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Fixed points of generalized rational (α, β, Z)-contraction mappings under simulation functions

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Abstract

In this paper, we combine the (α, β)-admissible mappings and simulation function in order to obtain the generalized form of rational (α, β, Z)-contraction mapping. Further this concept is used in the setting of b-metric space in order to obtain some fixed point theorems. Suitable examples are also established to verify the validity of the results obtained.

Keywords: Fixed points, generalized rational (α, β, Z)-contraction mapping, (α, β)-admissible mappings, simulation function, b-metric space.

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1. Introduction

Samet et al. [22] introduced α-ψ-contractive type mapping and α-admissible mappings. The concept is further generalized by Karapinar and Samet[15] by introducing generalized α-ψ-contractive type mapping. The concept of cyclic (α, β)-admissible mapping was introduced by Alizadeh et al. [2] by generalizing the concept of α-admissible mapping [22]. Khojastesh et al. [17] introduced simulation function and the notion of Z-contraction with respect to simulation function to generalize Banach contraction principle. The concept of Khojastesh et al. [17] is further modified by Argoubi et al. [5]. In this paper, we introduce cyclic (α, β)-admissible mapping in simulation function to result a generalized rational (α, β, Z)-contraction. Here, we use b-metric space [7, 10] in order to obtain fixed point theorems for generalized rational (α, β, Z)-contraction mappings. For more results in rational type contractions and Z-contractions we refer to the papers in [1, 3, 4, 6, 8, 9, 11–14, 16, 18–21, 23, 24] and references therein.

2. Preliminaries

Bakhtin [7] introduced the concept of b-metric space as follows.

Definition 2.1 ([7, 10])**.** Let W be a non empty set and the mapping $b : W \times W \to [0, +\infty)$ satisfies:

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1. $b(u,v) = 0$ if and only if $u = v$ for all $u, v \in W$;
2. $b(u,v) = b(v,u)$ for all $u, v \in W$;
3. there exists a real number $s \geqslant 1$ such that $b(u,v) \leqslant s[b(u,w) + b(w,v)]$ for all $u, v, w \in W$.

Then b is called a b-metric on W and (W, b) is called a b-metric space (in short bMS) with coefficient s.

Definition 2.2 ([7, 10])**.** Let (W, b) be a b-metric space, $\{u_n\}$ be a sequence in W and $x \in W$. Then

1. the sequence $\{u_n\}$ is said to be convergent in (W, b) and converges to u, if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $b(u_n, u) < \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to+\infty} u_n = u$ or $u_n \to u$ as $n \to +\infty$;
2. the sequence $\{u_n\}$ is said to be a Cauchy sequence in (W, b) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $b(u_n, u_m) < \epsilon$ for all $n, m > n_0$ or equivalently, if $\lim_{n,m\to+\infty} b(u_n, u_m) = 0$;
3. (W, b) is said to be a complete b-metric space if every Cauchy sequence in W converges to some $u \in W$.

It can be noted that a b-metric space need not be a continuous function.

Definition 2.3 ([2])**.** Let W be a nonempty set, f be a self-mapping on W and $\alpha, \beta : W \to [0, +\infty)$ be two mappings. We say that f is a cyclic (α, β)-admissible mapping if $u \in W$ with

$$\alpha(u) \geqslant 1 \text{ implies } \beta(fu) \geqslant 1$$

and $u \in W$ with

$$\beta(u) \geqslant 1 \text{ implies } \alpha(fu) \geqslant 1.$$

In 2015, Khojasteh et al. [17] introduced the class of simulation functions. Further, Argoubi et al. [5] modified the definition of simulation functions and defined as follows.

Definition 2.4 ([5])**.** A simulation function is a function $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ that satisfies the following conditions:

(1) $\zeta(q, p) < p - q$ for all $p, q > 0$;
(2) if $\{q_n\}$ and $\{p_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n\to+\infty} q_n = \lim_{n\to+\infty} p_n = l \in (0, +\infty)$, then

$$\lim_{n\to+\infty} \sup \zeta(q_n, p_n) < 0.$$

It is to be noted that any simulation function in the sense of Khojasteh et al. [17] is also a simulation function in the sense of Argoubi et al. [5]. The following function is a simulation function in the sense of Argoubi et al. [5]

Example 2.5 ([5])**.** *Define a function* $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ *by*

$$\zeta(q, p) = \begin{cases} 1, & \text{if } (p, q) = (0, 0), \\ \lambda p - q, & \text{otherwise}, \end{cases}$$

where $\lambda \in (0, 1)$*. Then* ζ *is a simulation function in the sense of Argoubi et al. [5].*

Theorem 2.6 ([17])**.** *Let* (W, b) *be a metric space and* $T : W \to W$ *be a* Z*-contraction with respect to a simulation function* ζ*; that is*

$$\zeta\Big(b(Tu, Tv), b(u, v)\Big) \geqslant 0, \text{ for all } u, v \in W.$$

Then T *has a unique fixed point.*

It is worth mentioning that the Banach contraction is an example of Z-contraction by defining $\zeta : [0,+\infty)\times[0,+\infty)\to\mathbb{R}$ via

$$\zeta(q,p)=\gamma p-q, \text{ for all } p,q\in[0,\infty),$$

where $\gamma\in[0,1)$.

Following lemma was proved by Qawagnesh [20] which is valid for complete b-metric space also.

Lemma 2.7 ([20]). *Let $A : W\to W$ be a cyclic (α,β)-admissible mapping. Assume that there exist $u_0,u_1\in W$ such that*

$$\alpha(u_0)\geqslant 1 \textit{ implies } \beta(u_1)\geqslant 1$$

and

$$\beta(u_0)\geqslant 1 \textit{ implies } \alpha(u_1)\geqslant 1.$$

Define a sequence $\{u_n\}$ by $u_{n+1}=Au_n$. Then

$$\alpha(u_n)\geqslant 1 \textit{ implies } \beta(u_m)\geqslant 1$$

and

$$\beta(u_n)\geqslant 1 \textit{ implies } \alpha(u_m)\geqslant 1$$

for all $m,n\in\mathbb{N}$ with $n<m$.

3. Main result

We start our result with the following definitions.

Definition 3.1. Let (W,b) be a complete b-metric space with $s\geqslant 1$, $A : W\to W$ be a mapping and $\alpha,\beta:\mathbb{R}\to[0,+\infty)$ be two functions. Then A is said to be a generalized (α,β,Z)-rational contraction mapping if A satisfies the following conditions:

(1) A is cyclic (α,β)-admissible;
(2) there exists simulation function $\zeta\in Z$ such that

$$\alpha(u)\beta(v)\geqslant 1 \text{ implies } \zeta(b(Au,Av),M(u,v))\geqslant 0$$

holds for all $u,v\in W$, where

$$M(u,v)=\max\left\{b(u,v),b(u,Au),b(v,Av),\frac{b(u,Au)\ b(u,Av)+b(v,Av)\ b(v,Au)}{1+s[b(u,Au)+b(v,Av)]},\right.$$
$$\left.\frac{b(u,Au)\ b(u,Av)+b(v,Av)\ b(v,Au)}{1+b(u,Av)+b(v,Au)}\right\}$$

Theorem 3.2. *Let (W,b) be a complete b-metric space with $s\geqslant 1$, $A : W\to W$ be a mapping and $\alpha,\beta : W\to[0,+\infty)$ be two functions. Suppose the following conditions hold:*

(1) *A is a generalized (α,β,Z)-rational contraction mapping;*
(2) *there exists an element $u_0\in W$ such that $\alpha(u_0)\geqslant 1$ and $\beta(u_0)\geqslant 1$;*
(3) *A is continuous.*

Then, A has a fixed point $u^*\in W$ such that $Au^*=u^*$.

Proof. Assume that there exists $u_0\in W$ such that $\alpha(u_0)\geqslant 1$. We divide our proof into the following steps.

Step 1: Define a sequence $\{u_n\}$ in W such that $u_{n+1}=Au_n$ for all $n\in\mathbb{N}\cup\{0\}$. If $u_n=u_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$, then A has a fixed point and proof is finished. Hence, we assume that $u_n\neq u_{n+1}$ for some $n\in\mathbb{N}\cup\{0\}$; that is, $b(u_n,u_{n+1})\neq 0$ for all $n\in\mathbb{N}\cup\{0\}$. Since A is a cyclic (α,β)-admissible mapping, we have

$$\alpha(u_0) \geqslant 1 \text{ implies } \beta(u_1) = \beta(Au_0) \geqslant 1 \text{ implies } \alpha(u_2) = \alpha(Au_1) \geqslant 1$$

and

$$\beta(u_0) \geqslant 1 \text{ implies } \alpha(u_1) = \alpha(Au_0) \geqslant 1 \text{ implies } \beta(u_2) = \beta(Au_1) \geqslant 1,$$

then by continuing the above process, we have

$$\alpha(u_n) \geqslant 1 \text{ and } \beta(u_n) \geqslant 1 \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Thus, $\alpha(u_n)\beta(u_{n+1}) \geqslant 1$, for all $n \in \mathbb{N} \cup \{0\}$. Therefore, we get

$$\zeta(b(Au_n, Au_{n+1}), M(u_n, u_{n+1})) \geqslant 0$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(u_n, u_{n+1}) &= \max\Big\{b(u_n, u_{n+1}), b(u_n, Au_n), b(u_{n+1}, Au_{n+1}), \\ &\quad \frac{b(u_n, Au_n)\ b(u_n, Au_{n+1}) + b(u_{n+1}, Au_{n+1})\ b(u_{n+1}, Au_n)}{1 + s[b(u_n, Au_n) + b(u_{n+1}, Au_{n+1})]}, \\ &\quad \frac{b(u_n, Au_n)\ b(u_n, Au_{n+1}) + b(u_{n+1}, Au_{n+1})\ b(u_{n+1}, Au_n)}{1 + b(u_n, Au_{n+1}) + b(u_{n+1}, Au_n)}\Big\} \\ &= \max\Big\{b(u_n, u_{n+1}), b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}), \\ &\quad \frac{b(u_n, u_{n+1})\ b(u_n, u_{n+2}) + b(u_{n+1}, u_{n+2})\ b(u_{n+1}, u_{n+1})}{1 + s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]}, \\ &\quad \frac{b(u_n, u_{n+1})\ b(u_n, u_{n+2}) + b(u_{n+1}, u_{n+2})\ b(u_{n+1}, u_{n+1})}{1 + b(u_n, u_{n+2}) + b(u_{n+1}, u_{n+1})}\Big\} \\ &= \max\Big\{b(u_n, u_{n+1}), b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2}), \\ &\quad \frac{b(u_n, u_{n+1})\ s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]}{1 + s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]}, \\ &\quad \frac{b(u_n, u_{n+1})\ s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]}{1 + s[b(u_n, u_{n+1}) + b(u_{n+1}, u_{n+2})]}\Big\} \\ &= \max\Big\{b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})\Big\}. \end{aligned}$$

It follows that

$$\zeta\Big(b(u_{n+1}, u_{n+2}), \max\{b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})\}\Big) \geqslant 0.$$

Condition (1) of Definition 2.4 implies that

$$\begin{aligned} 0 &\leqslant \zeta\Big(b(u_{n+1}, u_{n+2}), \max\{b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})\}\Big) \\ &< \max\{b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})\} - b(u_{n+1}, u_{n+2}). \end{aligned}$$

Thus, we conclude that

$$b(u_{n+1}, u_{n+2}) < \max\{b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})\}$$

for all $n \geqslant 1$. The last inequality implies that

$$b(u_{n+1}, u_{n+2}) < b(u_n, u_{n+1}), \quad \text{for all } n \geqslant 1.$$

It follows that the sequence $\{b(u_n, u_{n+1})\}$ is non increasing. Therefore, there exists $r \geqslant 0$ such that

$$\lim_{n\to+\infty} b(u_n, u_{n+1}) = r.$$

Note that if $r \neq 0$; that is $r > 0$, then by condition (2) of Definition 2.4, we have

$$0 \leqslant \lim_{n\to+\infty} \sup \zeta\Big(b(u_n, u_{n+1}), b(u_{n+1}, u_{n+2})\Big) < 0,$$

which is a contradiction. This implies that $r = 0$, that is

$$\lim_{n\to+\infty} b(u_n, u_{n+1}) = 0. \tag{3.1}$$

Step 2: Now, we prove that $\{u_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{u_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and two subsequences $\{u_{m(k)}\}$ and $\{u_{n(k)}\}$ of $\{u_n\}$ with $m(k) > n(k) > k$ and $m(k)$ is the smallest index in $\mathbb{N}$ such that

$$b(u_{n(k)}, u_{m(k)}) \geqslant \epsilon,$$

so,

$$b(u_{n(k)}, u_{m(k)-1}) < \epsilon.$$

Triangular inequality implies that

$$\epsilon \leqslant b(u_{n(k)}, u_{m(k)}) \leqslant s\ [b(u_{n(k)}, u_{m(k)-1}) + b(u_{m(k)-1}, u_{m(k)})] < s\ [\epsilon + b(u_{m(k)-1}, u_{m(k)})].$$

Taking $k \to +\infty$ in the above inequality and using (6), we get

$$\epsilon \leqslant \lim_{k\to+\infty} b(u_{n(k)}, u_{m(k)}) < s\epsilon \tag{3.2}$$

From triangular inequality, we have

$$b(u_{n(k)}, u_{m(k)}) \leqslant s\ [b(u_{n(k)}, u_{n(k)+1}) + b(u_{n(k)+1}, u_{m(k)})] \tag{3.3}$$

and

$$b(u_{n(k)+1}, u_{m(k)}) \leqslant s\ [b(u_{n(k)+1}, u_{n(k)}) + b(u_{n(k)}, u_{m(k)})]. \tag{3.4}$$

By taking the limit as $k \to +\infty$ in (3.3) and applying (3.1) and (3.2), we get

$$\epsilon \leqslant \lim_{k\to+\infty} \sup b(u_{n(k)}, u_{m(k)}) \leqslant s \lim_{k\to+\infty} \sup b(u_{n(k)+1}, u_{m(k)}).$$

Again, by taking the upper limit as $k \to +\infty$ in (3.4), we get

$$\begin{gathered}\lim_{k\to+\infty} \sup b(u_{n(k)+1}, u_{m(k)}) \leqslant s\ \Big(\lim_{k\to+\infty} \sup b(u_{n(k)}, u_{m(k)})\Big) \leqslant s.s\epsilon = s^2\epsilon, \\ \frac{\epsilon}{s} \leqslant \lim_{k\to+\infty} \sup b(u_{n(k)+1}, u_{m(k)}) \leqslant s^2\epsilon.\end{gathered} \tag{3.5}$$

Similarly,

$$\frac{\epsilon}{s} \leqslant \lim_{k\to+\infty} \sup b(u_{n(k)}, u_{m(k)+1}) \leqslant s^2\epsilon. \tag{3.6}$$

By triangular inequality, we have

$$b(u_{n(k)+1}, u_{m(k)}) \leqslant s\ \Big[b(u_{n(k)+1}, u_{m(k)+1}) + b(u_{m(k)+1}, u_{m(k)})\Big]. \tag{3.7}$$

On letting $k \to +\infty$ in (3.7) and using inequalities (3.1) and (3.5), we get

$$\frac{\epsilon}{s^2} \leqslant \lim_{k\to+\infty} \sup b(u_{n(k)+1}, u_{m(k)+1}). \tag{3.8}$$

Following the above process, we find

$$\lim_{k\to+\infty} \sup b(u_{n(k)+1}, u_{m(k)+1}) \leqslant s^3\epsilon. \tag{3.9}$$

From (3.8) and (3.9), we get

$$\frac{\epsilon}{s^2} \leqslant \lim_{k\to+\infty} \sup b(u_{n(k)+1}, u_{m(k)+1}) \leqslant s^3\epsilon.$$

Since $\alpha(u_0) > 1$ and $\beta(u_0) > 1$ by Lemma 2.7, we conclude that

$$\alpha(u_{n(k)})\beta(u_{m(k)}) \geqslant 1.$$

Since A is generalized (α, β, Z)-rational contraction, we have

$$\zeta\Big(b(Au_{n(k)}, Au_{m(k)}), M(u_{n(k)}, u_{m(k)})\Big) \geqslant 0$$

for all $u, v \in W$, where

$$\begin{aligned} M(u_{n(k)}, u_{m(k)}) &= \max\Big\{b(u_{m(k)}, u_{n(k)}), b(u_{n(k)}, Au_{n(k)}), b(u_{m(k)}, Au_{m(k)}), \\ &\qquad \frac{b(u_{n(k)}, Au_{n(k)})\ b(u_{n(k)}, Au_{m(k)}) + b(u_{m(k)}, Au_{m(k)})\ b(u_{m(k)}, Au_{n(k)})}{1 + s\ [b(u_{n(k)}, Au_{n(k)}) + b(u_{m(k)}, Au_{m(k)})]}, \\ &\qquad \frac{b(u_{n(k)}, Au_{n(k)})\ b(u_{n(k)}, Au_{m(k)}) + b(u_{m(k)}, Au_{m(k)})\ b(u_{m(k)}, Au_{n(k)})}{1 + b(u_{n(k)}, Au_{m(k)}) + b(u_{m(k)}, Au_{n(k)})}\Big\} \\ &= \max\Big\{b(u_{m(k)}, u_{n(k)}), b(u_{n(k)}, u_{n(k)+1}), (u_{m(k)}, u_{m(k)+1}), \\ &\qquad \frac{b(u_{n(k)}, u_{n(k)+1})\ b(u_{n(k)}, u_{m(k)+1}) + b(u_{m(k)}, u_{m(k)+1})\ b(u_{m(k)}, u_{n(k)+1})}{1 + s\ [b(u_{n(k)}, u_{n(k)+1}) + b(u_{m(k)}, u_{m(k)+1})]}, \\ &\qquad \frac{b(u_{n(k)}, u_{n(k)+1})\ b(u_{n(k)}, u_{m(k)+1}) + b(u_{m(k)}, u_{m(k)+1})\ b(u_{m(k)}, u_{n(k)+1})}{1 + b(u_{n(k)}, u_{m(k)+1}) + b(u_{m(k)}, u_{n(k)+1})}\Big\}. \end{aligned}$$

Taking the limit as $k \to +\infty$ and using (3.1), (3.2), (3.5), and (3.6), we get

$$\epsilon = \max\{\epsilon, 0, 0, 0, 0\} \leqslant \lim_{k\to+\infty} \sup M(u_{n(k)}, u_{m(k)}) \leqslant \max\{s\epsilon, 0, 0, 0, 0\} = s\epsilon.$$

Note that condition (2) of Definition 2.4, implies that

$$0 \leqslant \lim\sup \zeta\Big(b(Au_{n(k)}, Au_{m(k)}), M(u_{n(k)}, u_{m(k)})\Big) < 0,$$

which is a contradiction. Thus $\{u_n\}$ is a Cauchy sequence.

Step 3: Finally in this step we prove that A has a fixed point. Since $\{u_n\}$ is a Cauchy sequence in the complete b-metric space W, there exists $u^* \in W$ such that $u_n \to u^*$. The continuity of A implies that $Au_{2n} \to Au^*$. Since $u_{2n+1} = Au_{2n}$ and $u_{2n+1} \to u^*$, by uniqueness of limit, we have

$$Au^* = u^*.$$

So, u^* is a fixed point of A. This concludes the proof. □

Note that the continuity of the mapping A in Theorem 3.2 can be dropped if we replace condition (3) by a suitable one as in the following result.

Theorem 3.3. *Let (W, b) be a complete b-metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha, \beta : W \to [0, +\infty)$ be two functions. Suppose the following conditions hold:*

(1) *A is a generalized (α, β, Z)-rational contraction mapping;*
(2) *there exists an element $u_0 \in W$ such that $\alpha(u_0) \geqslant 1$ and $\beta(u_0) \geqslant 1$;*
(3) *if $\{u_n\}$ is a sequence in W converges to $u \in W$ with $\alpha(u_n) \geqslant 1$ (or $\beta(u_n) \geqslant 1$) for all $n \in \mathbb{N}$, then $\beta(u) \geqslant 1$ (or $\alpha(u) \geqslant 1$) for all $n \in \mathbb{N}$.*

Then, A has a fixed point.

Proof. Following the same steps as in the proof of Theorem 3.2 we construct a sequence $\{u_n\}$ in W by $u_{n+1} = Au_n$ for all $n \in \mathbb{N}$ such that $u_n \to u^* \in W$, $\alpha(u_n) \geqslant 1$, $\beta(u_n) \geqslant 1$ for all $n \in \mathbb{N}$. By condition (3), we have $\alpha(u^*) \geqslant 1$ and $\beta(u^*) \geqslant 1$. So, $\alpha(u^*)\beta(u^*) \geqslant 1$.

Claim: $Au^* = u^*$. Suppose not; that is $Au^* \neq u^*$. Therefore $b(Au^*, u^*) \neq 0$ and

$$\lim_{n\to\infty} b(u_{n+1}, Au^*) \neq 0. \tag{3.10}$$

Since A is a generalized (α, β, Z)-rational contraction mapping, we have

$$\zeta\Big(b(Au_n, Au^*), M(u_n, u^*)\Big) = \zeta\Big(b(u_{n+1}, Au^*), M(u_n, u^*)\Big) \geqslant 0 \tag{3.11}$$

for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} M(u_n, u^*) = \max\Big\{&b(u_n, u^*), b(u_n, Au_n), b(u^*, Au^*), \\ &\frac{b(u_n, Au_n)\ b(u_n, Au^*) + b(u^*, Au^*)\ b(u^*, Au_n)}{1 + s\ [b(u_n, Au_n) + b(u^*, Au^*)]}, \\ &\frac{b(u_n, Au_n)\ b(u_n, Au^*) + b(u^*, Au^*)\ b(u^*, Au_n)}{1 + b(u_n, Au^*) + b(u^*, Au_n)}\Big\} \\ = \max\Big\{&b(u_n, u^*), b(u_n, u_{n+1}), b(u^*, Au^*), \\ &\frac{b(u_n, u_{n+1})\ b(u_n, Au^*) + b(u^*, Au^*)\ b(u^*, u_{n+1})}{1 + s\ [b(u_n, u_{n+1}) + b(u^*, Au^*)]}, \\ &\frac{b(u_n, u_{n+1})\ b(u_n, Au^*) + b(u^*, Au^*)\ b(u^*, u_{n+1})}{1 + b(u_n, Au^*) + b(u^*, u_{n+1})}\Big\}. \end{aligned} \tag{3.12}$$

Letting $n \to +\infty$ in (3.12), we obtain

$$\lim_{n\to+\infty} M(u_n, u^*) = b(u^*, Au^*) \neq 0. \tag{3.13}$$

By using (3.10), (3.11), and (3.13), then condition (2) of Definition 2.4 implies that

$$0 \leqslant \lim_{n\to+\infty} \sup \zeta\Big(b(u_{n+1}, Au^*), M(u_n, u^*)\Big) < 0,$$

which is a contradiction. So $Au^* = u^*$. Thus, u^* is a fixed point of A. This concludes the proof. □

Now, we introduce an example to show that if A satisfies all hypothesis of Theorems 3.2 or 3.3, then fixed point of A is not necessarily to be unique.

Example 3.4. *Let $W = [0,1]$ and $s = 2$. Define $b : W \times W \to \mathbb{R}$ by $b(u,v) = |u - v|$. Also define the mapping $A : W \to W$ by $Au = u^2$. Define the function $\alpha, \beta : W \to \mathbb{R}$ by*

$$\alpha(u) = \beta(u) = \begin{cases} 1, & \text{if } u = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\zeta : (q,p) = \frac{p}{p+1} - q.$$

Then, we have the following:

(1) *A is continuous;*
(2) *there exists $u_0 \in W$ such that $\alpha(u_0) \geqslant 1$ and $\beta(u_0) \geqslant 1$;*
(3) *A is cyclic (α, β)-admissible mapping;*
(4) *for any $u, v \in W$ with $\alpha(u)\beta(v) \geqslant 1$, then*

$$\zeta\Big(b(Au, Av), M(u,v)\Big) \geqslant 0,$$

where

$$M(u,v) = \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\};$$

(5) *if $\{u_n\}$ is a sequence in W converges to $u \in W$ with $\alpha(u_n) \geqslant 1$ for all $n \in \mathbb{N}$, then $\beta(u) \geqslant 1$.*

Proof. Proof of (1) and (2) are clear. To prove (3), let $u \in W$. If $\alpha(u) \geqslant 1$ then $u = 0$. So, $A(u) = A(0) = 0$ and $\beta(Au) = \beta(0) = 1 \geqslant 1$. If $\beta(u) \geqslant 1$, then $u = 0$. So, $A(u) = A(0) = 0$ and $\alpha(Au) = \alpha(0) = 1 \geqslant 1$. So, A is cyclic (α, β)-admissible mapping. To prove (4), let $u, v \in W$ with $\alpha(u)\beta(u) \geqslant 1$. Then $u = v = 0$. So, $A(u) = A(v) = 0$. Therefore, we have

$$\begin{aligned} M(u,v) &= \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\} \\ &= \max\{b(0,0), b(0,0), b(0,0), b(0,0), b(0,0)\} = 0. \end{aligned}$$

So,

$$\zeta\Big(b(Au, Av), M(u,v)\Big) = \zeta(0,0) = \frac{0}{1+0} - 0 = 0 \geqslant 0.$$

To prove (5), let $\{u_n\}$ is a sequence in W such that $u_n \to u$, with $\alpha(u_n) \geqslant 1$. Then $u_n = 0$ for all $n \in \mathbb{N}$. So $u = 0$. Hence $\beta(u) = \beta(0) = 1 \geqslant 1$. Note that A satisfies all the conditions of Theorem 3.2 and 3.3. Hence, $0, 1$ are fixed points of A. So, the fixed points of A is not unique. □

Next, we gave some corollaries.

Corollary 3.5. *Let (W, b) be a complete b-metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha : W \times W \to [0, +\infty)$ be a function. Suppose that the following conditions hold:*

(1) *there exists $\zeta \in Z$ such that if $u, v \in W$ with $\alpha(u,v) \geqslant 1$, then $\zeta\Big(b(Au, Av), M(u,v)\Big) \geqslant 0$, where*

$$M(u,v) = \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\};$$

(2) *A is α-admissible;*
(3) *there exists $u_0 \in W$ such that $\alpha(u_0, Au_0) \geqslant 1$;*
(4) *A is continuous.*

Then A has a fixed point.

Proof. It follows from Theorem 3.2 by taking the function $\beta : W \times W \to [0, +\infty)$ to be α. □

Corollary 3.6. *Let (W, b) be a complete b-metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha : W \times W \to [0, +\infty)$ be a function. Suppose that the following conditions hold:*

(1) *there exists $\zeta \in Z$ such that if $u, v \in W$ with $\alpha(u, v) \geqslant 1$, then $\zeta\Big(b(Au, Av), M(u, v)\Big) \geqslant 0$, where*

$$M(u,v) = \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\};$$

(2) *A is α-admissible;*
(3) *there exists $u_0 \in W$ such that $\alpha(u_0, Au_0) \geqslant 1$;*
(4) *if $\{u_n\}$ is a sequence in W that converges to $u \in W$ with $\alpha(u_n, u_{n+1}) \geqslant 1$ for all $n \in \mathbb{N}$ and $u_n \to u \in W$ as $n \to +\infty$, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\alpha(u_{n_k}, u) \geqslant 1$ for all k.*

Then A has a fixed point.

Proof. It follows from Theorem 3.3 by taking the function $\beta : W \times W \to [0, +\infty)$ to be α. □

Corollary 3.7. *Let (W, b) be a complete b-metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha, \beta : W \to [0, +\infty)$ be two functions. Assume the following conditions hold:*

(1) *A is (α, β)-cyclic;*
(2) *there exists $u_0 \in W$ such that $\alpha(u_0) \geqslant 1$ and $\beta(u_0) \geqslant 1$;*
(3) *there exists $k \in [0, 1)$ such that if $u, v \in W$ with $\alpha(u)\beta(v) \geqslant 1$, then*

$$b(Au, Av) \leqslant k \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\};$$

(4) *A is continuous.*

Then A has a fixed point $u^* \in W$.

Proof. Suppose there exists $k \in [0, 1)$ such that condition (2) holds. Define the simulation function $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ by $\zeta(q, p) = kp - q$. Note that if $u, v \in W$ with $\alpha(u)\beta(v) \geqslant 1$, then

$$\zeta\Big(b(Au,Av), \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\}\Big) \geqslant 0.$$

The last inequality together with condition (1) ensure that A is generalized (α, β, Z)-rational contraction. Thus, A satisfies all conditions of Theorem 3.2 and hence A has a fixed point. The continuity of A in Corollary 3.7 can be replaced by a new suitable condition. □

Corollary 3.8. *Let (W, b) be a complete b-metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha, \beta : W \to [0, +\infty)$ be two functions. Assume the following conditions hold:*

(1) *A is (α, β)-cyclic;*
(2) *there exists $u_0 \in W$ such that $\alpha(u_0) \geqslant 1$ and $\beta(u_0) \geqslant 1$;*
(3) *there exists $k \in [0, 1)$ such that if $u, v \in W$ with $\alpha(u)\beta(v) \geqslant 1$, then*

$$b(Au, Av) \leqslant k \max\Big\{b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + s\ [b(u, Au) + b(v, Av)]}, \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + b(u, Av) + b(v, Au)}\Big\};$$

(4) *if $\{u_n\}$ is a sequence in W converges to $u \in W$ with $\alpha(u_n) \geqslant 1$ (or $\beta(u_n) \geqslant 1$) for all $n \in \mathbb{N}$, then $\beta(u) \geqslant 1$ (or $\alpha(u) \geqslant 1$) for all $n \in \mathbb{N}$.*

Then A has a fixed point $u^* \in W$.

Proof. Follows from Theorem 3.3 by following the same technique of the proof of Corollary 3.7. □

Corollary 3.9. *Let (W, b) be a complete b metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha, \beta : [0, +\infty) \to \mathbb{R}$ be two functions. Assume the following conditions are satisfied:*

(1) *A is (α, β)-cyclic;*
(2) *there exists $u_0 \in W$ such that $\alpha(u_0) \geqslant 1$ and $\beta(u_0) \geqslant 1$;*
(3) *there exists a lower semi-continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(q) > 0$ for all $q > 0$ and $\phi(0) = 0$ such that if $u, v \in W$ with $\alpha(u)\beta(v) \geqslant 1$, then*

$$\begin{aligned} b(Au, Av) \leqslant{}& \max\Big\{b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + s\ [b(u, Au) + b(v, Av)]}, \\ & \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + b(u, Av) + b(v, Au)}\Big\} \\ & - \phi\Big(\max\Big\{b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + s\ [b(u, Au) + b(v, Av)]}, \\ & \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + b(u, Av) + b(v, Au)}\Big\}\Big); \end{aligned}$$

(4) *A is continuous.*

Then A has a fixed point $u^* \in W$.

Proof. Follows from Theorem 3.2 by defining $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ i.e. via $\zeta(q, p) = p - \phi(p) - q$ and following the same technique as in Corollary 3.7. □

Corollary 3.10. *Let (W, b) be a complete b-metric space with $s \geqslant 1$, $A : W \to W$ be a mapping and $\alpha, \beta : [0, +\infty) \to \mathbb{R}$ be two functions. Assume the following conditions are satisfied:*

(1) *A is (α, β)-cyclic;*
(2) *there exists $u_0 \in W$ such that $\alpha(u_0) \geqslant 1$ and $\beta(u_0) \geqslant 1$;*
(3) *there exists a lower semi-continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(q) > 0$ for all $q > 0$ and $\phi(0) = 0$ such that if $u, v \in W$ with $\alpha(u)\beta(v) \geqslant 1$, then*

$$\begin{aligned} b(Au, Av) \leqslant{}& \max\Big\{b(u, v), b(u, Au), b(v, Av), \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + s\ [b(u, Au) + b(v, Av)]}, \\ & \frac{b(u, Au)\ b(u, Av) + b(v, Av)\ b(v, Au)}{1 + b(u, Av) + b(v, Au)}\Big\} \end{aligned}$$

$$- \phi\Big(\max\Big\{ b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)} \Big\}\Big);$$

(4) *if* $\{u_n\}$ *is a sequence in* W *converges to* $u \in W$ *with* $\alpha(u_n) \geqslant 1$ *(or* $\beta(u_n) \geqslant 1$*) for all* $n \in \mathbb{N}$*, then* $\beta(u) \geqslant 1$ *(or* $\alpha(u) \geqslant 1$*) for all* $n \in \mathbb{N}$*.*

Then A *has a fixed point* $u^* \in W$*.*

Proof. It follows from Theorem 3.3 by defining $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ via $\zeta(q,p) = p - \phi(p) - q$ and following the same technique as in Corollary 3.7. □

Corollary 3.11. *Let* (W, b) *be a complete* b*-metric space with* $s \geqslant 1$*,* $A : W \to W$ *be a mapping and* $\alpha, \beta : [0, +\infty) \to \mathbb{R}$ *be two functions. Assume the following conditions are satisfied:*

(1) A *is* (α, β)*-cyclic;*
(2) *there exists* $u_0 \in W$ *such that* $\alpha(u_0) \geqslant 1$ *and* $\beta(u_0) \geqslant 1$*;*
(3) *there exists a lower semi-continuous function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(q) < q$ *for all* $q > 0$ *and* $\phi(0) = 0$ *such that if* $u, v \in W$ *with* $\alpha(u)\beta(v) \geqslant 1$*, then*

$$b(Au, Av) \leqslant \phi \max\Big\{ b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)} \Big\};$$

(4) A *is continuous.*

Then A *has a fixed point* $u^* \in W$*.*

Proof. It follows from Theorem 3.2 by defining the simulation function $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ via $\zeta(q,p) = \phi(p) - q$ and following the same technique as in Corollary 3.7. □

Corollary 3.12. *Let* (W, b) *be a complete* b*-metric space with* $s \geqslant 1$*,* $A : W \to W$ *be a mapping and* $\alpha, \beta : [0, +\infty) \to \mathbb{R}$ *be two functions. Assume the following conditions are satisfied:*

(1) A *is* (α, β)*-cyclic;*
(2) *there exists* $u_0 \in W$ *such that* $\alpha(u_0) \geqslant 1$ *and* $\beta(u_0) \geqslant 1$*;*
(3) *there exists a lower semi-continuous function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(q) < q$ *for all* $q > 0$ *and* $\phi(0) = 0$ *such that if* $u, v \in W$ *with* $\alpha(u)\beta(v) \geqslant 1$*, then*

$$b(Au, Av) \leqslant \phi \max\Big\{ b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)} \Big\};$$

(4) *if* $\{u_n\}$ *is a sequence in* W *converges to* $u \in W$ *with* $\alpha(u_n) \geqslant 1$ *(or* $\beta(u_n) \geqslant 1$*) for all* $n \in \mathbb{N}$*, then* $\beta(u) \geqslant 1$ *(or* $\alpha(u) \geqslant 1$*) for all* $n \in \mathbb{N}$*.*

Then A *has a fixed point* $u^* \in W$*.*

Proof. It follows from Theorem 3.3 by defining the simulation function $\zeta : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ via $\zeta(q,p) = \phi(p) - q$ and following the same technique as in Corollary 3.7. □

Example 3.13. *Let* $W = [-1,1]$ *and* $s = 2$. *Define* $b : W \times W \to \mathbb{R}$ *by* $b(u,v) = |u - v|$. *Also, define the mapping* $A : W \to W$, *two functions* $\alpha, \beta : W \to [0,+\infty)$ *and the function* $\zeta : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ *as follows:*

$$Au = \begin{cases} \frac{u}{2}, & \text{if } u \in [0,1], \\ \frac{1}{2}, & \text{otherwise}, \end{cases} \qquad \alpha(u) = \begin{cases} \frac{u+3}{2}, & \text{if } u \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\beta(u) = \begin{cases} \frac{u+5}{3}, & \text{if } u \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \qquad \zeta(q,p) = \frac{p}{p+1} - q.$$

Then, we have the following:

(1) (W, b) *is a complete* b*-metric space;*
(2) ζ *is a simulation function;*
(3) *there exists* $u_0 \in W$ *such that* $\alpha(u_0) \geqslant 1$ *and* $\beta(u_0) \geqslant 1$;
(4) A *is continuous;*
(5) A *is cyclic* (α, β)*-admissible mapping;*
(6) *for* $u, v \in W$ *with* $\alpha(u)\beta(v) \geqslant 1$, *we have* $\zeta\Big(b(Au, Av), M(u,v)\Big) \geqslant 0$, *where*

$$M(u,v) = \max\Big\{b(u,v), b(u,Au), b(v,Av), \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + s\ [b(u,Au) + b(v,Av)]}, \frac{b(u,Au)\ b(u,Av) + b(v,Av)\ b(v,Au)}{1 + b(u,Av) + b(v,Au)}\Big\};$$

Proof. The proof of (1), (2), (3), (4) are clear. To prove (5), let $u \in W$. If $\alpha(u) \geqslant 1$, then $u \in [0,1]$. So,

$$\beta(Au) = \beta(\frac{u}{2}) = \frac{u+10}{6} \geqslant 1.$$

If $\beta(u) \geqslant 1$, then $u \in [0,1]$. So,

$$\alpha(Au) = \alpha(\frac{u}{2}) = \frac{u+6}{4} \geqslant 1.$$

So, A is cyclic (α, β)-admissible. To prove (6), let $u, v \in W$ with $\alpha(u)\beta(u) \geqslant 1$. Then $u, v \in [0,1]$, therefore, we have

$$\begin{aligned} \zeta\Big(b(Au,Av), M(u,v)\Big) &= \frac{M(u,v)}{1+M(u,v)} - b(Au,Av) \\ &= \frac{M(u,v)}{1+M(u,v)} - |\frac{1}{2}u - \frac{1}{2}v| \\ &\geqslant \frac{b(u,v)}{1+b(u,v)} - |\frac{1}{2}u - \frac{1}{2}v| \\ &= \frac{|u-v|}{1+|u-v|} - |\frac{1}{2}u - \frac{1}{2}v| \\ &= \frac{|u-v| - |u-v|^2}{2(1+|u-v|)} \geqslant 0. \end{aligned}$$

So, A is a generalized (α, β, Z)-contraction. Example 3.13 satisfies all the conditions of Theorem 3.2. So, A has fixed point. Here 0 is the fixed point of A. □

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