Fixed points of generalized rational $(\alpha, \beta, Z)$-contraction mappings under simulation functions

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Abstract

In this paper, we combine the $(\alpha, \beta)$-admissible mappings and simulation function in order to obtain the generalized form of rational $(\alpha, \beta, Z)$-contraction mapping. Further this concept is used in the setting of $b$-metric space in order to obtain some fixed point theorems. Suitable examples are also established to verify the validity of the results obtained.

Keywords: Fixed points, generalized rational $(\alpha, \beta, Z)$-contraction mapping, $(\alpha, \beta)$-admissible mappings, simulation function, $b$-metric space.


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1. Introduction

Samet et al. [22] introduced $\alpha$-$\psi$-contractive type mapping and $\alpha$-admissible mappings. The concept is further generalized by Karapinar and Samet[15] by introducing generalized $\alpha$-$\psi$-contractive type mapping. The concept of cyclic $(\alpha, \beta)$-admissible mapping was introduced by Alizadeh et al. [2] by generalizing the concept of $\alpha$-admissible mapping [22]. Khojasteh et al. [17] introduced simulation function and the notion of $Z$-contraction with respect to simulation function to generalize Banach contraction principle. The concept of Khojasteh et al. [17] is further modified by Argouib et al. [5]. In this paper, we introduce cyclic $(\alpha, \beta)$-admissible mapping in simulation function to result a generalized rational $(\alpha, \beta, Z)$-contraction. Here, we use $b$-metric space [7, 10] in order to obtain fixed point theorems for generalized rational $(\alpha, \beta, Z)$-contraction mappings. For more results in rational type contractions and $Z$-contractions we refer to the papers in [1, 3, 4, 6, 8, 9, 11–14, 16, 18–21, 23, 24] and references therein.

2. Preliminaries

Bakhtin [7] introduced the concept of $b$-metric space as follows.

Definition 2.1 ([7, 10]). Let $W$ be a non empty set and the mapping $b : W \times W \to [0, +\infty)$ satisfies:

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1. \( b(u, v) = 0 \) if and only if \( u = v \) for all \( u, v \in W \);
2. \( b(u, v) = b(v, u) \) for all \( u, v \in W \);
3. there exists a real number \( s \geq 1 \) such that \( b(u, v) \leq s[b(u, w) + b(w, v)] \) for all \( u, v, w \in W \).

Then \( b \) is called a \( b \)-metric on \( W \) and \( (W, b) \) is called a \( b \)-metric space (in short \( b \)MS) with coefficient \( s \).

**Definition 2.2** ([7, 10]). Let \( (W, b) \) be a \( b \)-metric space, \( \{u_n\} \) be a sequence in \( W \) and \( x \in W \). Then

1. the sequence \( \{u_n\} \) is said to be convergent in \( (W, b) \) and converges to \( u \), if for every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( b(u_n, u) < \epsilon \) for all \( n > n_0 \) and this fact is represented by \( \lim_{n \to +\infty} u_n = u \) or \( u_n \to u \) as \( n \to +\infty \);
2. the sequence \( \{u_n\} \) is said to be a Cauchy sequence in \( (W, b) \) if for every \( \epsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) such that \( b(u_n, u_m) < \epsilon \) for all \( n, m > n_0 \) or equivalently, if \( \lim_{n,m \to +\infty} b(u_n, u_m) = 0 \);
3. \( (W, b) \) is said to be a complete \( b \)-metric space if every Cauchy sequence in \( W \) converges to some \( u \in W \).

It can be noted that a \( b \)-metric space need not be a continuous function.

**Definition 2.3** ([2]). Let \( W \) be a nonempty set, \( f \) be a self-mapping on \( W \) and \( \alpha, \beta : W \to [0, +\infty) \) be two mappings. We say that \( f \) is a cyclic \((\alpha, \beta)\)-admissible mapping if \( u \in W \) with

\[
 \alpha(u) \geq 1 \text{ implies } \beta(fu) \geq 1
\]

and \( u \in W \) with

\[
 \beta(u) \geq 1 \text{ implies } \alpha(fu) \geq 1.
\]

In 2015, Khojasteh et al. [17] introduced the class of simulation functions. Further, Argoubi et al. [5] modified the definition of simulation functions and defined as follows.

**Definition 2.4** ([5]). A simulation function is a function \( \zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) that satisfies the following conditions:

1. \( \zeta(p, q) < p - q \) for all \( p, q > 0 \);
2. if \( \{q_n\} \) and \( \{p_n\} \) are sequences in \( (0, +\infty) \) such that \( \lim_{n \to +\infty} q_n = \lim_{n \to +\infty} p_n = l \in (0, +\infty) \), then

\[
 \lim_{n \to +\infty} \sup_{n} \zeta(q_n, p_n) < 0.
\]

It is to be noted that any simulation function in the sense of Khojasteh et al. [17] is also a simulation function in the sense of Argoubi et al. [5]. The following function is a simulation function in the sense of Argoubi et al. [5]

**Example 2.5** ([5]). Define a function \( \zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) by

\[
 \zeta(p, q) = \begin{cases} 
 1, & \text{if } (p, q) = (0, 0), \\
 \lambda p - q, & \text{otherwise},
\end{cases}
\]

where \( \lambda \in (0, 1) \). Then \( \zeta \) is a simulation function in the sense of Argoubi et al. [5].

**Theorem 2.6** ([17]). Let \( (W, b) \) be a metric space and \( T : W \to W \) be a \( Z \)-contraction with respect to a simulation function \( \zeta \); that is

\[
 \zeta\left(b(Tu, Tv), b(u, v)\right) \geq 0, \text{ for all } u, v \in W.
\]

Then \( T \) has a unique fixed point.
It is worth mentioning that the Banach contraction is an example of $Z$-contraction by defining $\zeta : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$ via
\[
\zeta(q, p) = \gamma p - q, \text{ for all } p, q \in [0, +\infty),
\]
where $\gamma \in [0, 1)$.

Following lemma was proved by Qawagnesh [20] which is valid for complete b-metric space also.

**Lemma 2.7 ([20]).** Let $A : W \rightarrow W$ be a cyclic $(\alpha, \beta)$-admissible mapping. Assume that there exist $u_0, u_1 \in W$ such that

\[
\alpha(u_0) \geq 1 \implies \beta(u_1) \geq 1
\]

and

\[
\beta(u_0) \geq 1 \implies \alpha(u_1) \geq 1.
\]

Define a sequence $\{u_n\}$ by $u_{n+1} = Au_n$. Then

\[
\alpha(u_n) \geq 1 \implies \beta(u_m) \geq 1
\]

and

\[
\beta(u_n) \geq 1 \implies \alpha(u_m) \geq 1
\]

for all $m, n \in \mathbb{N}$ with $n < m$.

### 3. Main result

We start our result with the following definitions.

**Definition 3.1.** Let $(W, b)$ be a complete b-metric space with $s \geq 1$, $A : W \rightarrow W$ be a mapping and $\alpha, \beta : \mathbb{R} \rightarrow [0, +\infty)$ be two functions. Then $A$ is said to be a generalized $(\alpha, \beta, Z)$-rational contraction mapping if $A$ satisfies the following conditions:

1. $A$ is cyclic $(\alpha, \beta)$-admissible;
2. There exists simulation function $\zeta \in Z$ such that

\[
\alpha(u)\beta(v) \geq 1 \implies \zeta(b(Au, Av), M(u, v)) \geq 0
\]

holds for all $u, v \in W$, where

\[
M(u, v) = \max\left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)} \right\}
\]

**Theorem 3.2.** Let $(W, b)$ be a complete b-metric space with $s \geq 1$, $A : W \rightarrow W$ be a mapping and $\alpha, \beta : [0, +\infty)$ be two functions. Suppose the following conditions hold:

1. $A$ is a generalized $(\alpha, \beta, Z)$-rational contraction mapping;
2. There exists an element $u_0 \in W$ such that $\alpha(u_0) \geq 1$ and $\beta(u_0) \geq 1$;
3. $A$ is continuous.

Then, $A$ has a fixed point $u^* \in W$ such that $Au^* = u^*$.

**Proof.** Assume that there exists $u_0 \in W$ such that $\alpha(u_0) \geq 1$. We divide our proof into the following steps.

**Step 1:** Define a sequence $\{u_n\}$ in $W$ such that $u_{n+1} = Au_n$ for all $n \in \mathbb{N} \cup \{0\}$. If $u_n = u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then $A$ has a fixed point and proof is finished. Hence, we assume that $u_n \neq u_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$; that is, $b(u_n, u_{n+1}) \neq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $A$ is a cyclic $(\alpha, \beta)$-admissible mapping, we
have
\[ \alpha(u_0) \geq 1 \implies \beta(u_1) = \beta(Au_0) \geq 1 \implies \alpha(u_2) = \alpha(Au_1) \geq 1 \]
and
\[ \beta(u_0) \geq 1 \implies \alpha(u_1) = \alpha(Au_0) \geq 1 \implies \beta(u_2) = \beta(Au_1) \geq 1, \]
then by continuing the above process, we have
\[ \alpha(u_n) \geq 1 \quad \text{and} \quad \beta(u_n) \geq 1 \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}. \]

Thus, \( \alpha(u_n)\beta(u_{n+1}) \geq 1 \), for all \( n \in \mathbb{N} \cup \{0\} \). Therefore, we get
\[ \zeta(b(Au_n, Au_{n+1}), M(u_n, u_{n+1})) \geq 0 \]
for all \( n \in \mathbb{N} \), where
\[
M(u_n, u_{n+1}) = \max \left\{ b(u_n, u_{n+1}), b(u_n, Au_n), b(u_{n+1}, Au_n), \frac{b(u_n, Au_n) b(u_n, Au_{n+1}) + b(u_{n+1}, Au_{n+1}) b(u_{n+1}, Au_n)}{1 + s[b(u_n, Au_n) + b(u_{n+1}, Au_{n+1})]} \right\}
\]
\[
= \max \left\{ b(u_n, u_{n+1}), b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}), \frac{b(u_n, u_{n+1}) b(u_n, u_{n+2}) + b(u_{n+1}, u_{n+2}) b(u_{n+1}, u_{n+1})}{1 + s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]} \right\}
\]
\[
= \max \left\{ b(u_n, u_{n+1}), b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}), \frac{b(u_n, u_{n+1}) s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]}{1 + s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]} \right\}
\]
\[
= \max \left\{ b(u_n, u_{n+1}), \max(b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})) \right\}.
\]

It follows that
\[ \zeta(b(u_{n+1}, u_{n+2}), \max(b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}))) \geq 0. \]

Condition (1) of Definition 2.4 implies that
\[ 0 \leq \zeta(b(u_{n+1}, u_{n+2}), \max(b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}))) < \max(b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})) - b(u_{n+1}, u_{n+2}). \]

Thus, we conclude that
\[ b(u_{n+1}, u_{n+2}) < \max(b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})) \]
for all \( n \geq 1 \). The last inequality implies that
\[ b(u_{n+1}, u_{n+2}) < b(u_n, u_{n+1}), \quad \text{for all} \quad n \geq 1. \]
It follows that the sequence \{b(u_n, u_{n+1})\} is non increasing. Therefore, there exists \( r \geq 0 \) such that
\[
\lim_{n \to +\infty} b(u_n, u_{n+1}) = r.
\]
Note that if \( r \neq 0 \); that is \( r > 0 \), then by condition (2) of Definition 2.4, we have
\[
0 \leq \lim_{n \to +\infty} \sup \left( b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}) \right) < 0,
\]
which is a contradiction. This implies that \( r = 0 \), that is
\[
\lim_{n \to +\infty} b(u_n, u_{n+1}) = 0. \tag{3.1}
\]

**Step 2:** Now, we prove that \( \{u_n\} \) is a Cauchy sequence. Suppose to the contrary that \( \{u_n\} \) is not a Cauchy sequence. Then there exists \( \epsilon > 0 \) and two subsequences \( \{u_{m(k)}\} \) and \( \{u_{n(k)}\} \) of \( \{u_n\} \) with \( m(k) > n(k) > k \) and \( m(k) \) is the smallest index in \( \mathbb{N} \) such that
\[
b(u_{n(k)}, u_{m(k)}) \geq \epsilon,
\]
so,
\[
b(u_{n(k)}, u_{m(k)-1}) < \epsilon.
\]

Triangular inequality implies that
\[
\epsilon \leq b(u_{n(k)}, u_{m(k)}) \leq s \left[ b(u_{n(k)}, u_{m(k)-1}) + b(u_{m(k)-1}, u_{m(k)}) \right] < s \left[ \epsilon + b(u_{m(k)-1}, u_{m(k)}) \right].
\]
Taking \( k \to +\infty \) in the above inequality and using (6), we get
\[
\epsilon \leq \lim_{k \to +\infty} \sup b(u_{n(k)}, u_{m(k)}) < s \epsilon. \tag{3.2}
\]

From triangular inequality, we have
\[
b(u_{n(k)}, u_{m(k)}) \leq s \left[ b(u_{n(k)}, u_{n(k)+1}) + b(u_{n(k)+1}, u_{m(k)}) \right] \tag{3.3}
\]
and
\[
b(u_{n(k)+1}, u_{m(k)}) \leq s \left[ b(u_{n(k)+1}, u_{n(k)}) + b(u_{n(k)}, u_{m(k)}) \right]. \tag{3.4}
\]
By taking the limit as \( k \to +\infty \) in (3.3) and applying (3.1) and (3.2), we get
\[
\epsilon \leq \lim_{k \to +\infty} \sup b(u_{n(k)}, u_{m(k)}) \leq s \lim_{k \to +\infty} \sup b(u_{n(k)+1}, u_{m(k)}).
\]
Again, by taking the upper limit as \( k \to +\infty \) in (3.4), we get
\[
\lim_{k \to +\infty} \sup b(u_{n(k)+1}, u_{m(k)}) \leq s \left( \lim_{k \to +\infty} \sup b(u_{n(k)}, u_{m(k)}) \right) \leq s \epsilon = s^2 \epsilon,
\]
\[
\frac{\epsilon}{s} \leq \lim_{k \to +\infty} \sup b(u_{n(k)+1}, u_{m(k)}) \leq s^2 \epsilon. \tag{3.5}
\]
Similarly,
\[
\frac{\epsilon}{s} \leq \lim_{k \to +\infty} \sup b(u_{n(k)}, u_{m(k)+1}) \leq s^2 \epsilon. \tag{3.6}
\]
By triangular inequality, we have
\[
b(u_{n(k)+1}, u_{m(k)}) \leq s \left[ b(u_{n(k)+1}, u_{m(k)+1}) + b(u_{m(k)+1}, u_{m(k)}) \right]. \tag{3.7}
\]
On letting \( k \to +\infty \) in (3.7) and using inequalities (3.1) and (3.5), we get
\[
\frac{\epsilon}{s^k} \leq \lim_{k \to +\infty} \sup_{n(k)+1} b(u_{n(k)+1}, u_{m(k)+1}). \tag{3.8}
\]
Following the above process, we find
\[
\lim_{k \to +\infty} \sup_{n(k)+1} b(u_{n(k)+1}, u_{m(k)+1}) \leq s^3 \epsilon. \tag{3.9}
\]
From (3.8) and (3.9), we get
\[
\frac{\epsilon}{s^k} \leq \lim_{k \to +\infty} \sup_{n(k)+1} b(u_{n(k)+1}, u_{m(k)+1}) \leq s^3 \epsilon.
\]
Since \( \alpha(u_0) > 1 \) and \( \beta(u_0) > 1 \) by Lemma 2.7, we conclude that
\[
\alpha(u_{n(k)}) \beta(u_{m(k)}) \geq 1.
\]
Since \( A \) is generalized \((\alpha, \beta, Z)\)-rational contraction, we have
\[
\zeta \left( b(Au_{n(k)}, Au_{m(k)}), M(u_{n(k)}, u_{m(k)}) \right) \leq 0
\]
for all \( u, v \in W \), where
\[
M(u_{n(k)}, u_{m(k)}) = \max \left\{ b(u_{n(k)}, u_{n(k)}), b(u_{n(k)}, Au_{n(k)}), b(u_{m(k)}, Au_{m(k)}), \right. \nonumber
\]
\[
\frac{b(u_{n(k)}, Au_{n(k)}) b(u_{n(k)}, Au_{m(k)}) + b(u_{m(k)}, Au_{n(k)}) b(u_{m(k)}, Au_{m(k)})}{1 + s [b(u_{n(k)}, Au_{n(k)}) + b(u_{m(k)}, Au_{m(k)})]}
\]
\[
\frac{b(u_{n(k)}, Au_{n(k)}) b(u_{n(k)}), u_{m(k)+1}) + b(u_{m(k)}, u_{n(k)+1}) b(u_{m(k)}, u_{n(k)+1})}{1 + s [b(u_{n(k)}, u_{n(k)+1}) + b(u_{m(k)}, u_{m(k)+1})]}
\]
\[
\frac{b(u_{n(k)}, u_{n(k)+1}) b(u_{n(k)}, u_{m(k)+1}) + b(u_{m(k)}, u_{n(k)+1}) b(u_{m(k)}, u_{m(k)+1})}{1 + s [b(u_{n(k)}, u_{n(k)+1}) + b(u_{m(k)}, u_{m(k)+1})]}
\]
\[
\max \left\{ b(u_{n(k)}, u_{n(k)+1}), b(u_{n(k)}, u_{m(k)+1}), (u_{m(k)}, u_{m(k)+1}), \right. \nonumber
\]
\[
\left. b(u_{n(k)}, u_{n(k)+1}) b(u_{n(k)}, u_{m(k)+1}) + b(u_{m(k)}, u_{n(k)+1}) b(u_{m(k)}, u_{n(k)+1}), \right. \nonumber
\]
\[
\left. b(u_{n(k)}, u_{n(k)+1}) b(u_{n(k)}, u_{m(k)+1}) + b(u_{m(k)}, u_{n(k)+1}) b(u_{m(k)}, u_{m(k)+1}) \right\}
\]
Taking the limit as \( k \to +\infty \) and using (3.1), (3.2), (3.5), and (3.6), we get
\[
\epsilon = \max \{ \epsilon, 0, 0, 0 \} \leq \lim_{k \to +\infty} \sup_{n(k)+1} M(u_{n(k)}, u_{m(k)}) \leq \max \{ \epsilon s, 0, 0, 0 \} = s \epsilon.
\]
Note that condition (2) of Definition 2.4, implies that
\[
0 \leq \lim \sup M(u_{n(k)}, u_{m(k)}) < 0,
\]
which is a contradiction. Thus \( \{u_n\} \) is a Cauchy sequence.

**Step 3:** Finally in this step we prove that \( A \) has a fixed point. Since \( \{u_n\} \) is a Cauchy sequence in the complete \( b\)-metric space \( W \), there exists \( u^* \in W \) such that \( u_n \to u^* \). The continuity of \( A \) implies that \( Au_n \to Au^* \). Since \( u_{2n+1} = Au_n \) and \( u_{2n+1} \to u^* \), by uniqueness of limit, we have
\[
Au^* = u^*.
\]
So, \( u^* \) is a fixed point of \( A \). This concludes the proof. \( \square \)
Note that the continuity of the mapping \( A \) in Theorem 3.2 can be dropped if we replace condition (3) by a suitable one as in the following result.

**Theorem 3.3.** Let \((W, b)\) be a complete \( b \)-metric space with \( s \geq 1 \), \( A : W \to W \) be a mapping and \( \alpha, \beta : W \to [0, +\infty) \) be two functions. Suppose the following conditions hold:

1. \( A \) is a generalized \((\alpha, \beta, Z)\)-rational contraction mapping;
2. there exists an element \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. if \( \{u_n\} \) is a sequence in \( W \) converges to \( u \in W \) with \( \alpha(u_n) \geq 1 \) (or \( \beta(u_n) \geq 1 \)) for all \( n \in \mathbb{N} \), then \( \beta(u) \geq 1 \) (or \( \alpha(u) \geq 1 \)) for all \( n \in \mathbb{N} \).

Then, \( A \) has a fixed point.

**Proof.** Following the same steps as in the proof of Theorem 3.2 we construct a sequence \( \{u_n\} \) in \( W \) by \( u_{n+1} = Au_n \) for all \( n \in \mathbb{N} \) such that \( u_n \to u^* \in W \), \( \alpha(u_n) \geq 1 \) and \( \beta(u_n) \geq 1 \) for all \( n \in \mathbb{N} \). By condition (3), we have \( \alpha(u^*) \geq 1 \) and \( \beta(u^*) \geq 1 \). So, \( \alpha(u^*) \beta(u^*) \geq 1 \).

**Claim:** \( Au^* = u^* \). Suppose not, that is \( Au^* \neq u^* \). Therefore \( b(Au^*, u^*) \neq 0 \) and

\[
\lim_{n \to +\infty} b(u_{n+1}, Au^*) \neq 0. \tag{3.10}
\]

Since \( A \) is a generalized \((\alpha, \beta, Z)\)-rational contraction mapping, we have

\[
\lim_{n \to +\infty} b(u_{n+1}, Au^*) \neq 0, \tag{3.11}
\]

for all \( n \in \mathbb{N} \). Now,

\[
M(u_n, u^*) = \max \left\{ \frac{b(u_n, u^*)}{1 + s \left[ b(u_n, Au_n) + b(u^*, Au^*_n) \right]}, \frac{b(u_n, Au_n) + b(u^*, Au_n)}{1 + b(u_n, Au^*_n) + b(u^*, Au^*_n)} \right\}. \tag{3.12}
\]

Letting \( n \to +\infty \) in (3.12), we obtain

\[
\lim_{n \to +\infty} M(u_n, u^*) = b(u^*, Au^*) \neq 0. \tag{3.13}
\]

By using (3.10), (3.11), and (3.13), then condition (2) of Definition 2.4 implies that

\[
0 \leq \lim_{n \to +\infty} \sup_{n \to +\infty} \left( b(u_{n+1}, Au^*), M(u_n, u^*) \right) < 0,
\]

which is a contradiction. So \( Au^* = u^* \). Thus, \( u^* \) is a fixed point of \( A \). This concludes the proof. \( \square \)

Now, we introduce an example to show that if \( A \) satisfies all hypothesis of Theorems 3.2 or 3.3, then fixed point of \( A \) is not necessarily to be unique.
Example 3.4. Let \( W = [0, 1] \) and \( s = 2 \). Define \( b : W \times W \to \mathbb{R} \) by \( b(u, v) = |u - v| \). Also define the mapping \( A : W \to W \) by \( Au = u^2 \). Define the function \( \alpha, \beta : W \to \mathbb{R} \) by

\[
\alpha(u) = \beta(u) = \begin{cases} 1, & \text{if } u = 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Define \( \zeta : [0, \infty) \times [0, \infty) \to \mathbb{R} \) by

\[
\zeta(q, p) = \frac{p}{p + 1} - q.
\]

Then, we have the following:

1. \( A \) is continuous;
2. there exists \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. \( A \) is cyclic \((\alpha, \beta)\)-admissible mapping;
4. for any \( u, v \in W \) with \( \alpha(u)\beta(v) \geq 1 \), then

\[
\zeta(b(Au, Av), M(u, v)) \geq 0,
\]

where

\[
M(u, v) = \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]} \right\},
\]

\[
\frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)}
\]

\( = \max \{ b(0, 0), b(0, 0), b(0, 0), b(0, 0) \} = 0. \)

So,

\[
\zeta(b(Au, Av), M(u, v)) = \zeta(0, 0) = \frac{0}{1 + 0} - 0 = 0 \geq 0.
\]

To prove (5), let \( \{u_n\} \) is a sequence in \( W \) converges to \( u \in W \) with \( \alpha(u_n) \geq 1 \) for all \( n \in \mathbb{N} \), then \( \beta(u) \geq 1 \).

Proof. Proof of (1) and (2) are clear. To prove (3), let \( u \in W \). If \( \alpha(u) \geq 1 \) then \( u = 0 \). So, \( A(u) = A(0) = 0 \) and \( \beta(Au) = \beta(0) = 1 \geq 1 \). If \( \beta(u) \geq 1 \), then \( u = 0 \). So, \( A(u) = A(0) = 0 \) and \( \alpha(Au) = \alpha(0) = 1 \geq 1 \). So, \( A \) is cyclic \((\alpha, \beta)\)-admissible mapping. To prove (4), let \( u, v \in W \) with \( \alpha(u)\beta(v) \geq 1 \). Then \( u = v = 0 \). So, \( A(u) = A(v) = 0 \). Therefore, we have

\[
M(u, v) = \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]} \right\},
\]

\[
\frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)}
\]

\( = \max \{ b(0, 0), b(0, 0), b(0, 0), b(0, 0) \} = 0. \)

Next, we gave some corollaries.

Corollary 3.5. Let \((W, b)\) be a complete \(b\)-metric space with \( s \geq 1 \), \( A : W \to W \) be a mapping and \( \alpha : W \times W \to [0, +\infty) \) be a function. Suppose that the following conditions hold:

1. there exists \( \zeta \in \mathbb{Z} \) such that if \( u, v \in W \) with \( \alpha(u, v) \geq 1 \), then \( \zeta(b(Au, Av), M(u, v)) \geq 0 \), where

\[
M(u, v) = \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]} \right\},
\]

\[
\frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)}
\]

Then, it has a fixed point.
(2) \(A\) is \(\alpha\)-admissible;
(3) there exists \(u_0 \in W\) such that \(\alpha(u_0, Au_0) \geq 1\);
(4) \(A\) is continuous.

Then \(A\) has a fixed point.

**Proof.** It follows from Theorem 3.2 by taking the function \(\beta : W \times W \to [0, +\infty)\) to be \(\alpha\). \(\Box\)

**Corollary 3.6.** Let \((W, b)\) be a complete \(b\)-metric space with \(s \geq 1\), \(A : W \to W\) be a mapping and \(\alpha : W \times W \to [0, +\infty)\) be a function. Suppose that the following conditions hold:

1. there exists \(\zeta \in \mathbb{Z}\) such that if \(u, v \in W\) with \(\alpha(u, v) \geq 1\), then
   \[
   \zeta\left(\beta(Au, Av), M(u, v)\right) \geq 0,
   \]
   where
   \[
   M(u, v) = \max\left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]}, \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)}\right\},
   \]
2. \(A\) is \(\alpha\)-admissible;
3. there exists \(u_0 \in W\) such that \(\alpha(u_0, Au_0) \geq 1\);
4. if \(\{u_n\}\) is a sequence in \(W\) that converges to \(u \in W\) with \(\alpha(u_n, u_{n+1}) \geq 1\) for all \(n \in \mathbb{N}\) and \(u_n \to u \in W\) as \(n \to +\infty\), then there exists a subsequence \(\{u_{n_k}\}\) of \(\{u_n\}\) such that \(\alpha(u_{n_k}, u) \geq 1\) for all \(k\).

Then \(A\) has a fixed point.

**Proof.** It follows from Theorem 3.3 by taking the function \(\beta : W \times W \to [0, +\infty)\) to be \(\alpha\). \(\Box\)

**Corollary 3.7.** Let \((W, b)\) be a complete \(b\)-metric space with \(s \geq 1\), \(A : W \to W\) be a mapping and \(\alpha, \beta : W \to [0, +\infty)\) be two functions. Assume the following conditions hold:

1. \(A\) is \((\alpha, \beta)\)-cyclic;
2. there exists \(u_0 \in W\) such that \(\alpha(u_0) \geq 1\) and \(\beta(u_0) \geq 1\);
3. there exists \(k \in [0, 1)\) such that if \(u, v \in W\) with \(\alpha(u) \beta(v) \geq 1\), then
   \[
   b(Au, Av) \leq k \max\left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]}, \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)}\right\},
   \]
4. \(A\) is continuous.

Then \(A\) has a fixed point \(u^* \in W\).

**Proof.** Suppose there exists \(k \in [0, 1)\) such that condition (2) holds. Define the simulation function \(\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}\) by \(\zeta(q, p) = kp - q\). Note that if \(u, v \in W\) with \(\alpha(u) \beta(v) \geq 1\), then
\[
\zeta\left(\beta(Au, Av), \max\left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]}, \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Av) + b(v, Au)}\right\}\right) \geq 0.
\]

The last inequality together with condition (1) ensure that \(A\) is generalized \((\alpha, \beta, Z)\)-rational contraction. Thus, \(A\) satisfies all conditions of Theorem 3.2 and hence \(A\) has a fixed point. The continuity of \(A\) in Corollary 3.7 can be replaced by a new suitable condition. \(\Box\)
There exists a lower semi-continuous function \( \alpha, \beta : W \rightarrow [0, +\infty) \) be two functions. Assume the following conditions hold:

1. \( A \) is \( (\alpha, \beta) \)-cyclic;
2. there exists \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. there exists \( k < 1 \) such that if \( u, v \in W \) with \( \alpha(u) \beta(v) \geq 1 \), then
   \[
   b(Au, Av) \leq k \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Au) + b(v, Av)} \right\}.
   \]
4. if \( \{u_n\} \) is a sequence in \( W \) converges to \( u \in W \) with \( \alpha(u_n) \geq 1 \) (or \( \beta(u_n) \geq 1 \)) for all \( n \in \mathbb{N} \), then \( \beta(u) \geq 1 \) (or \( \alpha(u) \geq 1 \)) for all \( n \in \mathbb{N} \).

Then \( A \) has a fixed point \( u^* \in W \).

Proof. Follows from Theorem 3.3 by following the same technique of the proof of Corollary 3.7.

Corollary 3.9. Let \( (W, b) \) be a complete \( b \)-metric space with \( s \geq 1 \), \( A : W \rightarrow W \) be a mapping and \( \alpha, \beta : [0, +\infty) \rightarrow \mathbb{R} \) be two functions. Assume the following conditions are satisfied:

1. \( A \) is \( (\alpha, \beta) \)-cyclic;
2. there exists \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. there exists a lower semi-continuous function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \phi(q) > 0 \) for all \( q > 0 \) and \( \phi(0) = 0 \) such that if \( u, v \in W \) with \( \alpha(u) \beta(v) \geq 1 \), then
   \[
   b(Au, Av) \leq \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Au) + b(v, Av)} \right\}.
   \]
4. \( A \) is continuous.

Then \( A \) has a fixed point \( u^* \in W \).

Proof. Follows from Theorem 3.2 by defining \( \zeta : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R} \) i.e. via \( \zeta(q, p) = p - \phi(p) - q \) and following the same technique as in Corollary 3.7.

Corollary 3.10. Let \( (W, b) \) be a complete \( b \)-metric space with \( s \geq 1 \), \( A : W \rightarrow W \) be a mapping and \( \alpha, \beta : [0, +\infty) \rightarrow \mathbb{R} \) be two functions. Assume the following conditions are satisfied:

1. \( A \) is \( (\alpha, \beta) \)-cyclic;
2. there exists \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. there exists a lower semi-continuous function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \phi(q) > 0 \) for all \( q > 0 \) and \( \phi(0) = 0 \) such that if \( u, v \in W \) with \( \alpha(u) \beta(v) \geq 1 \), then
   \[
   b(Au, Av) \leq \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Au) + b(v, Av)} \right\}.
   \]
It follows from Theorem 3.2 by defining the simulation function

\[
A \left( \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]} \right\} \right)
\]

(4) if \{u_n\} is a sequence in W converges to \( u \in W \) with \( \alpha(u_n) \geq 1 \) (or \( \beta(u_n) \geq 1 \)) for all \( n \in \mathbb{N} \), then \( \beta(u) \geq 1 \) (or \( \alpha(u) \geq 1 \)) for all \( n \in \mathbb{N} \).

Then \( A \) has a fixed point \( u^* \in W \).

**Proof.** It follows from Theorem 3.3 by defining the simulation function \( \zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) via \( \zeta(q, p) = p - \phi(p) - q \) and following the same technique as in Corollary 3.7.

**Corollary 3.11.** Let \((W, b)\) be a complete b-metric space with \( s \geq 1 \), \( A : W \to W \) be a mapping and \( \alpha, \beta : [0, +\infty) \to \mathbb{R} \) be two functions. Assume the following conditions are satisfied:

1. \( A \) is \((\alpha, \beta)\)-cyclic;
2. there exists \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. there exists a lower semi-continuous function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \phi(q) < q \) for all \( q > 0 \) and \( \phi(0) = 0 \) such that if \( u, v \in W \) with \( \alpha(u)\beta(v) \geq 1 \), then

\[
b(Au, Av) \leq \phi \left( \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]} \right\} \right)
\]

(4) \( A \) is continuous.

Then \( A \) has a fixed point \( u^* \in W \).

**Proof.** It follows from Theorem 3.2 by defining the simulation function \( \zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) via \( \zeta(q, p) = \phi(p) - q \) and following the same technique as in Corollary 3.7.

**Corollary 3.12.** Let \((W, b)\) be a complete b-metric space with \( s \geq 1 \), \( A : W \to W \) be a mapping and \( \alpha, \beta : [0, +\infty) \to \mathbb{R} \) be two functions. Assume the following conditions are satisfied:

1. \( A \) is \((\alpha, \beta)\)-cyclic;
2. there exists \( u_0 \in W \) such that \( \alpha(u_0) \geq 1 \) and \( \beta(u_0) \geq 1 \);
3. there exists a lower semi-continuous function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \phi(q) < q \) for all \( q > 0 \) and \( \phi(0) = 0 \) such that if \( u, v \in W \) with \( \alpha(u)\beta(v) \geq 1 \), then

\[
b(Au, Av) \leq \phi \left( \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]} \right\} \right)
\]

(4) if \( \{u_n\} \) is a sequence in \( W \) converges to \( u \in W \) with \( \alpha(u_n) \geq 1 \) (or \( \beta(u_n) \geq 1 \)) for all \( n \in \mathbb{N} \), then \( \beta(u) \geq 1 \) (or \( \alpha(u) \geq 1 \)) for all \( n \in \mathbb{N} \).

Then \( A \) has a fixed point \( u^* \in W \).

**Proof.** It follows from Theorem 3.3 by defining the simulation function \( \zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) via \( \zeta(q, p) = \phi(p) - q \) and following the same technique as in Corollary 3.7.
Example 3.13. Let $W = [-1, 1]$ and $s = 2$. Define $b : W \times W \rightarrow \mathbb{R}$ by $b(u, v) = |u - v|$. Also, define the mapping $A : W \rightarrow W$, two functions $\alpha, \beta : W \rightarrow [0, +\infty)$ and the function $\zeta : [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$ as follows:

$$
A u = \begin{cases} 
\frac{u}{2}, & \text{if } u \in [0, 1], \\
\frac{1}{2}, & \text{otherwise},
\end{cases} \quad \alpha(u) = \begin{cases} 
\frac{u + 3}{2}, & \text{if } u \in [0, 1], \\
0, & \text{otherwise},
\end{cases} \quad \zeta(q, p) = \frac{p}{p + 1} - q.
$$

Then, we have the following:

1. $(W, b)$ is a complete $b$-metric space;
2. $\zeta$ is a simulation function;
3. there exists $u_0 \in W$ such that $\alpha(u_0) \geq 1$ and $\beta(u_0) \geq 1$;
4. $A$ is continuous;
5. $A$ is cyclic $(\alpha, \beta)$-admissible mapping;
6. for $u, v \in W$ with $\alpha(u)\beta(v) \geq 1$, we have $\zeta(b(Au, Av), M(u, v)) \geq 0$, where

$$
M(u, v) = \max \left\{ b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + s [b(u, Au) + b(v, Av)]}, \frac{b(u, Au) b(u, Av) + b(v, Av) b(v, Au)}{1 + b(u, Au) + b(v, Av)} \right\}.
$$

Proof. The proof of (1), (2), (3), (4) are clear. To prove (5), let $u \in W$. If $\alpha(u) \geq 1$, then $u \in [0, 1]$. So,

$$
\beta(Au) = \beta \left(\frac{u}{2}\right) = \frac{u + 10}{6} \geq 1.
$$

If $\beta(u) \geq 1$, then $u \in [0, 1]$. So,

$$
\alpha(Au) = \alpha \left(\frac{u}{2}\right) = \frac{u + 6}{4} \geq 1.
$$

So, $A$ is cyclic $(\alpha, \beta)$-admissible. To prove (6), let $u, v \in W$ with $\alpha(u)\beta(u) \geq 1$. Then, $u, v \in [0, 1]$, therefore, we have

$$
\zeta(b(Au, Av), M(u, v)) = \frac{M(u, v)}{1 + M(u, v)} - b(Au, Av)
\geq \frac{b(u, v)}{1 + b(u, v)} - \frac{1}{2} |u - \frac{1}{2} v| \geq \frac{b(u, v)}{1 + b(u, v)} - \frac{1}{2} |u - \frac{1}{2} v| = \frac{|u - v|}{1 + |u - v|} - \frac{1}{2} |u - \frac{1}{2} v| = \frac{|u - v| - |u - v|^2}{2(1 + |u - v|)} \geq 0.
$$

So, $A$ is a generalized $(\alpha, \beta, Z)$-contraction. Example 3.13 satisfies all the conditions of Theorem 3.2. So, $A$ has fixed point. Here $0$ is the fixed point of $A$.

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