}(v_1) = \arg\min\{\|v_1 - v_2\| : v_2 \in \mathcal{K}\}.
\]

Next, we list some of the important properties of the projection mapping.

Lemma 2.1 ([3]). Suppose that \( P_{\mathcal{K}} : \mathcal{E} \to \mathcal{K} \) is a metric projection. Then, we have

(i) \( v_3 = P_{\mathcal{K}}(v_1) \) if and only if \( \langle v_1 - v_3, v_2 - v_3 \rangle \leq 0, \quad \forall v_2 \in \mathcal{K} \);
(ii) \( \|v_1 - P_{\mathcal{K}}(v_2)\|^2 + \|P_{\mathcal{K}}(v_2) - v_2\|^2 \leq \|v_1 - v_2\|^2, \quad v_1 \in \mathcal{K}, v_2 \in \mathcal{E} \);
(iii) \( \|v_1 - P_{\mathcal{K}}(v_1)\| \leq \|v_1 - v_2\|, \quad v_2 \in \mathcal{K}, \quad v_1 \in \mathcal{E} \).

Lemma 2.2 ([31]). Let \( \{p_n\} \subset [0, +\infty) \) be a sequence satisfying the following inequality

\[
p_{n+1} \leq (1 - q_n)p_n + q_nr_n, \quad \forall n \in \mathbb{N}.
\]

Furthermore, \( \{q_n\} \subset (0, 1) \) and \( \{r_n\} \subset \mathbb{R} \) be two sequences such that

\[
\lim_{n \to +\infty} q_n = 0, \quad \sum_{n=1}^{+\infty} q_n = +\infty \quad \text{and} \quad \limsup_{n \to +\infty} r_n \leq 0.
\]

Then, \( \lim_{n \to +\infty} p_n = 0 \).

Lemma 2.3 ([14]). Suppose that \( \{p_n\} \) is a sequence of real numbers such that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that

\[
p_{n_i} < p_{n_{i+1}}, \quad \forall \quad i \in \mathbb{N}.
\]

Then, there is a nondecreasing sequence \( m_k \subset \mathbb{N} \) such that \( m_k \to +\infty \) as \( k \to +\infty \), and meet the following requirements for numbers \( k \in \mathbb{N} \):

\[
p_{m_k} \leq p_{m_{k+1}} \quad \text{and} \quad p_k \leq p_{m_{k+1}}.
\]

Indeed, \( m_k = \max\{j \leq k : p_j \leq p_{j+1}\} \).

Next, we list some of the important identities that were used to prove the convergence analysis.

Lemma 2.4 ([3]). For any \( v_1, v_2 \in \mathcal{E} \) and \( \ell \in \mathbb{R} \), the following inequalities are holds.

(i) \( \|\ell v_1 + (1 - \ell)v_2\|^2 = \ell \|v_1\|^2 + (1 - \ell)\|v_2\|^2 - \ell (1 - \ell)\|v_1 - v_2\|^2 \);
(ii) \( \|v_1 + v_2\|^2 \leq \|v_1\|^2 + 2\langle v_2, v_1 + v_2 \rangle \).

Lemma 2.5 ([30]). Assume that \( S : \mathcal{K} \to \mathcal{E} \) is a pseudomonotone and continuous mapping. Then, \( u^* \) is a solution of the problem (VIP) if and only if \( u^* \) is a solution of the following problem.

\[
\text{Find } u \in \mathcal{K} \text{ such that } \langle S(v), v - u \rangle \geq 0, \quad \forall v \in \mathcal{K}.
\]

3. Main Results

In this section, we present a new inertial-type subgradient extragradient method that includes the new stepsize rule and inertial term, as well as the strong convergence theorem. The main result is as follows.
Algorithm 1

**Step 0:** Let \( u_{-1}, u_0 \in \mathcal{K}, \alpha > 0, \mu \in (0, 1), \rho_0 > 0 \) and select a nonnegative real sequence \( \{\varphi_n\} \) such that \( \sum_{n=1}^{+\infty} \varphi_n < +\infty \). Moreover, choose \( \{\gamma_n\} \subset (0, 1) \) satisfying the following conditions:

\[
\lim_{n \to +\infty} \gamma_n = 0 \text{ and } \sum_{n=1}^{+\infty} \gamma_n = +\infty.
\]

**Step 1:** Compute

\[
\chi_n = \left[ u_n + \alpha_n(u_n - u_{n-1}) \right] - \gamma_n \left[ u_n + \alpha_n(u_n - u_{n-1}) \right],
\]

where \( \alpha_n \) such that

\[
0 \leq \alpha_n \leq \alpha_n^* \quad \text{and} \quad \alpha_n^* = \begin{cases} \min \left\{ \alpha, \frac{\varepsilon_n}{\|u_n - u_{n-1}\|} \right\} & \text{if } u_n \neq u_{n-1}, \\ \alpha & \text{otherwise}, \end{cases}
\]

where \( \varepsilon_n = o(\gamma_n) \) is a positive sequence such that \( \lim_{n \to +\infty} \frac{\varepsilon_n}{\gamma_n} = 0 \).

**Step 2:** Compute

\[
v_n = P_{\mathcal{K}}(\chi_n - \rho_nS(\chi_n)).
\]

If \( \chi_n = v_n \), then STOP and \( v_n \) is a solution. Otherwise, go to **Step 3**.

**Step 3:** Compute

\[
u_{n+1} = P_{\mathcal{E}_n}(\chi_n - \rho_nS(v_n)),
\]

where

\[
\varepsilon_n = \{z \in \mathcal{E} : \langle \chi_n - \rho_nS(\chi_n) - v_n, z - v_n \rangle \leq 0\}.
\]

**Step 4:** Compute

\[
\rho_{n+1} = \begin{cases} \min \left\{ \rho_n + \varphi_n, \frac{\mu\|\chi_n - v_n\|}{\|S(\chi_n) - S(v_n)\|} \right\} & \text{if } S(\chi_n) - S(v_n) \neq 0, \\ \rho_n + \varphi_n & \text{otherwise.} \end{cases}
\]

Set \( n = n + 1 \) and go back to **Step 1**.

**Lemma 3.1.** The sequence \( \{\rho_n\} \) generated by (3.2) is convergent to \( \rho \) and satisfy the following inequality

\[
\min \left\{ \frac{\mu}{\|L\|}, \rho_0 \right\} \leq \rho \leq \rho_0 + P, \quad \text{where} \quad P = \sum_{n=1}^{+\infty} \varphi_n.
\]

**Proof.** Due to the Lipschitz continuity of a mapping \( S \), there exists a fixed number \( L > 0 \). Suppose that \( S(\chi_n) - S(v_n) \neq 0 \) such that

\[
\frac{\mu\|\chi_n - v_n\|}{\|S(\chi_n) - S(v_n)\|} \geq \frac{\mu\|\chi_n - v_n\|}{L\|\chi_n - v_n\|} = \frac{\mu}{L}.
\]

By using mathematical induction on the definition of \( \rho_{n+1} \), we have

\[
\min \left\{ \frac{\mu}{L}, \rho_0 \right\} \leq \rho \leq \rho_0 + P.
\]

Let \( [\rho_{n+1} - \rho_n]^+ = \max \{0, \rho_{n+1} - \rho_n\} \) and \( [\rho_{n+1} - \rho_n]^− = \max \{0, -(\rho_{n+1} - \rho_n)\} \). From the definition of \( \{\rho_n\} \), we have

\[
\sum_{n=1}^{+\infty} (\rho_{n+1} - \rho_n)^+ = \sum_{n=1}^{+\infty} \max \{0, \rho_{n+1} - \rho_n\} \leq P < +\infty.
\]
That is, the series \( \sum_{n=1}^{+\infty} (\rho_{n+1} - \rho_n)^+ \) is convergent. Next, we need to prove the convergence of \( \sum_{n=1}^{+\infty} (\rho_{n+1} - \rho_n)^- \). Let \( \sum_{n=1}^{+\infty} (\rho_{n+1} - \rho_n)^- = +\infty \). Due to the reason that \( \rho_{n+1} - \rho_n = (\rho_{n+1} - \rho_n)^+ - (\rho_{n+1} - \rho_n)^- \), thus, we have

\[
\rho_{k+1} - \rho_0 = \sum_{n=0}^{k} (\rho_{n+1} - \rho_n) = \sum_{n=0}^{k} (\rho_{n+1} - \rho_n)^+ - \sum_{n=0}^{k} (\rho_{n+1} - \rho_n)^- .
\]

(3.3)

By allowing \( k \to +\infty \) in (3.3), we have \( \rho_k \to -\infty \) as \( k \to \infty \). This is a contradiction. Due to the convergence of the series \( \sum_{n=0}^{k} (\rho_{n+1} - \rho_n)^+ \) and \( \sum_{n=0}^{k} (\rho_{n+1} - \rho_n)^- \) taking \( k \to +\infty \) in (3.3), we obtain \( \lim_{n \to \infty} \rho_n = \rho \). This completes the proof.

\[\square\]

**Lemma 3.2.** Let \( S : E \to E \) be an operator satisfies the conditions (S1)-(S4). Then, for \( u^* \in \Pi \neq \emptyset \), we have

\[
\|u_{n+1} - u^*\|^2 \leq \|x_n - u^*\|^2 - (1 - \frac{\mu_0}{\rho_{n+1}})\|x_n - v_n\|^2 - (1 - \frac{\mu_0}{\rho_{n+1}})\|u_{n+1} - v_n\|^2 .
\]

**Proof.** Consider that

\[
\|u_{n+1} - u^*\|^2 = \|P_{E_n}[x_n - \rho_nS(v_n)] - u^*\|^2
\]

\[
= \|P_{E_n}[x_n - \rho_nS(v_n)] + [x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)] - u^*\|^2
\]

\[
= \|[x_n - \rho_nS(v_n)] - u^*\|^2 + \|P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)]\|^2
\]

\[
+ 2\langle P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)], [x_n - \rho_nS(v_n)] - u^*\rangle .
\]

(3.4)

It is given that \( u^* \in \Pi \subset K \subset E_n \), we obtain

\[
\|P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)]\|^2
\]

\[
+ \langle P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)], [x_n - \rho_nS(v_n)] - u^*\rangle
\]

\[
= \langle [x_n - \rho_nS(v_n)] - P_{E_n}[x_n - \rho_nS(v_n)], u^* - P_{E_n}[x_n - \rho_nS(v_n)]\rangle \leq 0,
\]

that implies that

\[
\langle P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)], [x_n - \rho_nS(v_n)] - u^*\rangle
\]

\[
\leq -\|P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)]\|^2 .
\]

(3.5)

Combining the expressions (3.4) and (3.5), we have

\[
\|u_{n+1} - u^*\|^2 \leq \|x_n - \rho_nS(v_n) - u^*\|^2 - \|P_{E_n}[x_n - \rho_nS(v_n)] - [x_n - \rho_nS(v_n)]\|^2
\]

\[
\leq \|x_n - u^*\|^2 - \|x_n - u_{n+1}\|^2 + 2\rho_n\langle S(v), u^* - u_{n+1}\rangle .
\]

(3.6)

Since \( u^* \) is the solution of problem (VIP), we have

\[
\langle S(u^*), v - u^* \rangle \geq 0 , \text{ for all } v \in K .
\]

Due to the pseudomonotonicity of \( S \) on \( K \), we obtain

\[
\langle S(v), v - u^* \rangle \geq 0 , \text{ for all } v \in K .
\]
By substituting $v = v_n \in \mathcal{K}$, we get
\[ \langle S(v_n), v_n - u^* \rangle \geq 0. \]

Thus, we have
\[ \langle S(v_n), u^* - u_{n+1} \rangle = \langle S(v_n), u^* - v_n \rangle + \langle S(v_n), v_n - u_{n+1} \rangle \leq \langle S(v_n), v_n - u_{n+1} \rangle. \tag{3.7} \]

By use of expressions (3.6) and (3.7), we have
\[
\begin{align*}
\| u_{n+1} - u^* \|^2 &\leq \| X_n - u^* \|^2 - \| X_n - u_{n+1} \|^2 + 2 \rho_n \langle S(v_n), v_n - u_{n+1} \rangle \\
&\leq \| X_n - u^* \|^2 - \| X_n - v_n + v_n - u_{n+1} \|^2 + 2 \rho_n \langle S(v_n), v_n - u_{n+1} \rangle \\
&\leq \| X_n - u^* \|^2 - \| X_n - v_n \|^2 - \| v_n - u_{n+1} \|^2 + 2 \langle X_n - \rho_n S(v_n) - v_n, u_{n+1} - v_n \rangle. \tag{3.8}
\end{align*}
\]

By use of the definition $u_{n+1} = P_{E_n} [X_n - \rho_n S(v_n)]$ and $\rho_{n+1}$, we have
\[
2 \langle X_n - \rho_n S(v_n) - v_n, u_{n+1} - v_n \rangle \\
= 2 \langle X_n - \rho_n S(v_n) - v_n, u_{n+1} - v_n \rangle + 2 \rho_n \langle X_n - S(v_n), u_{n+1} - v_n \rangle \\
= 2 \frac{\rho_n}{\rho_{n+1}} \rho_{n+1} \| S(X_n) - S(v_n) \| \| u_{n+1} - v_n \| \\
\leq \frac{\mu \rho_n}{\rho_{n+1}} \| X_n - v_n \|^2 + \frac{\| u_{n+1} - v_n \|^2}{\rho_{n+1}}. \tag{3.9}
\]

Combining expressions (3.8) and (3.9), we get
\[
\begin{align*}
\| u_{n+1} - u^* \|^2 &\leq \| X_n - u^* \|^2 - \| X_n - v_n \|^2 - \| v_n - u_{n+1} \|^2 + \frac{\rho_n}{\rho_{n+1}} \| X_n - v_n \|^2 + \frac{\| u_{n+1} - v_n \|^2}{\rho_{n+1}} \\
&\leq \| X_n - u^* \|^2 - \left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) \| X_n - v_n \|^2 - \left(1 - \frac{\| u_{n+1} - v_n \|^2}{\rho_{n+1}}\right) \| u_{n+1} - v_n \|^2. \tag{3.10}
\end{align*}
\]

\[
\square
\]

**Theorem 3.3.** Let $(u_n)$ be a sequence generated by Algorithm 1 and satisfies the conditions (3.1)-(3.4). Then, $(u_n)$ strongly converges to $u^* \in \Pi$ and $P_\Pi(0) = u^*$.

**Proof.** It is given that $\rho_n \to \rho$ such that $\epsilon \in (0, 1 - \mu)$ and
\[
\lim_{n \to \infty} \left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) = 1 - \mu \epsilon > 0.
\]

Then, there exists a finite number $n_1 \in \mathbb{N}$ such that
\[
\left(1 - \frac{\mu \rho_n}{\rho_{n+1}}\right) > \epsilon > 0, \forall n \geq n_1. \tag{3.11}
\]

Combining expression (3.10) and (3.11), we obtain
\[
\| u_{n+1} - u^* \|^2 \leq \| X_n - u^* \|^2, \forall n \geq n_1. \tag{3.12}
\]

It is given in expression (3.1) that
\[
\lim_{n \to +\infty} \frac{\alpha_n}{\gamma_n} \| u_n - u_{n-1} \| \leq \lim_{n \to +\infty} \frac{\epsilon_n}{\gamma_n} \| u_n - u_{n-1} \| = 0. \tag{3.13}
\]

By the use of definition of $(\chi_n)$ and expression (3.13), we obtain
\[
\| X_n - u^* \| = \| u_n + \alpha_n (u_n - u_{n-1}) - \gamma_n u_n - \alpha_n \gamma_n (u_n - u_{n-1}) - u^* \|
\]
\[
\begin{align*}
\| (1 - \gamma_n) & (u_n - u^*) + (1 - \gamma_n) \alpha_n (u_n - u_{n-1}) - \gamma_n u_* \| \\
\leq & (1 - \gamma_n) \| u_n - u^* \| + (1 - \gamma_n) \alpha_n \| u_n - u_{n-1} \| + \gamma_n \| u_* \| \\
\leq & (1 - \gamma_n) \| u_n - u^* \| + \gamma_n M_1,
\end{align*}
\] (3.14)

where
\[
(1 - \gamma_n) \frac{\alpha_n}{\gamma_n} \| u_n - u_{n-1} \| + \| u_* \| \leq M_1.
\]

Combining expressions (3.12) with (3.15), we obtain
\[
\| u_{n+1} - u^* \| \leq (1 - \gamma_n) \| u_n - u^* \| + \gamma_n M_1 \\
\leq \max \{ \| u_n - u^* \|, M_1 \} \\
\vdots \\
\leq \max \{ \| u_1 - u^* \|, M_1 \}.
\]

Thus, we conclude that the \{u_n\} is a bounded sequence. Indeed, by expression (3.15) we have
\[
\| x_n - u^* \|^2 \leq (1 - \gamma_n)^2 \| u_n - u^* \|^2 + \gamma_n^2 M_1^2 + 2M_1 \gamma_n (1 - \gamma_n) \| u_n - u^* \| \\
\leq \| u_n - u^* \|^2 + \gamma_n \left[ \gamma_n M_1^2 + 2M_1 (1 - \gamma_n) \| u_n - u^* \| \right] \\
\leq \| u_n - u^* \|^2 + \gamma_n M_2,
\] (3.16)

where \( \gamma_n M_1^2 + 2M_1 (1 - \gamma_n) \| u_n - u^* \| \leq M_2 \) for some \( M_2 > 0 \). From (3.10) and (3.16), we have
\[
\| u_{n+1} - u^* \|^2 \leq \| u_n - u^* \|^2 + \gamma_n M_2 - \left( 1 - \frac{\mu \rho_n}{\rho_{n+1}} \right) \| x_n - v_n \|^2 - \left( 1 - \frac{\mu \rho_n}{\rho_{n+1}} \right) \| u_{n+1} - v_n \|^2. \] (3.17)

The rest of the proof classified into two parts.

**Case 1:** Suppose that a fixed number \( n_2 \in \mathbb{N} (n_2 \geq n_1) \) such that
\[
\| u_{n_1} - u^* \| < \| u_n - u^* \|, \forall n \geq n_2.
\]

The above expression implies that \( \lim_{n \to +\infty} \| u_n - u^* \| \) exists and let \( \lim_{n \to +\infty} \| u_n - u^* \| = 1 \) for some \( l \geq 0 \). From the expression (3.17), we have
\[
\left( 1 - \frac{\mu \rho_n}{\rho_{n+1}} \right) \| x_n - v_n \|^2 + \left( 1 - \frac{\mu \rho_n}{\rho_{n+1}} \right) \| u_{n+1} - v_n \|^2 \\
\leq \| u_n - u^* \|^2 + \gamma_n M_2 - \| u_{n+1} - u^* \|^2. \] (3.18)

Due to existence of a limit of sequence \( \| u_n - u^* \| \) and \( \gamma_n \to 0 \), we deduce that
\[
\| x_n - v_n \| \to 0 \quad \text{and} \quad \| u_{n+1} - v_n \| \to 0 \quad \text{as} \quad n \to +\infty. \] (3.19)

By the use of expression (3.19), we have
\[
\lim_{n \to +\infty} \| x_n - u_{n+1} \| \leq \lim_{n \to +\infty} \| x_n - v_n \| + \lim_{n \to +\infty} \| v_n - u_{n+1} \| = 0. \] (3.20)

Next, we have to compute
\[
\| x_n - u^* \| = \| u_n + \alpha_n (u_n - u_{n-1}) - \gamma_n [u_n + \alpha_n (u_n - u_{n-1})] - u_n \| \\
\leq \alpha_n \| u_n - u_{n-1} \| + \gamma_n \| u_n \| + \alpha_n \gamma_n \| u_n - u_{n-1} \| \\
= \gamma_n \frac{\alpha_n}{\gamma_n} \| u_n - u_{n-1} \| + \gamma_n \| u_n \| + \gamma_n \frac{\alpha_n}{\gamma_n} \| u_n - u_{n-1} \| \to 0 \quad \text{as} \quad n \to \infty.
\]
The last two expressions implies that
\[
\lim_{n \to +\infty} \|u_n - u_{n+1}\| \leq \lim_{n \to +\infty} \|u_n - \chi_n\| + \lim_{n \to +\infty} \|\chi_n - u_{n+1}\| = 0.
\]

The above explanation implies that the sequences \(\{\chi_n\}\) and \(\{v_n\}\) are bounded. By the use of reflexivity of \(E\) and the boundedness of sequence \(\{u_n\}\) there exists a subsequence \(\{u_{n_k}\}\) such that \(\{u_{n_k}\} \rightharpoonup \hat{u} \in \mathcal{E}\) as \(k \to +\infty\). Next, we have to prove that \(\hat{u} \in \Pi\). It is given that
\[
v_{n_k} = P_{\mathcal{K}}[\chi_{n_k} - \rho \mathcal{S}(\chi_{n_k})],
\]
that is equivalent to
\[
\langle \chi_{n_k} - \rho \mathcal{S}(\chi_{n_k}) - v_{n_k}, v - v_{n_k} \rangle \leq 0, \quad \forall v \in \mathcal{K}.
\]
The above inequality implies that
\[
\langle \chi_{n_k} - v_{n_k}, v - v_{n_k} \rangle \leq \rho \langle \mathcal{S}(\chi_{n_k}), v - v_{n_k} \rangle, \quad \forall v \in \mathcal{K}.
\]

Furthermore, we obtain
\[
\frac{1}{\rho} \langle \chi_{n_k} - v_{n_k}, v - v_{n_k} \rangle + \langle \mathcal{S}(\chi_{n_k}), v_{n_k} - \chi_{n_k} \rangle \leq \langle \mathcal{S}(\chi_{n_k}), v - \chi_{n_k} \rangle, \quad \forall v \in \mathcal{K}. \tag{3.21}
\]

Due to boundedness of the sequence \(\{\chi_{n_k}\}\) implies that \(\mathcal{S}(\chi_{n_k})\) is also bounded. By the use of \(\lim_{k \to \infty} \|\chi_{n_k} - v_{n_k}\| = 0\) and \(k \to \infty\) in (3.21), we obtain
\[
\liminf_{k \to \infty} \langle \mathcal{S}(\chi_{n_k}), v - \chi_{n_k} \rangle \geq 0, \quad \forall v \in \mathcal{K}.
\]

Furthermore, we have
\[
\langle \mathcal{S}(v_{n_k}), v - v_{n_k} \rangle = \langle \mathcal{S}(v_{n_k}) - \mathcal{S}(\chi_{n_k}), v - \chi_{n_k} \rangle + \langle \mathcal{S}(\chi_{n_k}), v - \chi_{n_k} \rangle + \langle \mathcal{S}(v_{n_k}), \chi_{n_k} - v_{n_k} \rangle. \tag{3.22}
\]

By the use of \(\lim_{k \to \infty} \|\chi_{n_k} - v_{n_k}\| = 0\) and \(\mathcal{S}\) is \(L\)-Lipschitz continuous on \(\mathcal{E}\) implies that
\[
\lim_{k \to \infty} \|\mathcal{S}(\chi_{n_k}) - \mathcal{S}(v_{n_k})\| = 0, \tag{3.23}
\]
which together with (3.22) and (3.23), we obtain
\[
\liminf_{k \to \infty} \langle \mathcal{S}(v_{n_k}), v - v_{n_k} \rangle \geq 0, \quad \forall v \in \mathcal{K}.
\]

Let consider a sequence of positive numbers \(\{\epsilon_k\}\) that is decreasing and converges to zero. For each \(k\), we denote \(m_k\) by the smallest positive integer such that
\[
\langle \mathcal{S}(\chi_{n_i}), v - \chi_{n_i} \rangle + \epsilon_k \geq 0, \quad \forall i \geq m_k. \tag{3.24}
\]

Due to \(\{\epsilon_k\}\) is decreasing, thus \(\{m_k\}\) is increasing.

**Case A:** If there exists a subsequence \(\{\chi_{n_{mj}}\}\) of \(\{\chi_{n_{mk}}\}\) such that \(\mathcal{S}(\chi_{n_{mj}}) = 0\), for all \(j\). Let \(j \to \infty\), we obtain
\[
\langle \mathcal{S}(\hat{u}), v - \hat{u} \rangle = \lim_{j \to \infty} \langle \mathcal{S}(\chi_{n_{mj}}), v - \hat{u} \rangle = 0.
\]

Hence \(\hat{u} \in \mathcal{K}\), therefore we obtain \(\hat{u} \in \Pi\).
**Case B:** If there exists \( n_0 \in \mathbb{N} \) such that for all \( n_{m_k} \geq n_0 \), \( S(\chi_{n_{m_k}}) \neq 0 \). Suppose that

\[
\Sigma_{n_{m_k}} = \frac{S(\chi_{n_{m_k}})}{\|S(\chi_{n_{m_k}})\|^2}, \quad \forall n_{m_k} \geq n_0.
\]

On the basis of the above definition, we can obtain

\[
\langle S(\chi_{n_{m_k}}), \Sigma_{n_{m_k}} \rangle = 1, \quad \forall n_{m_k} \geq n_0.
\] (3.25)

By using expressions (3.24) and (3.25) for all \( n_{m_k} \geq n_0 \), we have

\[
\langle S(\chi_{n_{m_k}}), v + \epsilon_k \Sigma_{n_{m_k}} - \chi_{n_{m_k}} \rangle \geq 0.
\]

Due to the pseudomonotonicity of \( S \) for \( n_{m_k} \geq n_0 \), we have

\[
\langle S(v + \epsilon_k \Sigma_{n_{m_k}}), v + \epsilon_k \Sigma_{n_{m_k}} - \chi_{n_{m_k}} \rangle \geq 0.
\]

For all \( n_{m_k} \geq n_0 \), we have

\[
\langle S(v), v - \chi_{n_{m_k}} \rangle \geq \langle S(v) - S(v + \epsilon_k \Sigma_{n_{m_k}}), v + \epsilon_k \Sigma_{n_{m_k}} - \chi_{n_{m_k}} \rangle - \epsilon_k \langle S(v), \Sigma_{n_{m_k}} \rangle.
\] (3.26)

Due to \( \{\chi_{n_k}\} \) weakly converges to \( \hat{u} \in \mathcal{K} \) with \( S \) is sequentially weakly continuous on the set \( \mathcal{K} \), we get \( \{S(\chi_{n_k})\} \) weakly converges to \( S(\hat{u}) \). Suppose that \( S(\hat{u}) \neq 0 \), we have

\[
\|S(\hat{u})\| \leq \liminf_{k \to \infty} \|S(\chi_{n_k})\|.
\]

Since \( \{\chi_{n_{m_k}}\} \subset \{\chi_{n_k}\} \) and \( \lim_{k \to \infty} \epsilon_k = 0 \), we have

\[
0 \leq \lim_{k \to \infty} \|\epsilon_k \Sigma_{n_{m_k}}\| = \lim_{k \to \infty} \frac{\epsilon_k}{\|S(\chi_{n_{m_k}})\|} \leq \frac{0}{\|S(\hat{u})\|} = 0.
\]

Consider that \( k \to \infty \) in (3.26), we obtain

\[
\langle S(v), v - \hat{u} \rangle \geq 0, \quad \forall v \in \mathcal{K}.
\]

By the use of Minty Lemma 2.5, we infer that \( \hat{u} \in \Pi \). By using the Lipschitz continuity and pseudomonotonicity of a mapping \( S \) implies that the solution set \( \Pi \) is a closed and convex set. It is given that \( u^* = P_\Pi(0) \) and by using Lemma 2.1 (ii), we have

\[
\langle 0 - u^*, v - u^* \rangle \leq 0, \quad \forall v \in \Pi.
\]

Next, we have to evaluate

\[
\limsup_{n \to +\infty} \langle u^*, u^* - u_n \rangle = \lim_{k \to +\infty} \langle u^*, u^* - u_{n_k} \rangle = \langle u^*, u^* - \hat{u} \rangle \leq 0.
\] (3.27)

By the use of \( \lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0 \) and expression (3.27) implies that

\[
\limsup_{n \to +\infty} \langle u^*, u^* - u_{n+1} \rangle \leq \limsup_{n \to +\infty} \langle u^*, u^* - u_n \rangle + \limsup_{n \to +\infty} \langle u^*, u_n - u_{n+1} \rangle \leq 0.
\] (3.28)

Take into account the expression (3.14), we have

\[
\|X_n - u^*\|^2 = \|u_n + \alpha_n(u_n - u_{n-1}) - \gamma_n u_n - \alpha_n \gamma_n (u_n - u_{n-1}) - u^*\|^2
\]
From expressions (3.12) and (3.29), we obtain

\[
\begin{align*}
&=(1-\gamma_n)\|u_n-u^*\|^2 + 2\gamma_n\|u_n-u^*\\|\|\chi_n-u_n\| \\
&= (1-\gamma_n)^2\|u_n-u^*\|^2 + 2(1-\gamma_n)^2\alpha_n\|u_n-u_{n-1}\|^2 \\
&\quad + 2\alpha_n(1-\gamma_n)^2\|u_n-u^*\\|\|u_n-u_{n-1}\| + 2\gamma_n\|\chi_n-u_n\| + 2\gamma_n\|u_n-u^*\\|\|u_n-u_{n-1}\|.
\end{align*}
\]

(3.29)

By the use of (3.20), (3.28), (3.30), and applying Lemma 2.2 conclude that \( \lim_{n \to +\infty} \|u_n-u^*\| = 0. \)

**Case 2:** Suppose that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that

\[ \|u_{n_i}-u^*\| \leq \|u_{n_i+1}-u^*\|, \quad \forall i \in \mathbb{N}. \]

By using Lemma 2.3 there exists a sequence \( \{m_k\} \subset \mathbb{N} \) as \( m_k \to +\infty \) such that

\[ \|u_{m_k}-u^*\| \leq \|u_{m_{k+1}}-u^*\| \quad \text{and} \quad \|u_{k}-u^*\| \leq \|u_{m_{k+1}}-u^*\|, \quad \forall k \in \mathbb{N}. \]

(3.31)

As similar to the Case 1, the expression (3.18) implies that

\[
\left(1 - \frac{\mu\rho_{m_k}}{\rho_{m_{k+1}}}\right)\|\chi_{m_k} - v_{m_k}\|^2 + \left(1 - \frac{\mu\rho_{m_k}}{\rho_{m_{k+1}}}\right)\|u_{m_{k+1}} - v_{m_k}\|^2 \leq \|u_{m_k} - u^*\|^2 + \gamma_{m_k}M_2 - \|u_{m_{k+1}} - u^*\|^2.
\]

Due to \( \gamma_{m_k} \to 0 \), we deduce the followings:

\[
\lim_{k \to +\infty} \|\chi_{m_k} - v_{m_k}\| = \lim_{k \to +\infty} \|u_{m_{k+1}} - v_{m_k}\| = 0.
\]

It continues from that

\[
\lim_{k \to +\infty} \|u_{m_{k+1}} - \chi_{m_k}\| \leq \lim_{k \to +\infty} \|u_{m_{k+1}} - v_{m_k}\| + \lim_{k \to +\infty} \|v_{m_k} - \chi_{m_k}\| = 0.
\]

Next, we will evaluate

\[
\begin{align*}
\|\chi_{m_k} - u_{m_k}\| &= \|u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_{k-1}}) - \gamma_{m_k}\left[u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_{k-1}})\right] - u_{m_k}\| \\
&\leq \alpha_{m_k}\|u_{m_k} - u_{m_{k-1}}\| + \gamma_{m_k}\|u_{m_k}\| + \alpha_{m_k}\gamma_{m_k}\|u_{m_k} - u_{m_{k-1}}\| \\
&= \gamma_{m_k}\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_{k-1}}\| + \gamma_{m_k}\|u_{m_k}\| + \gamma_{m_k}\frac{\alpha_{m_k}}{\gamma_{m_k}}\|u_{m_k} - u_{m_{k-1}}\| \to 0.
\end{align*}
\]

The above implies that

\[
\lim_{k \to +\infty} \|u_{m_k} - u_{m_{k+1}}\| \leq \lim_{k \to +\infty} \|u_{m_k} - \chi_{m_k}\| + \lim_{k \to +\infty} \|\chi_{m_k} - u_{m_{k+1}}\| = 0.
\]
By using the same argument as in Case 1, we obtain
\[
\limsup_{k \to +\infty} \|u_k - u^*\| \leq 0. \tag{3.32}
\]
Combining expressions (3.30) and (3.31), we obtain
\[
\begin{align*}
\|u_{m+1} - u^*\|^2 &\leq (1 - \gamma_{m,k}) \|u_{m,k} - u^*\|^2 + \gamma_{m,k} \left[ \alpha_{m,k} \|u_{m,k} - u_{m,k-1}\| \frac{\alpha_{m,k}}{\gamma_{m,k}} \|u_{m,k} - u_{m,k-1}\| \right. \\
& \quad + 2(1 - \gamma_{m,k}) \|u_{m,k} - u^*\| \frac{\alpha_{m,k}}{\gamma_{m,k}} \|u_{m,k} - u_{m,k-1}\| + 2\|u^*\| \|X_{m,k} - u_{m,k+1}\| + 2\{u^*, u^* - u_{m,k+1}\} \\
&\leq (1 - \gamma_{m,k}) \|u_{m,k+1} - u^*\|^2 + \gamma_{m,k} \left[ \alpha_{m,k} \|u_{m,k} - u_{m,k-1}\| \frac{\alpha_{m,k}}{\gamma_{m,k}} \|u_{m,k} - u_{m,k-1}\| \right. \\
& \quad + 2(1 - \gamma_{m,k}) \|u_{m,k} - u^*\| \frac{\alpha_{m,k}}{\gamma_{m,k}} \|u_{m,k} - u_{m,k-1}\| + 2\|u^*\| \|X_{m,k} - u_{m,k+1}\| + 2\{u^*, u^* - u_{m,k+1}\}.
\end{align*}
\tag{3.33}
\]
The above expression implies that
\[
\begin{align*}
\|u_{m,k+1} - u^*\|^2 &\leq \left[ \alpha_{m,k} \|u_{m,k} - u_{m,k-1}\| \frac{\alpha_{m,k}}{\gamma_{m,k}} \|u_{m,k} - u_{m,k-1}\| \right. \\
& \quad + 2(1 - \gamma_{m,k}) \|u_{m,k} - u^*\| \frac{\alpha_{m,k}}{\gamma_{m,k}} \|u_{m,k} - u_{m,k-1}\| + 2\|u^*\| \|X_{m,k} - u_{m,k+1}\| + 2\{u^*, u^* - u_{m,k+1}\}.
\end{align*}
\]
Since \(\gamma_{m,k} \to 0\), and \(\|u_{m,k} - u^*\|\) is a bounded, expressions (3.32) and (3.33) imply that
\[
\|u_{m,k+1} - u^*\|^2 \to 0, \text{ as } k \to +\infty.
\]
It implies that
\[
\lim_{n \to +\infty} \|u_k - u^*\|^2 \leq \lim_{n \to +\infty} \|u_{m,n+1} - u^*\|^2 \leq 0.
\]
As a consequence \(u_n \to u^*\). This completes the proof of the theorem.

4. Numerical illustrations

This section examines three numerical examples to show the efficiency of the proposed methods. Any of these numerical experiments provide a detailed understanding of how better control parameters can be chosen. Some of them show the advantages of the proposed methods compared to existing ones in the literature.

Example 4.1. First consider the HqHard problem that is taken from [6]. Let \(\mathcal{S} : \mathbb{R}^N \to \mathbb{R}^N\) be an operator defined by \(\mathcal{S}(u) = Mu + q\), where \(q \in \mathbb{R}^N\) and \(M = AA^T + B + D\),
\[
M = AA^T + B + D,
\]
where \(A\) is an \(N \times N\) matrix, \(B\) is an \(N \times N\) skew-symmetric matrix and \(D\) is an \(N \times N\) positive definite diagonal matrix. The set \(\mathcal{K}\) is taken in the following way:
\[
\mathcal{K} = \{u \in \mathbb{R}^N : -100 \leq u_i \leq 100\}.
\]
It is clear that \(\mathcal{S}\) is monotone and Lipschitz continuous through \(L = \|M\|\). The control condition are taken as follows: (1) Algorithm 3.1 in [32]: \(\rho_0 = 0.15, \mu = 0.75, \alpha_n = 0.65\); (2) Algorithm 1: \(\rho_0 = 0.15, \mu = 0.75, \alpha = 0.65, \epsilon_n = \frac{1}{(n+1)^2}, \gamma_n = \frac{1}{10(n+2)}, \varphi_n = \frac{10}{10(n+1)}\). During this experiment, the initial point is \(u_{-1} = u_0 = (2, 2, \ldots, 2)\) and \(D_n = \|x_n - v_n\| \leq 10^{-4}\). The numerical results of these methods are shown in Table 1.
It is easy to verify that $Q$ is symmetric and positive definite on $\mathbb{R}^4$ and consequently $f$ is pseudo-convex on $\mathcal{K}$. Hence, $\nabla f$ is pseudo-monotone. Using the quotient rule, we obtain
\[
\nabla f(u) = \frac{(b^T u + b_0)(2Qu + a) - b(u^T Q + a^T u + a_0)}{(b^T u + b_0)^2}.
\] (4.1)

In this point of view, we can set $\delta = \nabla f$ in Theorem 3.3. We minimize $f$ over $\mathcal{K} = \{u \in \mathbb{R}^4 : 1 \leq u_i \leq 10, i = 1, 2, 3, 4\}$. This problem has a unique solution $u^* = (1, 1, 1, 1)^T \in \mathcal{K}$. The control condition are taken as follows: (1) Algorithm 2 in [32]: $\rho_0 = 0.25, \mu = 0.70, \alpha_1 = 0.65$; (2) Algorithm 1: $\rho_0 = 0.25, \mu = 0.70, \alpha = 0.65, \varepsilon_n = 10^{-(n+1)/\tau}, \phi_n = 10^{-(n+1)/\tau}, \gamma_n = 10^{-(n+1)/\tau}$. During this experiment, the initial points are different and $D_n = \|X_n - v_n\| \leq 10^{-4}$. The numerical results of these methods are shown in Tables 2-5.

### Table 1: Numerical illustrations for both methods.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Number of Iterations</th>
<th>Elapsed Time</th>
<th>Number of Iterations</th>
<th>Elapsed Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>25</td>
<td>0.123786</td>
<td>13</td>
<td>0.086372</td>
</tr>
<tr>
<td>20</td>
<td>41</td>
<td>0.283782</td>
<td>20</td>
<td>0.226123</td>
</tr>
<tr>
<td>50</td>
<td>123</td>
<td>0.998352</td>
<td>41</td>
<td>0.641284</td>
</tr>
<tr>
<td>100</td>
<td>311</td>
<td>2.241831</td>
<td>45</td>
<td>1.263114</td>
</tr>
<tr>
<td>200</td>
<td>225</td>
<td>7.462846</td>
<td>53</td>
<td>3.462451</td>
</tr>
<tr>
<td>500</td>
<td>119</td>
<td>12.46582</td>
<td>62</td>
<td>5.568421</td>
</tr>
</tbody>
</table>

**Example 4.2.** Consider the quadratic fractional programming problem in the following form [7]:
\[
\begin{align*}
\min f(u) &= u^T Q u + a^T u + a_0 \\
\text{subject to } u &\in \mathcal{K} = \{u \in \mathbb{R}^4 : b^T u + b_0 > 0\},
\end{align*}
\]

where
\[
Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad a_0 = -2, \quad \text{and} \quad b_0 = 4.
\]

### Table 2: Example 4.2: numerical study of Algorithm 2 in [32] and Algorithm 1

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.7362105292642</td>
<td>0.833094809285095</td>
<td>9.93024123820372</td>
<td>1.04929132798160</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>2</td>
<td>9.659890882229999</td>
<td>0.999443650015545</td>
<td>9.8307401956143</td>
<td>1.029984710307986</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>3</td>
<td>9.57174064515717</td>
<td>1.06492890311881</td>
<td>9.75902590955452</td>
<td>1.00885135708242</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>4</td>
<td>9.4955289901919</td>
<td>1.12929808105285</td>
<td>9.68783246075971</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>5</td>
<td>9.4073941326346</td>
<td>1.19231579167986</td>
<td>9.61648289403968</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>6</td>
<td>9.3259697958624</td>
<td>1.25416303576667</td>
<td>9.54565902780142</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>7</td>
<td>9.24348825257906</td>
<td>1.31478659951554</td>
<td>9.47487029657638</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>8</td>
<td>9.16162563841652</td>
<td>1.3741932423997</td>
<td>9.40428659097328</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>9</td>
<td>9.0799080317225</td>
<td>1.4329065685804</td>
<td>9.33391113686474</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
<tr>
<td>10</td>
<td>8.99831103926803</td>
<td>1.48938649679281</td>
<td>9.2637417307480</td>
<td>1.00000000000000</td>
<td>1.00000000000000</td>
</tr>
</tbody>
</table>

**CPU time** is seconds 0.657/61
Table 3: Example 4.2: numerical study of Algorithm 1 and \( u_0 = u_1 = [2, -5, 5, -2]^T \).

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.5377533757632</td>
<td>-2.90587269134480</td>
<td>8.61905436436968</td>
<td>1.720381664648790</td>
</tr>
<tr>
<td>2</td>
<td>4.4104024375028</td>
<td>1</td>
<td>7.4502669420132</td>
<td>1.1942560870334</td>
</tr>
<tr>
<td>4</td>
<td>3.05882604768791</td>
<td>-7.9235662619279</td>
<td>7.05918169089723</td>
<td>-2.216031462490</td>
</tr>
<tr>
<td>5</td>
<td>2.1344465314296</td>
<td>-7.4999847685431</td>
<td>6.31036694537031</td>
<td>1.1022327884896</td>
</tr>
<tr>
<td>6</td>
<td>1.87015108945082</td>
<td>1</td>
<td>6.02864753785448</td>
<td>0.9550242010157</td>
</tr>
<tr>
<td>7</td>
<td>1.46861344878822</td>
<td>1.15019734000242</td>
<td>5.438154442871</td>
<td>0.236813108691433</td>
</tr>
<tr>
<td>8</td>
<td>1.38899856735670</td>
<td>0.490171438084139</td>
<td>3.438154442871</td>
<td>1.28802829936381</td>
</tr>
<tr>
<td>9</td>
<td>1.29986269980453</td>
<td>0.213085356892448</td>
<td>6.02864753785448</td>
<td>0.9550242010157</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.218463

Table 4: Example 4.2: numerical study of Algorithm 2 in [32] and \( u_0 = u_1 = [1, -2, 5, -4]^T \).

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.7763733455425</td>
<td>-0.0495715784342590</td>
<td>9.2904270258555</td>
<td>0.811840145149117</td>
</tr>
<tr>
<td>2</td>
<td>8.69495843974052</td>
<td>9.9978633212697</td>
<td>9.10036150591126</td>
<td>0.999888330072762</td>
</tr>
<tr>
<td>3</td>
<td>8.61358693872012</td>
<td>1</td>
<td>9.10035792482012</td>
<td>0.999890341430832</td>
</tr>
<tr>
<td>4</td>
<td>8.53228969183012</td>
<td>1</td>
<td>9.10036662421324</td>
<td>0.999890341430832</td>
</tr>
<tr>
<td>5</td>
<td>8.45108721657438</td>
<td>1</td>
<td>9.1004014710912</td>
<td>0.99987666312513</td>
</tr>
<tr>
<td>6</td>
<td>8.3699396345207</td>
<td>1</td>
<td>9.10040076423535</td>
<td>0.999877350082930</td>
</tr>
<tr>
<td>7</td>
<td>8.28902146271176</td>
<td>1</td>
<td>9.1003588494207</td>
<td>0.99989507927591</td>
</tr>
<tr>
<td>8</td>
<td>8.2081793122734</td>
<td>1</td>
<td>9.1003405336513</td>
<td>0.99989507927591</td>
</tr>
<tr>
<td>9</td>
<td>8.1274622756739</td>
<td>1</td>
<td>9.1003405336513</td>
<td>0.99989507927591</td>
</tr>
<tr>
<td>10</td>
<td>8.0469206065515</td>
<td>1</td>
<td>9.1003405336513</td>
<td>0.99989507927591</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.438657
Algorithm 1: The control conditions are taken as follows: (1) Algorithm 2 in [32]: 

It is easy to see that the initial points are different and

Example 4.3. Suppose that the non-linear complementarity problem of Kojima–Shindo while the feasible set $\mathcal{K}$ is

Let the mapping $S : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is defined by

It is easy to see that $S$ is not monotone on the set $\mathcal{K}$. By using the Monte-Carlo approach [7] it can be shown that $S$ is pseudomonotone on $\mathcal{K}$. This problem has a unique solution $u^* = (5, 5, 5, 5)^T$. Actually, in general, it is a very difficult task to check the pseudomonotonicity of any mapping $S$ in practice. We here employ the Monte-Carlo approach according to the definition of pseudomonotonicity: Generate a large number of pairs of points $u$ and $v$ uniformly in $\mathcal{K}$ satisfying $S(u)^T(v-u) \geq 0$ and then check if $S(v)^T(v-u) \geq 0$. The control conditions are taken as follows: (1) Algorithm 2 in [32]: $\rho_0 = 0.35, \mu = 0.33, \alpha_n = 0.53$; (2) Algorithm 1: $\rho_0 = 0.35, \mu = 0.33, \alpha = 0.53, \epsilon_n = \frac{1}{(n+1)^{1/2}}, \varphi_n = \frac{100}{(n+1)^{1/2}}, \gamma_n = \frac{1}{(n+2)^{1/2}}$. During this experiment, the initial points are different and $D_n = \|x_n - v_n\| \leq 10^{-4}$. The numerical results of these methods are shown in Tables 6-9.

Table 5: Example 4.2: numerical study of Algorithm 1 and $u_0 = u_1 = [1, -2, 5, 4]^T$.  

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.625403169183484</td>
<td>0.0390003399813842</td>
<td>6.32280270247068</td>
<td>-0.00324645602473037</td>
</tr>
<tr>
<td>2</td>
<td>2.35518326143277</td>
<td>1.0000000000000005618</td>
<td>5.05780638110935</td>
<td>-0.69837169171649</td>
</tr>
<tr>
<td>3</td>
<td>1.71910476701455</td>
<td>0.953162043340826</td>
<td>4.297714937510</td>
<td>0.99999999525121</td>
</tr>
<tr>
<td>4</td>
<td>1.15470777251028</td>
<td>0.895316405359908</td>
<td>3.59372414213764</td>
<td>0.99999998377171</td>
</tr>
<tr>
<td>5</td>
<td>1.12364324376447</td>
<td>0.659026290197076</td>
<td>2.75601348181125</td>
<td>0.90277544700774</td>
</tr>
<tr>
<td>6</td>
<td>1.23231081634727</td>
<td>0.698969138725128</td>
<td>2.24361021384599</td>
<td>0.93239978539485</td>
</tr>
<tr>
<td>7</td>
<td>1.09676739975337</td>
<td>0.93889369343763</td>
<td>1.952423852901</td>
<td>0.91881729612742</td>
</tr>
<tr>
<td>8</td>
<td>1.60994694902141</td>
<td>0.94146755684416</td>
<td>1.67181167392232</td>
<td>0.988172769211274</td>
</tr>
<tr>
<td>9</td>
<td>1.10537058979123</td>
<td>0.897210645546332</td>
<td>1.39018239991049</td>
<td>0.98138582259744</td>
</tr>
<tr>
<td>10</td>
<td>1.06931813997595</td>
<td>0.948386311174977</td>
<td>1.22427688163233</td>
<td>0.99543685765338</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.160578

Table 6: Example 4.3: numerical study of Algorithm 2 in [32] and $u_0 = u_1 = [1, 1, -2, 5, 4]^T$.  

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.625403169183484</td>
<td>0.0390003399813842</td>
<td>6.32280270247068</td>
<td>-0.00324645602473037</td>
</tr>
<tr>
<td>2</td>
<td>2.35518326143277</td>
<td>1.0000000000000005618</td>
<td>5.05780638110935</td>
<td>-0.69837169171649</td>
</tr>
<tr>
<td>3</td>
<td>1.71910476701455</td>
<td>0.953162043340826</td>
<td>4.297714937510</td>
<td>0.99999999525121</td>
</tr>
<tr>
<td>4</td>
<td>1.15470777251028</td>
<td>0.895316405359908</td>
<td>3.59372414213764</td>
<td>0.99999998377171</td>
</tr>
<tr>
<td>5</td>
<td>1.12364324376447</td>
<td>0.659026290197076</td>
<td>2.75601348181125</td>
<td>0.90277544700774</td>
</tr>
<tr>
<td>6</td>
<td>1.23231081634727</td>
<td>0.698969138725128</td>
<td>2.24361021384599</td>
<td>0.93239978539485</td>
</tr>
<tr>
<td>7</td>
<td>1.09676739975337</td>
<td>0.93889369343763</td>
<td>1.952423852901</td>
<td>0.91881729612742</td>
</tr>
<tr>
<td>8</td>
<td>1.60994694902141</td>
<td>0.94146755684416</td>
<td>1.67181167392232</td>
<td>0.988172769211274</td>
</tr>
<tr>
<td>9</td>
<td>1.10537058979123</td>
<td>0.897210645546332</td>
<td>1.39018239991049</td>
<td>0.98138582259744</td>
</tr>
<tr>
<td>10</td>
<td>1.06931813997595</td>
<td>0.948386311174977</td>
<td>1.22427688163233</td>
<td>0.99543685765338</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.160578

Example 4.3. Suppose that the non-linear complementarity problem of Kojima–Shindo while the feasible set $\mathcal{K}$ is

Let the mapping $S : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be defined by

It is easy to see that $S$ is not monotone on the set $\mathcal{K}$. By using the Monte-Carlo approach [7] it can be shown that $S$ is pseudomonotone on $\mathcal{K}$. This problem has a unique solution $u^* = (5, 5, 5, 5)^T$. Actually, in general, it is a very difficult task to check the pseudomonotonicity of any mapping $S$ in practice. We here employ the Monte-Carlo approach according to the definition of pseudomonotonicity: Generate a large number of pairs of points $u$ and $v$ uniformly in $\mathcal{K}$ satisfying $S(u)^T(v-u) \geq 0$ and then check if $S(v)^T(v-u) \geq 0$. The control conditions are taken as follows: (1) Algorithm 2 in [32]: $\rho_0 = 0.35, \mu = 0.33, \alpha_n = 0.53$; (2) Algorithm 1: $\rho_0 = 0.35, \mu = 0.33, \alpha = 0.53, \epsilon_n = \frac{1}{(n+1)^{1/2}}, \varphi_n = \frac{100}{(n+1)^{1/2}}, \gamma_n = \frac{1}{(n+2)^{1/2}}$. During this experiment, the initial points are different and $D_n = \|x_n - v_n\| \leq 10^{-4}$. The numerical results of these methods are shown in Tables 6-9.
Table 7: Example 4.3: numerical study of Algorithm 1 and \( u_0 = u_1 = [1, -1, 3, 5]^T \).

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-22.4790198515975</td>
<td>45.1962163525125</td>
<td>61.2227868529569</td>
<td>64.3371909274850</td>
</tr>
<tr>
<td>2</td>
<td>-22.725625769091</td>
<td>45.8890698132465</td>
<td>61.9346930512654</td>
<td>65.0528149718569</td>
</tr>
<tr>
<td>3</td>
<td>-46.9174363358688</td>
<td>45.3948805004849</td>
<td>49.2538917010001</td>
<td>50.2790369811032</td>
</tr>
<tr>
<td>4</td>
<td>4.764708048016</td>
<td>31.6637577598404</td>
<td>6.571701330197401</td>
<td>0.238784700196093</td>
</tr>
<tr>
<td>5</td>
<td>4.8293734933528</td>
<td>4.9939226879006</td>
<td>5.0102669319225</td>
<td>0.3663967892221</td>
</tr>
<tr>
<td>6</td>
<td>34.3710564606076</td>
<td>43.4227198682972</td>
<td>34.4353964718562</td>
<td>0.6299429612540</td>
</tr>
<tr>
<td>7</td>
<td>45.0594866099192</td>
<td>45.1112144480466</td>
<td>45.1291165685866</td>
<td>1.9164510896386</td>
</tr>
<tr>
<td>8</td>
<td>50.4501740621952</td>
<td>50.5024262296367</td>
<td>50.5103528904701</td>
<td>-1.9716915135855</td>
</tr>
<tr>
<td>9</td>
<td>50.5314043274895</td>
<td>50.5834557546292</td>
<td>50.5624446015856</td>
<td>-2.0062892664043</td>
</tr>
<tr>
<td>10</td>
<td>50.640342132421</td>
<td>50.6917607820349</td>
<td>50.7043978558995</td>
<td>-5.1207188674962</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.232178

Table 8: Example 4.3: numerical study of Algorithm 2 in [32] and \( u_0 = u_1 = [-1, 2, -3, 4]^T \).

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
<th>( u_3 )</th>
<th>( u_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-6.20601542924621</td>
<td>40.0843112633838</td>
<td>48.1337650982707</td>
<td>46.6230007827220</td>
</tr>
<tr>
<td>2</td>
<td>-6.34498204992797</td>
<td>40.5012112543060</td>
<td>48.5506649603177</td>
<td>47.0399008076869</td>
</tr>
<tr>
<td>3</td>
<td>-12.9307982468928</td>
<td>40.8471240829636</td>
<td>47.5033192826780</td>
<td>46.0967916997756</td>
</tr>
<tr>
<td>4</td>
<td>-12.824511099744</td>
<td>40.1793689297275</td>
<td>47.6299387228835</td>
<td>46.2241560861850</td>
</tr>
<tr>
<td>5</td>
<td>-33.1769125028857</td>
<td>34.7884187072125</td>
<td>38.168090616878</td>
<td>37.2122057191811</td>
</tr>
<tr>
<td>6</td>
<td>-3.6371526214896</td>
<td>20.5019731319843</td>
<td>-1.7618502948704</td>
<td>2.54573698430683</td>
</tr>
<tr>
<td>7</td>
<td>1.04302948168927</td>
<td>5.0545238960439</td>
<td>1.1065019817653</td>
<td>2.54725466388794</td>
</tr>
<tr>
<td>8</td>
<td>1.04978285120354</td>
<td>5.0000058170198</td>
<td>1.11279404365063</td>
<td>2.54921548666884</td>
</tr>
<tr>
<td>9</td>
<td>1.0568705719584</td>
<td>4.9999999821774</td>
<td>1.11938992728867</td>
<td>2.55121489438238</td>
</tr>
<tr>
<td>10</td>
<td>1.0634337268371</td>
<td>4.999995642519517</td>
<td>1.12554051292081</td>
<td>2.55325367456541</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.6462847
Table 9: Example 4.3: numerical study of Algorithm 1 and $u_0 = u_1 = [-1, 2, -3, 4]^T$.

<table>
<thead>
<tr>
<th>Iter (n)</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>67.76571431336 04</td>
<td>76.71455308199899</td>
<td>119.464591876179</td>
<td>136.782073520191</td>
</tr>
<tr>
<td>2</td>
<td>-68.2650687067241</td>
<td>78.1919764133559</td>
<td>120.945690214610</td>
<td>138.263931342072</td>
</tr>
<tr>
<td>3</td>
<td>-10.0905654923274</td>
<td>7.52762193553885</td>
<td>48.8773488496240</td>
<td>59.812015858899</td>
</tr>
<tr>
<td>4</td>
<td>1.58999071741342</td>
<td>0.271536323838546</td>
<td>5.4220385209038</td>
<td>5.83759718234703</td>
</tr>
<tr>
<td>5</td>
<td>1.76489239217402</td>
<td>0.956593352840844</td>
<td>5.0119076595474</td>
<td>4.97568499729916</td>
</tr>
<tr>
<td>7</td>
<td>-12.7857198680</td>
<td>-5.3218756241530</td>
<td>19.1290503062070</td>
<td>19.33455630365</td>
</tr>
<tr>
<td>9</td>
<td>5.00029614550570</td>
<td>4.99911277307117</td>
<td>5.00029489874547</td>
<td>5.00029489874547</td>
</tr>
<tr>
<td>10</td>
<td>5.00024019269176</td>
<td>4.99928042449310</td>
<td>5.00023918201651</td>
<td>5.00023918201651</td>
</tr>
<tr>
<td>11</td>
<td>5.00021568341312</td>
<td>4.99935386554261</td>
<td>5.00023803185504</td>
<td>5.00021477628717</td>
</tr>
<tr>
<td>12</td>
<td>5.00020993400317</td>
<td>4.99938686331783</td>
<td>5.00020905278423</td>
<td>5.00020905278423</td>
</tr>
</tbody>
</table>

CPU time is seconds 0.136686

Acknowledgment

The authors are heartily grateful to the reviewers for their valuable remarks which greatly improved the results and presentation of the paper.

This project was supported by Rajamangala University of Technology Phra Nakhon (RMUTP).

References