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Third Hankel determinant and Zalcman functional for a class of starlike functions with respect to symmetric points related with sine function

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Abstract

In this article we define a class of starlike functions with respect to symmetric points in the domain of sine function. Also, we investigate coefficients bounds and upper bounds for the third order Hankel determinant for this defined class. We also evaluate the Zalcman functional $|a_3^2 - a_5|$. Specializing the parameters, we improve Zalcman functional for the class of starlike functions.

Keywords: Analytic functions, subordinations, sine function, Hankel determinant, Zalcman functional.

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1. Introduction and Definitions

Let $\mathcal{A}$ represents the class of all analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{D}), \tag{1.1}$$

these functions are regular in the region $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We denote $\mathcal{S}$ as subclass of class $\mathcal{A}$, which are univalent in $\mathbb{D}$.

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For two analytic functions h_1 and h_2 we say that h_1 is subordinated to h_2 and symbolically is written as $h_1 \prec h_2$, if there is an analytic function $\nu(z)$ in $\mathbb{D}$, with the properties that $\nu(0) = 0$ and $|\nu(z)| < 1$, such that $h_1(z) = h_2(\nu(z))$, $z \in \mathbb{D}$. In case if $h_2(z) \in \mathcal{S}$ then the following relation holds;

$$h_1(z) \prec h_2(z), \quad z \in \mathbb{D} \quad \Longleftrightarrow \quad h_1(0) = h_2(0) \quad \& \quad h_1(\mathbb{D}) \subset h_2(\mathbb{D}).$$

Let $\mathcal{P}$ be the family of functions $h(z)$ that are holomorphic in $\mathbb{D}$ with $\Re(h(z)) > 0$ and have the power series of the form;

$$h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \ (z \in \mathbb{D}). \tag{1.2}$$

The class $\mathcal{S}^*$, also known as the class of starlike functions, is the most important subclass of $\mathcal{S}$, which can be defined as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \right\}.$$

A function $f(z)$ in class $\mathcal{A}$ is said to be starlike with respect to symmetric point, denoted by $\mathcal{S}_s^*$ which was introduced by Sakaguchi [28], if for any $r_0 < 1$ and for any z_0 on the circle $|z| = r_0$, where the angular velocity of $f(z)$ about the point $f(z_0)$ is positive at z_0 as z traverses the circle $|z| = r_0$ in the positive direction, mathematically

$$\Re\left(\frac{2zf'(z)}{f(z) - f(-z_0)}\right) > 0, \text{ for } z = z_0, \ |z| = r_0.$$

Cho et al. [6] introduced and studied the class of starlike functions $\mathcal{S}^*(1 + \sin(z))$ defined by

$$\frac{zf'(z)}{f(z)} \prec 1 + \sin(z), \ z \in \mathbb{D}.$$

Now we define the subclass $\mathcal{S}_s^*(1 + \sin z)$ of $\mathcal{S}_s^*$ in Sine domain as follow

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec 1 + \sin(z) = \Psi(z), \ z \in \mathbb{D}.$$

One of most attracting area of modern Geometric Function Theory is the Hankel determinant of coefficients of functions for various classes. For given parameters $j, k \in \mathbb{N} = \{1, 2, \ldots\}$, the Hankel determinant $H_{j,k}(f)$ was defined by Pommerenke [22, 23] for a function $f \in \mathcal{S}$ having series expansion (1.1) as follows:

$$H_{j,k}(f) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+j-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+j-1} & a_{k+j} & \cdots & a_{k+2j-2} \end{vmatrix}.$$

The Hankel determinants for different orders is obtained for different value of j and k. When $j = 2$ and $k = 1$, the determinant is

$$|H_{2,1}(f)| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left|a_3 - a_2^2\right|, \text{ where } a_1 = 1.$$

Note that $H_{2,1}(f) = a_3 - a_2^2$, is classical Fekete-Szegö functional. In year 1933, the maximum value of $|H_{2,1}(f)|$ was obtained for a class $\mathcal{S}$. For various subclasses of class $\mathcal{A}$, the maximum value of $|H_{2,1}(f)|$ was investigated by different authors, for details see [7, 16–19, 29–34]. Furthermore, second Hankel determinant when $j = 2$ and $k = 2$ is

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The upper bound of $|H_{2,2}(f)|$ has been studied by several authors. For instance readers are advised to see the work of Hayman [8], Noonan and Thomas [20], Janteng et al. [9], Raina et al. [25], and Orhan et al. [21]. The determinant

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_5\left(a_3 - a_2^2\right) - a_4\left(a_4 - a_2 a_3\right) + a_3\left(a_2 a_4 - a_3^2\right) \tag{1.3}$$

is known as third order Hankel determinant. Babalola [2] was the first person to study the upper bound of $H_{3,1}(f)$ for subclass of $\mathcal{S}$. For more details on this topic readers are advised to see the work of several researchers like Zaprawa [35], Raza et al. [27], Khan et al. [11], Cho et al. [5], Lecko et al. [13], Srivastava et al. [32], and Khan et al. [12].

Very recently, Arif et al. [1], investigated upper bounds for third hankel determinant for the class of functions $\mathcal{S}^*(\Psi)$ associated with trigonometric sine function. Stimulated by aforementioned work, we determine the upper bounds of the 3rd Hankel for the class $\mathcal{S}_s^*(\Psi)$ of symmetric points associated with sine function.

The following set of Lemmas are vital to our main results.

Lemma 1.1. *If $h \in \mathcal{P}$ is expressed in series expansion* (1.2), *then*

$$|p_n| \leqslant 2 \text{ for } n \geqslant 1, \tag{1.4}$$

$$\left|p_{i+j} - \mu p_i p_j\right| \leqslant 2 \text{ for } 0 \leqslant \mu \leqslant 1, \tag{1.5}$$

and for complex number ξ, we have

$$\left|p_2 - \xi p_1^2\right| \leqslant 2\max\{1, |2\xi - 1|\}. \tag{1.6}$$

where the inequalities (1.4) *and* (1.5), *are taken from [24] and* (1.6) *is obtained in [10].*

Lemma 1.2 ([1]). *Let $h \in \mathcal{P}$ has power series* (1.2), *then*

$$\left|\alpha p_1^3 - \beta p_1 p_2 + \gamma p_3\right| \leqslant 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|.$$

Lemma 1.3 ([26]). *Let m, n, l and a satisfy the inequalities $0 < m < 1, 0 < r < 1$, and*

$$8r(1-r)\left[(mn - 2l)^2 + (m(r+m) - n)^2\right] + m(1-m)(n - 2rm)^2 \leqslant 4m^2(1-m)^2 r(1-r).$$

If $h(z) \in \mathcal{P}$ and has power series (1.2), *then*

$$\left|lp_1^4 + rp_2^2 + 2mp_1p_3 - \frac{3}{2}np_1^2p_2 - p_4\right| \leqslant 2.$$

2. Coefficients estimates and Fekete-Szegö inequality

In this section we evaluate the coefficients estimates for the class $\mathcal{S}_s^*(\sin)$. Further we evaluate Fekete-Szegö functional for this class.

Theorem 2.1. *If $f(z)$ is of the form* (1.1) *and belongs to $\mathcal{S}_s^*(\Psi)$, then*

$$|a_2| \leqslant \frac{1}{2}, \quad |a_3| \leqslant \frac{1}{2}, \quad |a_4| \leqslant \frac{1}{4}, \quad |a_5| \leqslant \frac{3}{4}. \tag{2.1}$$

Proof. Since $f(z) \in \mathcal{S}_s^*(\Psi)$, then from subordination definition there exists a Schwarz function $\nu(z)$ with $\nu(0) = 0$ and $|\nu(z)| < 1$, such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \Psi(\nu(z)) = 1 + \sin(\nu(z)), \quad (z \in \mathbb{D}).$$

Since $f(z)$ of the form (1.1) so we have

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2 z + 2a_3 z^2 + (4a_4 - 2a_2 a_3) z^3 + (4a_5 - 2a_3^2) z^4 + \cdots . \tag{2.2}$$

For a function

$$h(z) = \frac{1+v(z)}{1-v(z)} = 1 + p_1 z + p_2 z^2 + \cdots .$$

We have $h(z) \in \mathcal{P}$ and by using (1.2)

$$v(z) = \frac{h(z)-1}{h(z)+1} = \frac{p_1 z + p_2 z^2 + p_3 z^3 + \cdots}{2 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots}.$$

This gives

$$\begin{aligned} 1 + \sin(v(z)) = 1 &+ \frac{1}{2} p_1 z + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right) z^2 + \left(\frac{5p_1^3}{48} - \frac{p_1 p_2}{2} + \frac{p_3}{2}\right) z^3 \\ &+ \left(-\frac{1}{32} p_1^4 + \frac{5}{16} p_1^2 p_2 - \frac{1}{2} p_3 p_1 - \frac{1}{4} p_2^2 + \frac{1}{2} p_4\right) z^4 + \cdots . \end{aligned} \tag{2.3}$$

By comparing (2.2) and (2.3), we get

$$a_2 = \frac{p_1}{4}, \tag{2.4}$$

$$a_3 = \frac{p_2}{4} - \frac{p_1^2}{8}, \tag{2.5}$$

$$a_4 = \frac{1}{96} p_1^3 - \frac{3}{32} p_1 p_2 + \frac{1}{8} p_3, \tag{2.6}$$

$$a_5 = \frac{3}{64} p_1^2 p_2 - \frac{1}{8} p_3 p_1 - \frac{1}{32} p_2^2 + \frac{1}{8} p_4. \tag{2.7}$$

Now putting (1.4) in (2.4), we obtain

$$|a_2| \leqslant \frac{1}{2}.$$

Now using (1.6) with (2.5), we get

$$|a_3| = \frac{1}{4}\left|p_2 - \frac{p_1^2}{2}\right| \leqslant \frac{1}{2} \max\left\{1, \left|2\left(\frac{1}{2}\right) - 1\right|\right\} = \frac{1}{2} \max\{1, 0\} = \frac{1}{2}.$$

Using triangle inequality and Lemma 1.2 in (2.6), leads us to

$$|a_4| \leqslant \frac{1}{4}.$$

By rearranging the equation (2.7), we get

$$|a_5| = \left|\frac{1}{8}\left(p_4 - \frac{1}{4} p_2^2\right) - \frac{1}{8} p_1 \left(p_3 - \frac{3}{8} p_1 p_2\right)\right|,$$

using (1.4) and (1.5), we get

$$|a_5| \leqslant \frac{3}{4}.$$

□

Theorem 2.2. *If* $f(z)$ *is of the form* (1.1) *and belongs to* $\mathcal{S}_s^*(\Psi)$, *then for any complex number* ξ,

$$\left|a_3 - \xi a_2^2\right| \leqslant \frac{1}{2}\max\left\{1, \frac{|\xi|}{2}\right\}.$$

Proof. Utilizing (2.4) and (2.5), we get

$$\left|a_3 - \xi a_2^2\right| = \left|\frac{p_2}{4} - \frac{p_1^2}{8} - \frac{\xi}{16}p_1^2\right|.$$

This gives

$$\left|a_3 - \xi a_2^2\right| = \frac{1}{4}\left|p_2 - \left(\frac{2+\xi}{4}\right)p_1^2\right|.$$

Application of (1.6), leads us to

$$\left|a_3 - \xi a_2^2\right| \leqslant \frac{1}{2}\max\left\{1, \frac{|\xi|}{2}\right\}.$$

□

Corollary 2.3. *If* $f \in \mathcal{S}_s^*(\Psi)$ *and* $\xi = 1$, *then*

$$\left|a_3 - a_2^2\right| \leqslant \frac{1}{2}. \tag{2.8}$$

3. Hankel determinants

Now we obtain other important results on the basis of which we will evaluate the third Hankel for this class.

Theorem 3.1. *If* f *is of the form* (1.1) *and belongs to* $\mathcal{S}_s^*(\Psi)$, *then*

$$|a_2 a_3 - a_4| \leqslant \frac{1}{4}. \tag{3.1}$$

Proof. From (2.4), (2.5), and (2.6), we have

$$|a_2a_3 - a_4| = \left|-\frac{1}{24}p_1^3 + \frac{5}{32}p_2p_1 - \frac{1}{8}p_3\right| = \left|\frac{1}{24}p_1^3 - \frac{5}{32}p_1p_2 + \frac{1}{8}p_3\right|.$$

Implementation of triangle inequality and Lemma 1.2, in (2.6), leads us to

$$|a_2a_3 - a_4| \leqslant \frac{1}{4}.$$

□

Theorem 3.2. *If* $f(z)$ *is of the form* (1.1) *and belongs to* $\mathcal{S}_s^*(\Psi)$, *then*

$$\left|a_2a_4 - a_3^2\right| \leqslant \frac{11}{16}. \tag{3.2}$$

Proof. Since from (2.4), (2.5), and (2.6), we have

$$\left|a_2a_4 - a_3^2\right| = \left|\frac{p_1p_3}{32} + \frac{5}{288}p_1^2p_2 - \frac{5}{389}p_1^4 - \frac{1}{16}p_2^2\right| = \left|\frac{5}{128}p_1^2\left(p_2 - \frac{1}{3}p_1^2\right) + \frac{1}{16}\left(p_1p_3 - p_2^2\right) - \frac{p_1p_3}{32}\right|.$$

Using (1.4) , (1.5), and (1.6), we get the desired result. □

Theorem 3.3. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *belongs to* $\mathcal{S}_s^*(\Psi)$, *then*

$$|H_{3,1}(f)| \leqslant \frac{25}{32} \simeq 0.781\,25.$$

Proof. Third order Hankel determinant from equation (1.3) can be written as,

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_2a_3\right) + a_5\left(a_3 - a_2^2\right),$$

where $a_1 = 1$. This provides that

$$|H_{3,1}(f)| \leqslant |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\left|a_4 - a_2a_3\right| + |a_5|\left|a_3 - a_2^2\right|.$$

By implementing (2.1), (2.8), (3.1), and (3.2), we obtain our desired result. □

4. Zalcman functional

In the field of Geometric function theory, one of the classical conjecture proposed by Lawrence Zalcman in 1960 is that the coefficients of class $\mathcal{S}$ satisfy the inequality,

$$|a_n^2 - a_{2n-1}| \leqslant (n-1)^2.$$

The above form holds equality only for the famous Koebe function $k(z) = \frac{z}{(1-z)^2}$ and its rotation. When $n = 2$, the equality holds for the famous Fekete-Szegö inequality. In literature, many researchers [3, 4, 15] studied about Zalcman functional.

Theorem 4.1. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *belongs to* $\mathcal{S}_s^*(\Psi)$, *then*

$$\left|a_3^2 - a_5\right| \leqslant \frac{1}{4} \simeq 0.25.$$

Proof. In order to find Zalcman functional, we use the equations (2.5) and (2.7), then we have

$$\begin{aligned}
\left|a_3^2 - a_5\right| &= \left|\left(\frac{p_2}{4} - \frac{p_1^2}{8}\right)^2 - \frac{3}{64}p_1^2p_2 + \frac{1}{8}p_3p_1 + \frac{1}{32}p_2^2 - \frac{1}{8}p_4\right| \\
&= \left|\frac{p_1^4}{64} + \frac{3p_2^2}{32} - \frac{7}{64}p_1^2p_2 + \frac{1}{8}p_3p_1 - \frac{1}{8}p_4\right| \\
&= \frac{1}{8}\left|\frac{p_1^4}{8} + \frac{3p_2^2}{4} - \frac{7}{8}p_1^2p_2 + 2\left(\frac{1}{2}\right)p_3p_1 - p_4\right|,
\end{aligned}$$

using Lemma 1.3, to the last term will give us required results. □

5. Conclusion

On the basis of various newly defined classes, we introduced a class of starlike functions with respect to symmetric point in the domain of sine function. For this defined class, known results such as coefficient bounds and upper bounds for third order Hankel determinant are evaluated. Also, more geometric results of interest for this class can be looked into.

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References

[1] M. Arif, M. Raza, H. Tang, S. Hussain, H. Khan, *Hankel determinant of order three for familiar subsets of analytic functions related with sine function*, Open Math., **17** (2019), 1615–1630. 1, 1.2

[2] K. O. Babalola, *On* $H_3(1)$ *Hankel determinant for some classes of univalent functions*, Inequal. Theory Appl., **6** (2007), 1–7. 1

[3] D. Bansal, J. Sokól, *Zalcman conjecture for some subclass of analytic functions*, J. Fract. Calc. Appl., **8** (2017), 1–5. 4

[4] J. E. Brown, A. Tsao, *On the Zalcman conjecture for starlike and typically real functions*, Math. Z., **191** (1986), 467–474. 4

[5] N. E. Cho, B. Kowalczyk, O. S. Kwon , A. Lecko, J. Sim, *Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha*, J. Math. Inequal., **11** (2017), 429–439. 1

[6] N. E. Cho, V. Kumar, S. S. Kumar, V. Ravichandran, *Radius problems for starlike functions associated with the Sine function*, Bull. Iranian Math. Soc., **45** (2019), 213–232. 1

[7] H. Ö. Güney, G. Murugusundaramoorthy, H. M. Srivastava, *The second hankel determinant for a certain class of bi-close-to-convex function*, Results Math., **74** (2019), 13 pages. 1

[8] W. K. Hayman, *On the second Hankel determinant of mean univalent functions*, Proc. London Math. Soc., **18** (1968), 77–94. 1

[9] A. Jangteng, S. A. Halim, M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math., **7** (2006), 1–5. 1

[10] F. R. Keogh, E. P. Merkes, *coefficient inequality for certain subclasses of analytic functions*, Proc. Amer. Math. Soc., **20** (1969), 8–12. 1.1

[11] M. G. Khan, B. Ahmad, G. Muraugusundaramoorthy, R. Chinram, W. K. Mashwani, *Applications of Modified Sigmoid Functions to a Class of Starlike Functions*, J. Funct. Spaces, **2020** (2020), 8 pages. 1

[12] M. G. Khan, B. Ahmad, J. Sokol, Z. Muhammad, W. K. Mashwani, R. Chinram, P. Petchkaew, *Coefficient problems in a class of functions with bounded turning associated with Sine function*, Eur. J. Pure Appl. Math., **14** (2021), 53–64. 1

[13] A. Lecko, Y. J. Sim, B. Śmiarowska, *The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2*, Complex Anal. Oper. Theory, **13** (2019), 2231–2238. 1

[14] R. J. Libera, E. J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivative in* $\mathcal{P}$, Proc. Amer. Math. Soc., **87** (1983), 251–257.

[15] W. C. Ma, *The Zalcman conjecture for close-to-convex functions*, Proc. Amer. Math. Soc., **104** (1988), 741–744. 4

[16] S. Mahmood, M. Jabeen, S. N. Malik, H. M. Srivastava, R. Manzoor, S. M. J. Riaz, *Some coefficient inequalities of* q*-starlike functions associated with conic domain defined by* q*-derivative*, J. Funct. Spaces, **2018** (2018), 1–13. 1

[17] S. Mahmood, I. Khan, H. M. Srivastava, S. N. Malik, *Inclusion relations for certain families of integral operators associated with conic regions*, J. Inequal. Appl., **59** (2019), 11 pages.

[18] S. Mahmood, H. M. Srivastava, N. Khan, Q. Z. Ahmad, B. Khan, I. Ali, *Upper bound of the third Hankel determinant for a subclass of* q*-starlike functions*, Symmetry **11** (2019), 1–13.

[19] S. Mahmood, H. M. Srivastava, S. N. Malik, *Some subclasses of uniformly univalent functions with respect to symmetric points*, Symmetry, **11** (2019), 14 pages. 1

[20] J. W. Noonan, D. K. Thomas, *On the second Hankel determinant of areally mean* p*-valent functions*, Trans. Amer. Math. Soc., **223** (1976), 337–346. 1

[21] H. Orhan, N. Magesh, J. Yamini, *Bounds for the second Hankel determinant of certain bi-univalent functions*, Turkish J. Math., **40** (2016), 679–687. 1

[22] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc., **41** (1966), 111–122. 1

[23] C. Pommerenke, *On the Hankel determinants of univalent functions*, Mathematika, **14** (1967), 108–112. 1

[24] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, (1975). 1.1

[25] R. K. Raina, J. Sokół, *On coefficient estimates for a certain class of starlike functions*, Hacet. J. Math. Stat., **44** (2015), 1427–1433. 1

[26] V. Ravichandran, S. Verma, *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris, **353** (2015), 505–510. 1.3

[27] M. Raza, S. N. Malik, *Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inequal. Appl., **2013** (2013), 8 pages. 1

[28] K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan, **11** (1959), 72–75. 1

[29] M. Shafiq, H. M. Srivastava, N. Khan, Q. Z. Ahmad, M. Darus, S. Kiran, *An upper bound of the third Hankel determinant for a subclass of* q*-starlike functions associated with k-Fibonacci numbers*, Symmetry, **12** (2020), 1–17. 1

[30] L. Shi, M. G. Khan, B. Ahmad, *Some geometric properties of a family of analytic functions involving a generalized q-operator*, Symmetry, **12** (2020), 11 pages.

[31] H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan, B. Khan, *Hankel and Toeplitz determinants for a subclass of* q*-starlike functions associated with a general conic domain*, Mathematics, **7** (2019), 15 pages.

[32] H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, N. Zaman, H. H. Shah, *Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli*, Mathematics, **7** (2019), 10 pages. 1

[33] H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, *Coefficient inequalities for* q*-starlike functions associated with the Janowski functions*, Hokkaido Math. J., **48** (2019), 407–425.

[34] H. M. Srivastava, B. Khan, N. Khan, M. Tahir, S. Ahmad, N. Khan, *Upper bound of the third Hankel determinant for a subclass of* q*-starlike functions associated with the* q*-exponential function*, Bull. Sci. Math., **167** (2021), 16 pages. 1

[35] P. Zaprawa, *Third Hankel determinants for subclasses of univalent functions*, Mediterr. J. Math., **14** (2017), 10 pages. 1