Journal of Mathematics and Computer Science

Generalizing Halbeisen's and Hungerbühler's optimal bounds for the length of Collatz cycles to $3n + c$ cycles

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Abstract

The next element in the $3n + 1$ sequence is defined to be $(3n + 1)/2$ if n is odd or $n/2$ otherwise. The Collatz conjecture states that no matter what initial value of n is chosen, the sequence always reaches 1 (where it goes into the repeating sequence $(1, 2, 1, 2, 1, 2, \ldots)$). The only known Collatz cycle is $(1, 2)$. Let c be an odd integer not divisible by 3. Similar cycles exist for the more general $3n + c$ sequence. The $3n + c$ cycles are commonly grouped according to their length and number of odd elements. The smallest odd element in one of these cycles is greater than the smallest odd elements of the other cycles in the group. A *parity vector* corresponding to a cycle consists of 0's for the even elements and 1's for the odd elements. A parity vector generated by the ceiling function is used to determine this smallest odd element. Similarly, the largest odd element in one of these cycles is less than the largest odd elements of the other cycles in the group. A parity vector generated by the floor function is used to determine this largest odd element. This smallest odd element and largest odd element appear to be in the same cycle. This means that the parity vector generated by the floor function can be rotated to match the parity vector generated by the ceiling function. Two linear congruences are involved in this rotation. The natural numbers generated by one of these congruences appear to be uniformly distributed (after sorting).

1. Introduction

Halbeisen and Hungerbühler [2] found new techniques which allow a refined analysis of rational (and hence integer) Collatz cycles. In particular, they prove optimal estimates for the length of a cycle having positive elements in terms of its minimum. Their main results are reproduced here since they are directly applicable to $3n + c$ cycles. Most lemmas are omitted.

2. Halbeisen and Hungerbühler's Results for Collatz Cycles

For $x \in \mathbb{R}$ let $g_0(x) = x/2$ and $g_1(x) = (3x+1)/2$. Let $\mathbb{Q}[(2)]$ denote the local ring of fractions of $\mathbb{Z}$ at the prime ideal(2), i.e. the domain of all rational numbers having (written in least terms) an odd denominator. A number $p/q \in \mathbb{Q}[(2)]$ with odd q is considered even or odd according to the parity of the numerator p. Then the Collatz sequence generated by $x_0 \in \mathbb{Q}[(2)]$ is defined by $x_n = g_0(x_{n-1})$ if x_{n-1} is even or $g_1(x_{n-1})$ if x_{n-1} is odd for $n \in \mathbb{N}$. Let $S_{l,n}$ denote the set of all 0-1 sequences of length l containing exactly n ones, $S_l = \cup_{n=0}^l S_{l,n}$ and $S = \cup_{l \in \mathbb{N}} S_l$. With every $s = (s_1, \ldots, s_l) \in S_l$ we associate the affine function $\phi_s : \mathbb{R} \to \mathbb{R}$, $\phi_s = g_{s_l} \circ \ldots \circ g_{s_2} \circ g_{s_1}$. A sequence $(x_0, \ldots, x_l)$ of real numbers x_i is called a *pseudo-cycle of length* l if there exists $s = (s_1, \ldots, s_l) \in S_l$ such that (1) $\phi_s(x_0) = x_0 \in \mathbb{Q}[(2)]$ and (2) $g_{s_{i+1}}(x_i) = x_{i+1}$ for $i = 0, \ldots, l-1$.

Notice that if $p/q \in \mathbb{Q}$ with $2^r | q$ then $2^r | \tilde{q}$ where $\tilde{q}$ denotes the denominator of $g_i(p/q)$ ($i = 0, 1$). Hence every element of a pseudo-cycle is in $\mathbb{Q}[(2)]$. Thus, if p/q and $g_i(p/q) = \tilde{p}/\tilde{q}$ are consecutive elements of a pseudo-cycle, then $i = 0$ if p is even (since else $\tilde{p}/\tilde{q} \notin \mathbb{Q}[(2)]$) or $i = 1$ if p is odd (since else $\tilde{p}/\tilde{q} \notin \mathbb{Q}[(2)]$). The conclusion of this observation is given by the following lemma 2.1.

Lemma 2.1. *The set of pseudo-cycles coincides with the set of Collatz cycles in* $\mathbb{Q}[(2)]$. *Cycles consist of either positive or negative elements.*

The function $\varphi : S \to \mathbb{N}$ will be defined recursively by $\varphi(\varnothing) = 0$, $\varphi(s0) = \varphi(s)$, and $\varphi(s1) = 3\varphi(s) + 2^{l(s)}$ where s denotes an arbitrary element of S and $l(s)$ the length of s. The function φ is computed explicitly by $\varphi(s) = \sum_{j=1}^{l(s)} s_j 3^{s_{j+1} + \ldots + s_{l(s)}} 2^{j-1}$.

A consequence of the above definition is the decomposition formula $\varphi(s\bar{s}) = 3^{n(\bar{s})} \varphi(s) + 2^{l(s)} \varphi(\bar{s})$. Here $s\bar{s}$ is the concatenation of s, $\bar{s} \in S$, and $n(s)$ denotes the number of 1's in the sequence s. The next lemma 2.2 shows how φ is used to explicitly compute the function ϕ_s.

Lemma 2.2 ([3, Lagarias]). *For arbitrary* $s \in S$, $\phi_s(x) = \frac{3^{n(s)} x + \varphi(s)}{2^{l(s)}}$ *and hence for every* $s \in S$ *there exists a unique* $x_0 \in \mathbb{Q}[(2)]$ *which generates a Collatz cycle in* $\mathbb{Q}[(2)]$ *of length* $l(s)$ *and which coincides with the pseudo-cycle generated by* s. *The value* x_0 *is given by* $x_0 = \frac{\varphi(s)}{2^{l(s)} - 3^{n(s)}}$.

Proof. The proof is by induction with respect to $l(s)$. (1) $l(s) = 1$: This is checked from the definition. (2) $l(s) > 1$: If $s = \bar{s}0$ then $\phi_{\bar{s}0}(x) = \frac{\phi_{\bar{s}}(x)}{2} = \frac{3^{n(\bar{s})} x + \varphi(\bar{s})}{2 \cdot 2^{l(\bar{s})}} = \frac{3^{n(s)} x + \varphi(s)}{2^{l(s)}}$. The case $s = \bar{s}1$ is analogous. $\square$

For $s \in S_l$ let $\sigma(s)$ denote the orbit of s in S_l generated by the left-shift permutation $\lambda_l : (s_1, \ldots, s_l) \to (s_2, \ldots, s_l, s_1)$, i.e. $\sigma(s) := \{\lambda_l^k(s) : k = 1, \ldots, l\}$. Furthermore, let $M_{l,n}$ denote $\max_{s \in S_{l,n}} \{\min_{t \in \sigma(s)} \varphi(t)\}$.

Now suppose the Collatz conjecture is verified for all initial values $x_0 \leqslant m$. If one can then show that $\forall n, l < L : \frac{M_{l,n}}{2^l - 3^n} \leqslant m$, it follows that the length of a Collatz cycle in $\mathbb{N}$ which does not contain 1 is at least L.

Let $\tilde{s}$ denote the sequence for which φ attains the value $M_{l,n}$.

Lemma 2.3. *Let* $n \leqslant l$ *be natural numbers. Let* $\tilde{s}_i := \lceil in/l \rceil - \lceil (i-1)n/l \rceil$ *(for* $1 \leqslant i \leqslant l$*). Then* $\varphi(\tilde{s}) = \min_{t \in \sigma(\tilde{s})} \{\varphi(t)\} = M_{l,n}$.

Corollary 2.4. *For every* l *and* $n \leqslant l$ *we have* $M_{l,n} = \sum_{j=1}^l (\lceil jn/l \rceil - \lceil (j-1)n/l \rceil) 2^{j-1} 3^{n - \lceil jn/l \rceil}$.

3. The Minimum Element in a $3n + c$ Cycle

Setting c to $2^{l(s)} - 3^{n(s)}$ in Lemma 2.2 gives integer $3n + c$ cycles. A staircase for $\tilde{s}$ where $l = 27$ and $n = 17$ along with a staircase representing the partial sums of $\lfloor in/l \rfloor - \lfloor (i-1)n/l \rfloor$ is given in Figure 1.
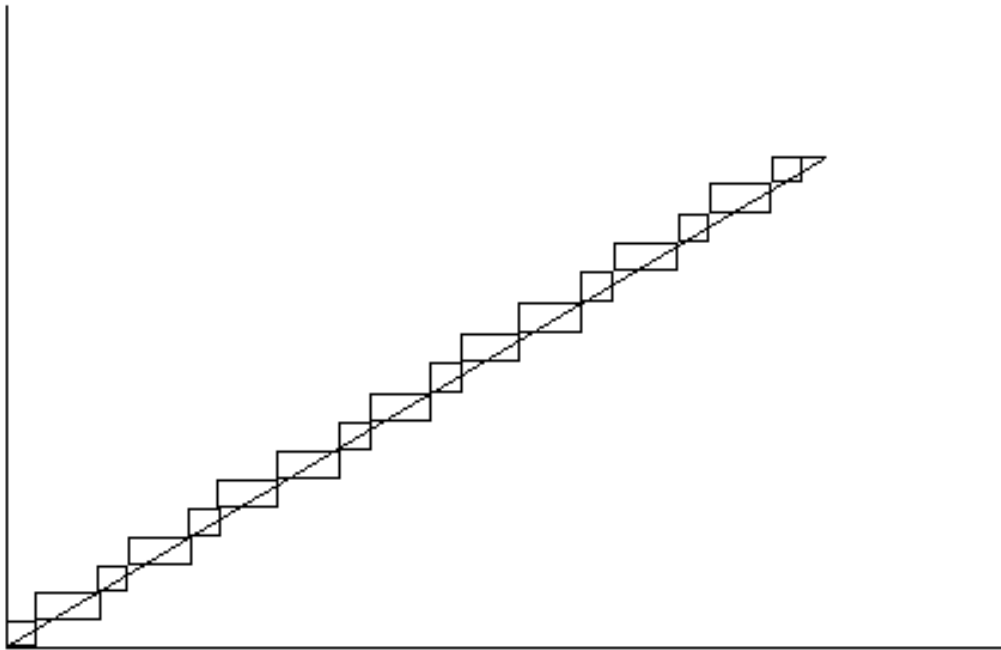
Figure 1: staircase for $\tilde{s}$ and the partial sums of $\lfloor in/l \rfloor - \lfloor (i-1)n/l \rfloor$

The staircase using the floor function can be viewed as being an upside-down staircase where Halbeisen and Hungerbühler's logic can be used to find a lower bound of the maximum odd element in a $3n + c$ cycle. Let $t_j = \lceil jn/l \rceil - \lceil (j-1)n/l \rceil$, $j = 1, \ldots, l$. This parity vector is an element of $S_{l,n}$. Let r denote $\gcd(l, n)$. The parity vector $\lfloor jn/l \rfloor - \lfloor (j-1)n/l \rfloor$, $j = 1, \ldots, l$, consists of r identical sub-vectors. Similarly, the parity vector t_j consists of r identical sub-vectors and each of these sub-vectors is the same as the corresponding sub-vector of $\lfloor jn/l \rfloor - \lfloor (j-1)n/l \rfloor$, $j = 1, \ldots, l$, except for the first and last elements. First suppose that l and n are relatively prime. When the parity vector $\lfloor jn/l \rfloor - \lfloor (j-1)n/l \rfloor$, $j = 1, \ldots, l$, is right-rotated by one position (corresponding to a multiplication by 2), it matches t_j except for the first two elements of each sub-vector. The first mismatch corresponds to a loss of 3^{n-1} and the second mismatch corresponds to a gain of $2 \cdot 3^{n-1}$. In general, the loss is $\sum_{i=0}^{r-1} 2^{i(l/r)} 3^{n-1-i(n/r)}$. Let $N_{l,n}$ denote $2M_{l,n} - \sum_{i=0}^{r-1} 2^{i(l/r)} 3^{n-1-i(n/r)}$. A primitive $3n + c$ cycle doesn't have any common divisors of its elements. A generalization of Halbeisen and Hungerbühler's result is given by Corollary 3.1:

Corollary 3.1. *If $c = 2^l - 3^n$, $M_{l,n}$ is greater than or equal to the minimum odd elements in the $3n + c$ cycles corresponding to $s \in S_{l,n}$ (not necessarily primitive) and $N_{l,n}$ is less than or equal to the maximum odd elements in the cycles.*

The elements of the $3n + c$ cycles are $\varphi(t)_{t \in \sigma(s)}$ where $s \in S_{l,n}$. From the definition of $N_{l,n}$, it is not apparent that it is in a cycle, but it appears to be in the same cycle as $M_{l,n}$.

For example, for $(l, n) = (6, 4)$, the parity vector for $M_{l,n}$ is $(1, 1, 0, 1, 1, 0)$, $c = -17$, and the odd elements of the cycle containing $M_{l,n}$ and $N_{l,n}$ are $(85, 119, 85, 119)$. There is only one more element in $S_{l,n}$ in this case and its odd elements are $(65, 89, 125, 179)$. The smallest odd element in the cycle containing $M_{l,n}$ (equal to 85) is greater than 65 and the largest odd element in the cycle containing $N_{l,n}$ (equal to 119) is less than 179. The cycle with odd elements of $(85, 119)$ is not primitive and reduces to a cycle with odd elements of $(5, 7)$ for $c = -1$. When l and n are not relatively prime and $c = 2^l - 3^n$, the cycles generated from $M_{l,n}$ are not primitive. This is due to the duplicated sub-vectors in the parity vector forming a geometric progression. This geometric progression is the same as in the expansion of $(a^x - b^x)/(a - b)$. Reducing the cycle generated from $M_{6,4}$ effectively divides $2^6 - 3^4$ by $(2^6 - 3^4)/(2^3 - 3^2)$.

When $l = 11$ and $n = 7$, $M_{l,n} = 3767$, $N_{l,m} = 6805$, and $2^l - 3^n = -139$. The quotient $3767/139$ (approximately equal to 27) is greater than the minimum odd element in the $c = -1$ cycle of $(34, 17, 25, 37, 55$

, $82, 41, 61, 91, 136, 68$) and $6805/139$ (approximately equal to 49) is less than the maximum odd element. For the $c = -1$ cycle of $(5, 7, 10)$, $M_{3,2} = 5$ and $N_{3,2} = 7$ (-1 equals $2^3 - 3^2$). For $c = -17$, the cycles are $(85, 119, 170, 85, 119, 170)$, $(103, 146, 73, 101, 143, 206)$, and $(65, 89, 125, 179, 260, 130)$ ($2^6 - 3^4 = -17$). The first cycle contains $M_{6,4}$ (equal to 85) and $N_{6,4}$ (equal to 119). As expected, 85 is greater than 73 and 65, and 119 is less than 143 and 179. The cycle $(85, 119, 170)$ is not primitive and reduces to the above $c = -1$ cycle. For the $c = 1$ cycle of $(2, 1)$, $2^2 - 3^1 = 1$ and $M_{2,1} = N_{2,1} = 1$. There can be no other such $c = 1$ cycles due to the Catalan conjecture (proved by Mihăilescu [5]). This theorem states that the only natural number solutions of $x^a - y^b$ are $x = 3$, $a = 2$, $y = 2$, and $b = 3$. This leaves the possibility of $3n + c$ cycles where $s \in S_{l,n}$ that are not primitive and reduce to $c = 1$ cycles.

All the parity vectors in S are used up by the $3n + c$ cycles where $c = 2^l - 3^n$. Two $3n + c$ cycles with different c values can't have the same parity vector. For example, the elements of a $c = 5$ cycle are $(19, 31, 49, 76, 38)$ and a $c = 7$ sequence having the same parity vector is $(65, 101, 155, 236, 118, \ldots)$. The ratios of the odd elements are 0.2923, 0.3069, and 0.3161 and would have to keep increasing to match the iterations of the $3n + 5$ cycle. So the unreduced $3n + c$ cycles where $c = 2^l - 3^n$ account for all possible primitive $3n + c$ cycles.

4. Statistical Results

Let d denote the rotation of the floor function parity vector required to match the ceiling function parity vector (measured in the clockwise direction). This quantity appears to satisfy the congruence $n \cdot d \equiv -1 \bmod l$ when $\gcd(l, n) = 1$. A similar congruence is $d(n - x) - (l - d)x \equiv -1 \bmod l$. This congruence was derived using the staircases and can be solved given the d value so that the values of $(n - x, x)$ are not specific to properties of $3n + c$ cycles.

For a real number x, let $[x]$ denote the integral part of x and $\{x\}$ the fractional part. The sequence $\omega = (x_n)$, $n = 1, 2, 3, \ldots$ of real numbers is said to be *uniformly distributed modulo* 1 (abbreviated u.d. mod 1) if for every pair a, b of real numbers with $0 \leqslant a < b \leqslant 1$ we have $\lim_{N \to \infty} \frac{A([a,b);N;\omega)}{N} = b - a$. The formal definition of u.d. mod 1 was given by Weyl [6, 7]. Let $\Delta : 0 = z_0 < z_1 < z_2 < \ldots$ be a subdivision of the interval $[0, \infty)$ with $\lim_{k \to \infty} z_k = \infty$. For $z_{k-1} \leqslant x < z_k$ put $[x]_\Delta = z_{k-1}$ and $\{x\}_\Delta = \frac{x - z_{k-1}}{z_k - z_{k-1}}$ so that $0 \leqslant \{x\}_\Delta < 1$. The sequence of (x_n), $n = 1, 2, 3, \ldots$ of non-negative real numbers is said to be *uniformly distributed modulo* Δ (abbreviated u.d. mod Δ) if the sequence $(\{x_n\}_\Delta)$, $n = 1, 2, 3, \ldots$ is u.d. mod 1. The notion of u.d. mod Δ was introduced by LeVeque [4].

If f is a function having a Riemann integral in the interval $[a, b]$, then its integral is the limit of Riemann sums taken by sampling the function f in a set of points chosen from a fine partition of the interval. This is then a criterion for determining if a sequence is uniformly distributed. A sequence of real numbers is uniformly distributed mod 1 if and only if for every Riemann-integrable function f on $[0, 1]$ one has $\lim_{N \to \infty} 1/N \sum_{n \leqslant N} f(\{x_n\}) = \int_0^1 f(x) dx$. In the following, evidence that $n - x$ and x are u.d. mod Δ is presented using this criterion and Weyl's criterion [6, 7].

Weyl's criterion is that the sequence (γ_n), $n = 1, 2, 3, \ldots$, is u.d. mod 1 if and only if

$$\lim_{N \to \infty} 1/N \sum_{n=1}^{N} e^{2\pi i m \gamma_n} = 0$$

for every integer $m \neq 0$. In the following, the z increments in the subdivision are set to $\sqrt{2}$ to avoid any aliasing with the integer $n - x$ and x values. A plot of the resulting sequence generated from the sorted $n - x$ values for $n = 1, \ldots, l - 1$ and $l = 997$ is given in Figure 2.
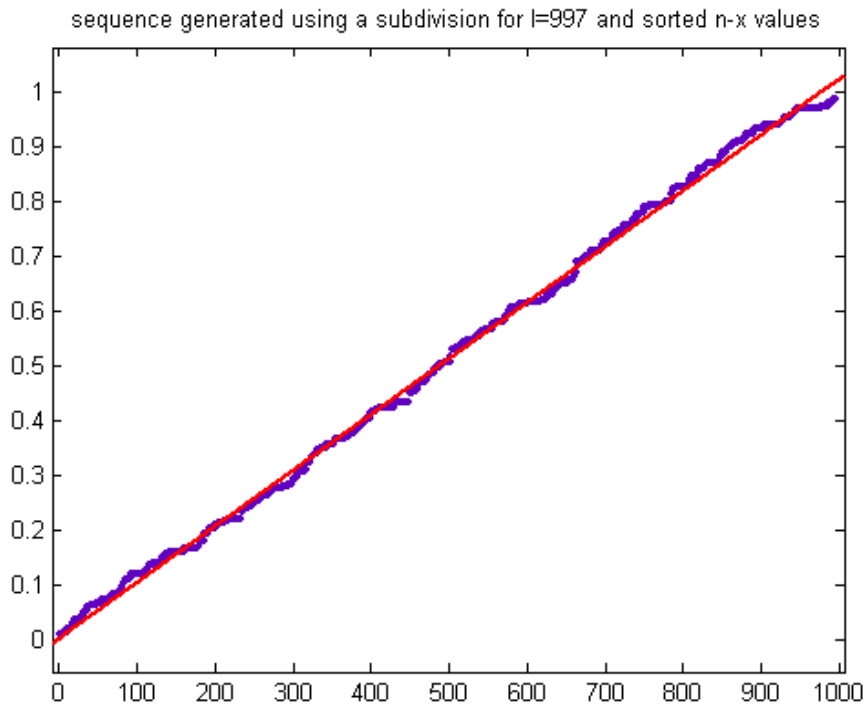
Figure 2: sequence generated from the sorted $n - x$ values for $n = 1, \ldots, l - 1$ and $l = 997$

The red curve corresponds to a linear least-squares fit of the data. In the following, (γ_n) is set to such sequences and the moduli of the complex-valued results are computed. The moduli for $l = 1999$, $n - x$, and $m = 1$ are given in Figure 3.
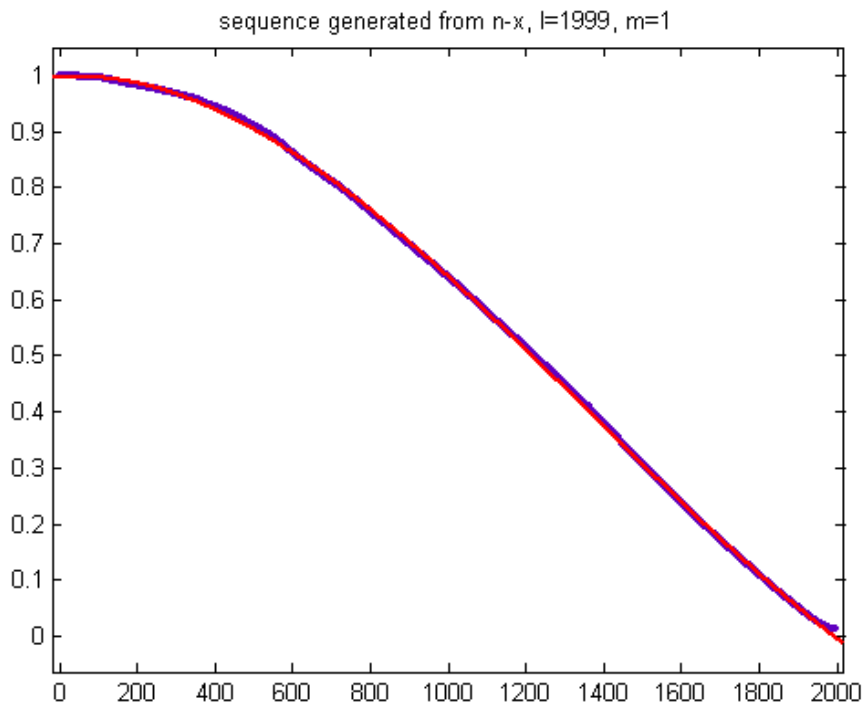

Figure 3: Moduli for $l = 1999$, $n - x$, and $m = 1$

A cubic least-squares fit of the curve (where R-squared=0.9999) is included (the red curve). The moduli

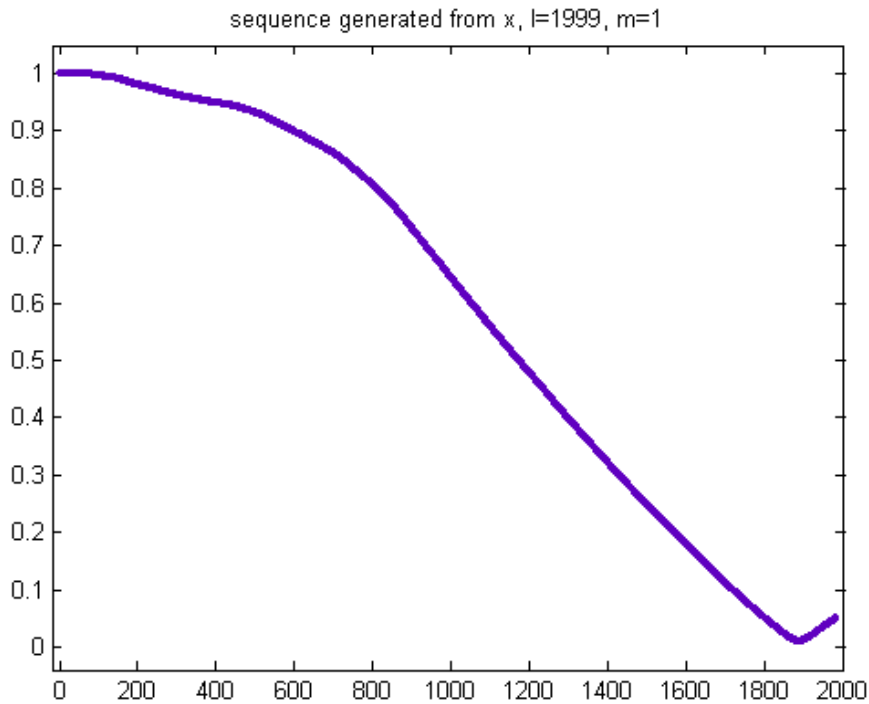for $l = 1999$, x, and $m = 1$ (excluding 16 values of zero in the input sequence) are given in Figure 4.

Figure 4: Moduli for $l = 1999$, x, and $m = 1$

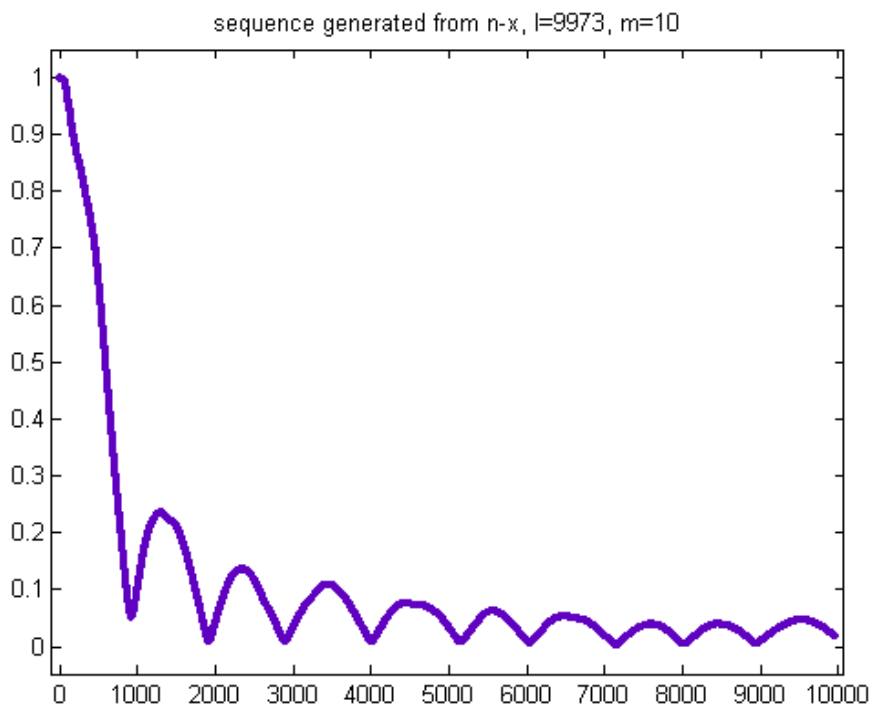The moduli for $l = 9973$, $n - x$, and $m = 10$ are given in Figure 5.

Figure 5: Moduli for $l = 9973$, $n - x$, and $m = 10$

The moduli for $l = 9973$, x, and $m = 10$ (excluding 18 values of zero in the input sequence) are given in Figure 6.
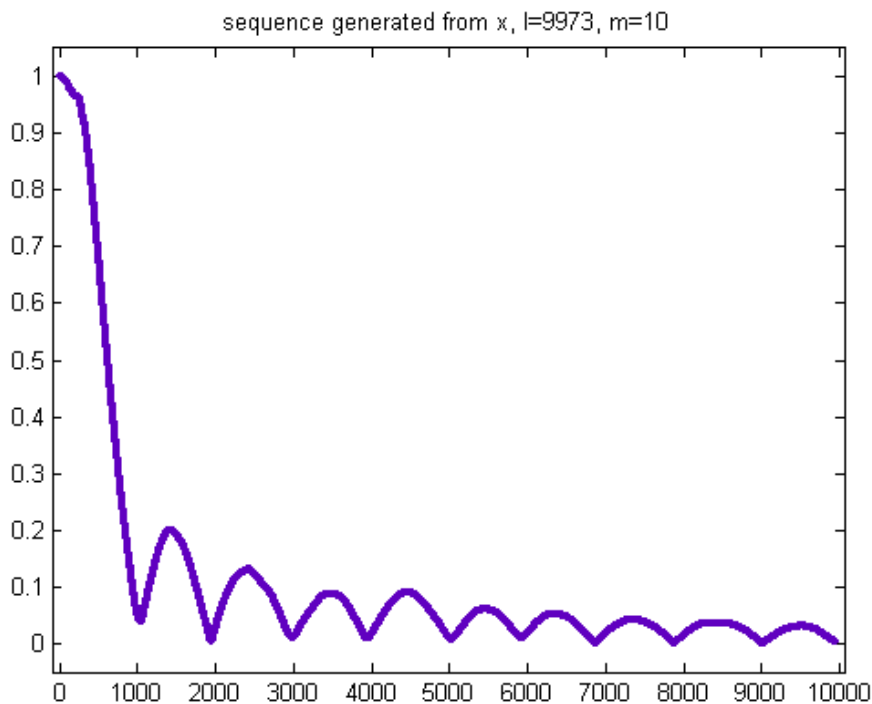

Figure 6: Moduli for $l = 9973$, x, and $m = 10$

In general, there are $m - 1$ oscillations in such curves.

The functions $f(x)$ to be considered are x, x^2, x^3, x^4, $\sqrt{x}$, $\sqrt{\sqrt{x}}$, $\log(x)$, e^x, $\sin(x)$, $\cos(x)$, $\tan(x)$, and $\frac{1}{a^2 + x^2}$. The values of $\int_0^1 f(x) dx$ are $1/2$, $1/3$, $1/4$, $1/5$, $2/3$, $4/5$, -1, 2.72, 0.84, 0.54, 1.56, and $\frac{1}{a} \tan^{-1} \frac{x}{a}$ (equal to 0.23 for $a = 2$ and 0.11 for $a = 3$) respectively. For $l = 997$ and the sequence generated from x, the results are 0.49, 0.31, 0.22, 0.17, 0.66, 0.80, -1.00, 2.69, 0.84, 0.56, 1.45, 0.23 (for $a = 2$), and 0.11 (for $a = 3$) respectively. For $l = 9973$ and the sequence generated from $n - x$, the results are 0.50, 0.34, 0.25, 0.20, 0.67, 0.80, -0.97, 2.74, 0.84, 0.54, 1.55, 0.23 (for $a = 2$), and 0.11 (for $a = 3$) respectively.

The trigonometric functions require a fixed amount to be added to the sequence values (apparently to change the phase). The exponential function also requires a fixed amount to be added to the sequence values - the same as for the cosine function. Apparently, this is due to Euler's formula $e^{ix} = \cos(x) + i \cdot \sin(x)$. Denote the amounts for sine and cosine by j and k respectively. These values satisfy the equation $j^2 + k^2 = \cos(1)$, similar to the formula $\sin(x)^2 + \cos(x)^2 = 1$. They also satisfy the equation $j/k = \sqrt{\tan(1)}$, similar to the formula $\sin(x)/\cos(x) = \tan(x)$. The amount required for the sine function is $\sin^{-1}(\cos(1))$. The amount required for the cosine function can be determined by using the formula $j^2 + k^2 = \cos(1)$. The amount required for the tangent function is $1/e$ ($e = \tan(1)/j$).

5. Conclusion

Halbeisen and Hungerbühler's function $M_{l,n}$ is applied to $3n + c$ cycles and the corresponding function $N_{l,n}$ is introduced. The congruences $n \cdot d \equiv -1 \bmod l$ and $d(n - x) - (l - d)x \equiv -1 \bmod l$, $\gcd(l, n) = 1$, are involved in rotating the floor function parity vector to match the ceiling function parity vector. All primitive $3n + c$ cycles are shown to be generated from cycles (possibly unreduced) where $c = 2^l - 3^n$. Future research would involve determining the properties of the purportedly uniformly distributed sequence mentioned above. Cox [1] investigated convolving the zeta function zeros with the Möbius function. This method is applicable to any uniformly distributed sequence.

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