J. Math. Computer Sci., 24 (2022), 256–272

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Degree of approximation for bivariate extension of blending type q-Durrmeyer operators based on Pólya distribution

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Abstract

In this paper we introduce a bivariate of q-Durrmeyer variant of generalized Bernstein operators by using Pólya distribution. The convergence rate of these operators is examined by means of the Lipschitz class and the modulus of continuity. Furthermore, we obtain a Voronovskaja type symptotic formula, error estimation in terms of the partial modulus of continuity and Peetre's K-functional.

Keywords: Durrmeyer operators, K-functional, modulus of continuity, Pólya distribution.

2020 MSC: 41A36, 40C15, 40G10, 46E30.

1. Introduction

In the last decade, the application of q-calculus in the approximation theory, mechanics and physics has been one of the main research areas. In approximation theory applications of q-calculus was initiated by Lupaş [12], who first introduced the q-Bernstein polynomials. Another q-based Bernstein operator was introduced in 1997 by Phillips in [15]. Thereafter, many authors studied new classes of q-generalized operators and established many interesting properties. Some important results in this direction are mention in the papers [1, 2, 4, 5, 7, 9, 11, 13, 14, 16, 17]. Details regarding the definitions and notions of q-calculus can be found at [3].

In 2008, Gupta [8] introduced the q-Durrmeyer operators $D_{n,q}(f)$ and studied some approximation properties of such operators. For $f \in C[0,1], x \in [0,1], n \in \mathbb{N}, 0 < q < 1$, operators introduced by Gupta [8] are defined as follow:

$$D_{n_q}(f;x) = [n+1]_q \sum_{k=0}^{n} q^{-k} p_{n,k}(q,x) \int_0^1 p_{n,k}(q,qt) f(t) d_q t, \tag{1.1}$$

where

$$p_{n,k}(q,x) = \binom{n}{k} x^k \prod_{s=0}^{n-k-1} (1-q^s).$$

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doi: 10.22436/jmcs.024.03.07

Received: 2021-02-01 *Revised:* 2021-02-17 *Accepted:* 2021-02-19

It is proved that for $q = 1$, the operators defined by (1.1) reduce to the well known Bernstein-Durrmeyer operators.

Stancu [17], by means of the probabilistic methods, introduced and investigated a linear operators $S_n^{(\alpha)}$ which maps the space $C([0,1])$ onto itself and it's defined by

$$S_n^{(\alpha)}(h;x) = \sum_{k=0}^{n} w_{n,k}(x,\eta)h\left(\frac{k}{n}\right),$$

where

$$w_{n,k}(z_1,\eta) = \begin{bmatrix} n \\ k \end{bmatrix} \frac{\prod_{i=0}^{k-1}(z_1+\eta i)\prod_{j=0}^{n-k-1}(1-z_1+j\eta)}{\prod_{s=0}^{n-1}(z_1+\eta s)},$$

with η being a parameter which may depend only on the natural number n. If η is non-negative, then these operators preserve the positivity of the function h.

Nowak [14], introduced the q-variant of the Lupaş operators based on Pólya distribution. For $h \in C(I)$ with $I = [0,1]$, $\eta > 0$, and $0 < q < 1$, the operators introduced by Nowak are defined by

$$L^{\eta}_{m_1,q}(h;z_1) = \sum_{k_1=0}^{m_1} p^{\eta}_{m_1,k_1}(z_1)h\left(\frac{[k_1]_q}{[m_1]_q}\right), \tag{1.2}$$

where $p^{\eta}_{m_1,k_1}(z_1)$ is the Pólya distribution with density function given by

$$p^{\eta}_{m_1,k_1}(z_1) = \begin{bmatrix} m_1 \\ k_1 \end{bmatrix}_q \frac{\prod_{i=0}^{k_1-1}(z_1+\eta[i]_q)\prod_{j=0}^{m_1-k_1-1}(1-z_1q^j+\eta[j]_q)}{\prod_{s=0}^{m_1-1}(z_1+\eta[s]_q)}.$$

As special cases, this class contains the following three well-known sequences.

(i) For $\eta = 0$, operators defined by (1.2) reduce to the well known q-Bernstein operators, introduced by [14].

(ii) For $q = 1$, operators $L^{\eta}_{n,q}(h;z_1)$ reduce to Bernstein-Stancu operators introduced by [4].

(iii) For $\eta = 0$ and $q = 1$, operators defined by (1.2) reduce to classical Bernstein operators.

In [10], the q-Durrmayer type modification of operator (1.2) was introduced and studied by Gupta et al. The q-Durrmeyer type operator based on Pólya distribution was defined by

$$\mathfrak{D}^{\frac{1}{[m_1]_{q_1}}}_{m_1,q_{m_1}}(h;z_1) = [m_1+1]_{q_1}\sum_{k_1=0}^{m_1} q_1^{-k_1}p^{\frac{1}{[m_1]_{q_1}}}_{m_1,k_1}(z_1)\int_0^1 p_{m_1,k_1}(q_1t_1)h(t_1)d_{q_1}t_1, \tag{1.3}$$

where

$$p^{\frac{1}{[m_1]_{q_1}}}_{m_1,k_1}(u) = \begin{bmatrix} m_1 \\ k_1 \end{bmatrix}_{q_1} \frac{\prod_{j=0}^{k_1-1}\left(u+\frac{[j]_{q_1}}{[m_1]_{q_1}}\right)\prod_{j=0}^{m_1-k_1-1}\left(1-u+\frac{[j]_{q_1}}{[m_1]_{q_1}}\right)}{\prod_{j=0}^{n_1-1}\left(u+\frac{[j]_{q_1}}{[m_1]_{q_1}}\right)},$$

and

$$p_{m_1,k_1}(q_1t) = \begin{bmatrix} m_1 \\ k_1 \end{bmatrix}_{q_1} q_1^{k_1}t^{k_1}\prod_{j=0}^{m_1-k_1-1}\left(1-q_1^{j+1}t\right).$$

In order to prove Lemmas 1.2 and 2.1, we will establish the following statements for operators defined by (1.3).

Lemma 1.1. *For each positive numbers* $C_{s,r}(q_1) > 0, s = \{0, 1, \ldots, r\}, r \in \mathbb{N}$ *the following statements holds*

$$\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_r; z_1) = \frac{[m_1+1]_{q_1}!}{[m_1+r+1]_{q_1}!}\sum_{k_1=0}^{r} l_i(r)[m_1]_{q_1}^{k_1} L_{m_1,q_1}^{1/[m_1]_{q_1}}(e_{k_1}; z_1),$$

where $e_r(z_1) = z_1^r$ *and* $\sum_{k_1=0}^{r} l_i(r)[m_1]_{q_1}^{k_1} = [m_1+1]_{q_1}[m_1+2]_{q_1}\cdots[m_1+r]_{q_1}$.

Proof. Indeed, we have

$$\begin{aligned}\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_r; z_1) &= [m_1+1]_{q_1}\sum_{k_1=0}^{m_1} q_1^{-k_1} p_{m_1,k_1}^{\frac{1}{[m_1]_{q_1}}}(z_1)\int_0^1 p_{m_1,k_1}(q_1 t_1) e_r(t_1) d_{q_1} t_1\\ &= [m_1+1]_{q_1}\sum_{k_1=0}^{m_1} q_1^{-k_1} p_{m_1,k_1}^{\frac{1}{[m_1]_{q_1}}}(z_1) q_1^{k_1}\frac{[m_1]_{q_1}![k_1+r]_{q_1}!}{[k_1]_{q_1}![m_1+r+1]_{q_1}!}.\end{aligned}$$

Using $[s+k_1]_{q_1} = [s]_{q_1} + q_1^s[k_1]_{q_1}$, we obtain

$$\frac{[k_1+r]_{q_1}!}{[k_1]_{q_1}!} = [k_1+1]_{q_1}[k_1+2]_{q_1}\cdots[k_1+r]_{q_1} = \prod_{s=1}^{r}([s]_{q_1} + q_1^s[k_1]_{q_1}) = \sum_{s=1}^{r} C_{s,r}(q_1)[k_1]_{q_1}^s,$$

where $C_{s,r}(q_1), s = 1, 2, \ldots, r$ are the constants independent of k_1. Hence

$$\begin{aligned}\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_r; z_1) &= \frac{[m_1+1]_{q_1}!}{[m_1+r+1]_{q_1}!}\sum_{k_1=0}^{m_1} p_{m_1,k_1}^{\frac{1}{[m1]_{q_1}}}(z_1)\sum_{s=1}^{r} C_{s,r}(q_{m_1})[k_1]_{q_1}^s\\ &= \frac{[m_1+1]_{q_1}!}{[m_1+r+1]_{q_1}!}\sum_{s=1}^{r} C_{s,r}(q_1)[m_1]_{q_1}^s L_{m_1,q_1}^{1/[m_1]_{q_1}}(e_s; z_1).\end{aligned}$$

□

Lemma 1.2 ([10]). *The q-Durrmayer operators defined by* (1.3) *satisfy the equalities*

i) $\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_0; z_1) = 1$;

ii) $\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_1; z_1) = \frac{q_1[m_1]_{q_1} z_1 + 1}{[m_1+2]_{q_1}}$;

iii)

$$\begin{aligned}\mathfrak{D}_{m_1,m_2,q_1,q_2}^{\frac{1}{[m_1]_{q_1}}\frac{1}{[m_2]_{q_2}}}(e_{20}; z_1, z_2) &= \frac{[m_1]_{q_1}^3 q_1^3}{(1+[m_1]_{q_1})[m_1+2]_{q_1}[m_1+3]_{q_1}}\left(z_1(z_1+\frac{1}{[m_1]_{q_1}}) + \frac{z_1(1-z_1)}{[m_1]_{q_1}}\right)\\ &\quad + \frac{q_1(2q_1+1)[m_1]_{q_1} z_1}{[m_1+2]_{q_1}[m_1+3]_{q_1}} + \frac{q_1+1}{[m_1+2]_{q_1}[m_1+3]_{q_1}};\end{aligned}$$

iv)

$$\begin{aligned}\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_{30}; z_1, z_2) &= \frac{q_1^5[m_1]_{q_1}^3}{\prod_{i=0}^{2}[m_1+i+2]_{q_1}}\frac{1}{\prod_{i=0}^{2}(1+[i]_{q_1}/[m_1]_{q_1})}\sum_{k=0}^{2}\frac{A_k(q_1, 1/[m_1]_{q_1}; z_1)}{[m_1]_{q_1}^k}\\ &\quad + \frac{(q_1^3+3q_1^4+2q_1^5)[m_1]_{q_1}^3}{(1+[m_1]_{q_1})\prod_{i=0}^{2}[m_1+i+2]_{q_1}}\left(z_1(z_1+1/[m_1]_{q_1}) + z_1(1-z_1)/[m_1]_{q_1}\right)\\ &\quad + \frac{(q_1+3q_1^2+4q_1^3+3q_1^4)[m_1]_{q_1} z_1}{\prod_{i=0}^{2}[m_1+i+2]_{q_1}} + \frac{1+2q_1+2q_1^2+2q_1^3}{\prod_{i=0}^{2}[m_1+i+2]_{q_1}};\end{aligned}$$

v)

$$
\begin{aligned}
&\mathfrak{D}_{m_1,q_1}^{\frac{1}{[m_1]_{q_1}}}(e_{40};z_1,z_2)\\
&=\frac{q_1^9[m_1]_{q_1}^4}{\prod_{i=0}^{3}[m_1+i+2]_{q_1}}\frac{1}{\prod_{i=0}^{3}(1+[i]_{q_1}/[m_1]_{q_1})}\sum_{k=0}^{3}\frac{A_k^*(q_1,1/[m_1]_{q_1};z_1)}{[m_1]_{q_1}^k}\\
&\quad+\frac{(q_1^5+q_1^6+2q_1^7+4q_1^8+2q_1^9)[m_1]_{q_1}^3}{\prod_{i=0}^{3}[m_1+i+2]_{q_1}}\frac{1}{\prod_{i=0}^{2}(1+[i]_{q_1}/[m_1]_{q_1})}\sum_{k=0}^{2}\frac{A_k(q_1,1/[m_1]_{q_1};z_1)}{[m_1]_{q_1}^k}\\
&\quad+\frac{(q_1^3+4q_1^4+7q_1^5+9q_1^6+9q_1^7+5q_1^8)[m_1]_{q_1}^3}{(1+[m_1]_{q_1})\prod_{i=0}^{3}[m_1+i+2]_{q_1}}\Big(z_1(z_1+1/[m_1]_{q_1})+z_1(1-z_1)/[m_1]_{q_1}\Big)\\
&\quad+\frac{(q_1+4q_1^2+8q_1^3+12q_1^4+12q_1^5+9q_1^6+5q_1^7)[m_1]_{q_1}z_1}{\prod_{i=0}^{3}[m_1+i+2]_{q_1}}+\frac{(1+3q_1+5q_1^2+7q_1^3+6q_1^4+4q_1^5+2q_1^6)}{\prod_{i=0}^{3}[m_1+i+2]_{q_1}};
\end{aligned}
$$

where

$$
\begin{aligned}
A_0(q_1;1/[m_1]_{q_1};z_1)&=z_1(z_1+1/[m_1]_{q_1})(z_1+[2]_{q_1}/[m_1]_{q_1}),\\
A_1(q_1;1/[m_1]_{q_1};z_1)&=z_1(1-z_1)(z_1+1/[m_1]_{q_1})(2+q_1),\\
A_2(q_1;1/[m_1]_{q_1};z_1)&=z_1(1-z_1)(1-[2]_{q_1}z_1),\\
A_0^*(q_1;1/[m_1]_{q_1};z_1)&=z_1(z_1+1/[m_1]_{q_1})(z_1+[2]_{q_1}/[m_1]_{q_1})(z_1+[3]_{q_1}/[m_1]_{q_1}),\\
A_1^*(q_1;1/[m_1]_{q_1};z_1)&=z_1(1-z_1)(z_1+1/[m_1]_{q_1})(z_1+[2]_{q_1}/[m_1]_{q_1})(q_1^2+2q_1+3),\\
A_2^*(q_1;1/[m_1]_{q_1};z_1)&=z_1(1-z_1)(z_1+1/[m_1]_{q_1})((q_1^2+3q_1+3)z_1(z_1+1/[m_1]_{q_1})\\
&\qquad-[2]_{q_1}^2(z_1+1/[m_1]_{q_1}-[2]_{q_1}z_1(1+[3]_{q_1}/[m_1]_{q_1}))),\\
A_3^*(q_1;1/[m_1]_{q_1};z_1)&=z_1(1-z_1)(z_1+1/[m_1]_{q_1})([2]_{q_1}z_1([3]_{q_1}z_1-q_1-2)+1-q_1/[m_1]_{q_1}).
\end{aligned}
$$

The aim of paper is to study properties of these operators, the study of the uniform convergence, as well as rate of convergence in terms of the Lipschitz class function. Furthermore, we obtain a Voronovskaja type symptotic formula, error estimation in terms of modulus of continuity and Petree K-functional.

2. Construction of the Bivariate q-Durrmeyer-Pólya operators

In what follows, let $I=[0,1]$, $I^2=I\times I$ and $\{q_{m_i}\}$ be the sequences of real numbers such that $0<q_{m_i}<1$, $\lim_{m_i\to\infty} q_{m_i}=1$ and $\lim_{m_i\to\infty} q_{m_i}^{m_i}=a$, $a\in[0,1)$, $i=1,2$. Further, let $\delta_{m_1}(z_1)=\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((u-z_1)^2,q_{m_1},z_1)$ and $\delta_{m_2}(z_2)=\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}}(v-z_2)^2,q_{m_2},z_2)$. Let $\mathfrak{D}_{m_i,q_{m_i}}^{\frac{1}{[m_i]_{q_{m_i}}}}:C(I)\to C(I), i=1,2$ defined for each positive integers m_1,m_2 and $0<q_{m_1},q_{m_2}<1$, and any $h\in C(I)$, $g\in C(I)$, respectively by

$$\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}(h;z_1)=[m_1+1]_{q_{m_1}}\sum_{k_1=0}^{m_1}q_{m_1}^{-k_1}p_{m_1,k_1}^{\frac{1}{[m_1]_{q_{m_1}}}}(z_1)\int_0^1 p_{m_1,k_1}(q_{m_1}t_1)h(t_1)d_{q_{m_1}}t_1, \tag{2.1}$$

$$\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}}(h;z_2)=[m_2+1]_{q_{m_2}}\sum_{k_2=0}^{m_2}q_{m_2}^{-k_2}p_{m_2,k_2}^{\frac{1}{[m_2]_{q_{m_2}}}}(z_2)\int_0^1 p_{m_2,k_2}(q_{m_2}t_2)h(t_2)d_{q_{m_2}}t_2. \tag{2.2}$$

The parametric extension of (2.1) and (2.2) are the operators $\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}, \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} : C(I^2) \to C(I^2)$, defined for each positive integers m_1, m_2 and $h \in C(I^2)$, as follows:

$$\mathfrak{D}_{m_1,q_{m_1}}^{z_1,\frac{1}{[m_1]_{q_{m_1}}}}(h;z_1) = [m_1+1]_{q_{m_1}} \sum_{k_1=0}^{m_1} q_{m_1}^{-k_1} p_{m_1,k_1}^{\frac{1}{[m_1]_{q_{m_1}}}}(z_1) \int_0^1 p_{m_1,k_1}(q_{m_1}t_1)h(t_1,v)d_{q_{m_1}}t_1,$$

$$\mathfrak{D}_{m_2,q_{m_2}}^{z_2,\frac{1}{[m_2]_{q_{m_2}}}}(h;z_2) = [m_2+1]_{q_{m_2}} \sum_{k_2=0}^{m_2} q_{m_2}^{-k_2} p_{m_2,k_2}^{\frac{1}{[m_2]_{q_{m_2}}}}(z_2) \int_0^1 p_{m_2,k_2}(q_{m_2}t_2)h(u,t_2)d_{q_{m_2}}t_2.$$

Motivated by Gupta et al. [10], for $h \in C(I^2)$ and $0 < q_{m_1}, q_{m_2} < 1$, we construct the bivariate extension of the univariate q-Durrmeyer operators based Pólya distribution for each positive integers m_1, m_2 as follows:

$$\begin{aligned}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(h;z_1,z_2) &= [m_1+1]_{q_{m_1}}[m_2+1]_{q_{m_2}} \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2} q_{m_1}^{-k_1} q_{m_2}^{-k_2} p_{m_1,k_1}^{\frac{1}{[m_1]_{q_{m_1}}}}(z_1) p_{m_2,k_2}^{\frac{1}{[m_2]_{q_{m_2}}}}(z_2) \\ &\quad \cdot \int_0^1\int_0^1 p_{m_1,k_1}(q_{m_1}t_1)p_{m_2,k_2}(q_{m_2}t_2)h(t_1,t_2)d_{q_{m_1}}t_1 d_{q_{m_2}}t_2.\end{aligned} \tag{2.3}$$

In what follows, let $e_{ij} : I^2 \to R$, $e_{ij}(z_1,z_2) = z_1^i z_2^j$, $(z_1,z_2) \in I^2, (i,j) \in N^0 \times N^0$ with $i+j \leqslant 4$ be the two dimensional test function. Further, we give some lemmas.

Lemma 2.1. *The bivariate q-Durrmayer operators based on Pólya distribution defined by (2.3), satisfy the equalities*

i) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(e_{00};z_1,z_2) = 1$;

ii) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(e_{10};z_1,z_2) = \frac{q_{m_1}[m_1]_{q_{m_1}}z_1+1}{[m_1+2]_{q_{m_1}}}$;

iii)

$$\begin{aligned}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(e_{20};z_1,z_2) &= \frac{[m_1]_{q_{m_1}}^3 q_{m_1}^3}{(1+[m_1]_{q_{m_1}})[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}}\left(z_1(z_1+\frac{1}{[m_1]_{q_{m_1}}})+\frac{z_1(1-z_1)}{[m_1]_{q_{m_1}}}\right) \\ &\quad + \frac{q_{m_1}(2q_{m_1}+1)[m_1]_{q_{m_1}}z_1}{[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}} + \frac{q_{m_1}+1}{[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}};\end{aligned}$$

iv) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(e_{11};z_1,z_2) = \frac{q_{m_1}[m_1]_{q_{m_1}}z_1+1}{[m_1+2]_{q_{m_1}}}\frac{q_{m_2}[m_2]_{q_{m_2}}z_2+1}{[m_2+2]_{q_{m_2}}}$;

v) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_2}}}\frac{1}{[m_2]_{q_{m_2}}}}(e_{01};z_1,z_2) = \frac{q_{m_2}[m_2]_{q_{m_2}}z_2+1}{[m_2+2]_{q_{m_2}}}$;

vi)

$$\begin{aligned}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(e_{02};z_1,z_2) &= \frac{[m_2]_{q_{m_2}}^3 q_{m_2}^3}{(1+[m_2]_{q_{m_2}})[m_2+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}}\left(z_2(z_2+\frac{1}{[m_2]_{q_{m_2}}})+\frac{z_2(1-z_2)}{[m_2]_{q_{m_2}}}\right) \\ &\quad + \frac{q_{m_2}(2q_{m_2}+1)[m_2]_{q_{m_2}}z_2}{[m_2+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}} + \frac{q_{m_2}+1}{[m_2+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}};\end{aligned}$$

vii)

$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(e_{30};z_1,z_2)$$

$$= \frac{q_{m_1}^5 [m_1]_{q_{m_1}}^3}{\prod_{i=0}^{2} [m_1+i+2]_{q_{m_1}}} \frac{1}{\prod_{i=0}^{2} (1+[i]_{q_{m_1}}/[m_1]_{q_{m_1}})} \sum_{k=0}^{2} \frac{A_k(q_{m_1}, 1/[m_1]_{q_{m_1}}; z_1)}{[m_1]_{q_{m_1}}}^k$$
$$+ \frac{(q_{m_1}^3 + 3q_{m_1}^4 + 2q_{m_1}^5)[m_1]_{q_{m_1}}^3}{(1+[m_1]_{q_{m_1}}) \prod_{i=0}^{2} [m_1+i+2]_{q_{m_1}}} \left(z_1(z_1 + 1/[m_1]_{q_{m_1}}) + z_1(1-z_1)/[m_1]_{q_{m_1}} \right)$$
$$+ \frac{(q_{m_1} + 3q_{m_1}^2 + 4q_{m_1}^3 + 3q_{m_1}^4)[m_1]_{q_{m_1}} z_1}{\prod_{i=0}^{2} [m_1+i+2]_{q_{m_1}}} + \frac{1 + 2q_{m_1} + 2q_{m_1}^2 + 2q_{m_1}^3}{\prod_{i=0}^{2} [m_1+i+2]_{q_{m_1}}};$$

viii)

$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{40}; z_1, z_2)$$
$$= \frac{q_{m_1}^9 [m_1]_{q_{m_1}}^4}{\prod_{i=0}^{3} [m_1+i+2]_{q_{m_1}}} \frac{1}{\prod_{i=0}^{3} (1+[i]_{q_{m_1}}/[m_1]_{q_{m_1}})} \sum_{k=0}^{3} \frac{A_k^*(q_{m_1}, 1/[m_1]_{q_{m_1}}; z_1)}{[m_1]_{q_{m_1}}^k}$$
$$+ \frac{(q_{m_1}^5 + q_{m_1}^6 + 2q_{m_1}^7 + 4q_{m_1}^8 + 2q_{m_1}^9)[m_1]_{q_{m_1}}^3}{\prod_{i=0}^{3} [m_1+i+2]_{q_{m_1}}} \frac{1}{\prod_{i=0}^{2} (1+[i]_{q_{m_1}}/[m_1]_{q_{m_1}})} \sum_{k=0}^{2} \frac{A_k(q_{m_1}, 1/[m_1]_{q_{m_1}}; z_1)}{[m_1]_{q_{m_1}}^k}$$
$$+ \frac{(q_{m_1}^3 + 4q_{m_1}^4 + 7q_{m_1}^5 + 9q_{m_1}^6 + 9q_{m_1}^7 + 5q_{m_1}^8)[m_1]_{q_{m_1}}^3}{(1+[m_1]_{q_{m_1}}) \prod_{i=0}^{3} [m_1+i+2]_{q_{m_1}}} \left(z_1(z_1 + 1/[m_1]_{q_{m_1}}) + z_1(1-z_1)/[m_1]_{q_{m_1}} \right)$$
$$+ \frac{(q_{m_1} + 4q_{m_1}^2 + 8q_{m_1}^3 + 12q_{m_1}^4 + 12q_{m_1}^5 + 9q_{m_1}^6 + 5q_{m_1}^7)[m_1]_{q_{m_1}} z_1}{\prod_{i=0}^{3} [m_1+i+2]_{q_{m_1}}}$$
$$+ \frac{(1 + 3q_{m_1} + 5q_{m_1}^2 + 7q_{m_1}^3 + 6q_{m_1}^4 + 4q_{m_1}^5 + 2q_{m_1}^6)}{\prod_{i=0}^{3} [m_1+i+2]_{q_{m_1}}};$$

where $A_i(q_{m_1}; 1/[m_1]_{q_{m_1}}; z_1)$ *and* $A_k^*(q_{m_1}, 1/[m_1]_{q_{m_1}}; z_1)$ $i = 1, 2, 3$ *are defined in Lemma 1.2.*

Proof. This lemma follows easily while applying previous lemmas, as well as taking into account the definition of q-Durrmeyer-Pólya operators (2.3), and obtains the following indentities:

$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{00}; z_1, z_2) = \mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}} (1; z_1) \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} (1; z_2);$$
$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{10}; z_1, z_2) = \mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}} (u; z_1) \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} (1; z_2);$$
$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{20}; z_1, z_2) = \mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}} (u^2; z_1) \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} (1; z_2);$$
$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{11}; z_1, z_2) = \mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}} (u; z_1) \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} (v; z_2);$$
$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{01}; z_1, z_2) = \mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}} (1; z_1) \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} (v; z_2);$$
$$\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}} \frac{1}{[m_2]_{q_{m_2}}}} (e_{02}; z_1, z_2) = \mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}} (1; z_1) \mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}} (v^2; z_2).$$

□

Lemma 2.2. *For operators defined by* (2.3)*, the following identities hold true*

i) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(t-z_1;z_1,z_2)=\frac{(q_{m_1}[m_1]_{q_{m_1}}-[m_1+2]_{q_{m_1}})z_1+1}{[m_1+2]_{q_{m_1}}}$;

ii)

$$\begin{aligned}
&\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}((t-z_1)^2;z_1,z_2)\\
&=\left(1+\frac{([m_1]_{q_{m_1}}-1)[m_1]^2_{q_{m_1}}q^3_{m_1}}{(1+[m_1]_{q_{m_1}})[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}}-\frac{2q_{m_1}[m_1]_{q_{m_1}}}{[m_1+2]_{q_{m_1}}}\right)z_1^2\\
&\quad+\left(\frac{2[m_1]^2_{q_{m_1}}q^3_{m_1}}{(1+[m_1]_{q_{m_1}})[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}}+\frac{q_{m_1}(2q_{m_1}+1)[m_1]_{q_{m_1}}}{[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}}-\frac{2}{[m_1+2]_{q_{m_1}}}\right)z_1\\
&\quad+\frac{q_{m_1}+1}{[m_1+2]_{q_{m_1}}[m_1+3]_{q_{m_1}}},
\end{aligned}$$

iii) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(s-z_2;z_1,z_2)=\frac{(q_{m_2}[m_2]_{q_{m_2}}-[m_2+2]_{q_{m_2}})z_2+1}{[m_2+2]_{q_{m_2}}}$;

iv)

$$\begin{aligned}
&\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}((s-z_2)^2;z_1,z_2)\\
&=\left(1+\frac{q_{m_2}([m_2]_{q_{m_2}}-1)[m_2]^2_{q_{m_2}}q^3_{m_2}}{(1+[m_2]_{q_{m_2}})[m_2+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}}-\frac{2q_{m_2}[m_2]_{q_{m_2}}}{[m_2+2]_{q_{m_2}}}\right)z_2^2\\
&\quad+\left(\frac{2[m_2]^2_{q_{m_2}}q^3_{m_2}}{(1+[m_2]_{q_{m_2}})[m_2+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}}+\frac{q_{m_2}(2q_{m_2}+1)[m_2]_{q_{m_2}}}{[m_1+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}}-\frac{2}{[m_2+2]_{q_{m_2}}}\right)z_2\\
&\quad+\frac{q_{m_2}+1}{[m_2+2]_{q_{m_2}}[m_2+3]_{q_{m_2}}}.
\end{aligned}$$

Lemma 2.3. *For* $0<q_{m_i}<1, i=1,2, z_1,z_2\in[0,1]$ *and* $m_1,m_2>3$*, we have*

i) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}((t-z_1)^2;z_1,z_2)\leqslant\frac{3}{[m_1+2]_{q_{m_1}}}\delta^2_{m_1}(z_1)$;

ii) $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}((s-z_2)^2;z_1,z_2)\leqslant\frac{3}{[m_2+2]_{q_{m_2}}}\delta^2_{m_2}(z_2)$,

where $\delta^2_{m_1}(z_1)=\psi^2(z_1)+\frac{3}{[m_1+2]_{q_{m_1}}}$, $\psi^2(z_1)=z_1(1-z_1)$, *and* $\delta^2_{m_2}(z_2)=\psi^2(z_2)+\frac{3}{[m_2+2]_{q_{m_2}}}$, $\psi^2(z_2)=z_2(1-z_2)$.

Corollary 2.4. *Taking into account Lemma* 2.2*, we get the following limits as follows,*

$$\lim_{m_1\to\infty}[m_1]_{q_{m_1}}\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(t-z_1;z_1,z_2)=1-(a+1)z_1,$$

$$\lim_{m_2\to\infty}[m_2]_{q_{m_2}}\mathfrak{D}_{m_2,m_2,q_{m_2},q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}\frac{1}{[m_2]_{q_{m_2}}}}(s-z_2;z_1,z_2)=1-(a+1)z_2,$$

$$\lim_{m_1\to\infty}[m_1]_{q_{m_1}}\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}((t-z_1)^2;z_1,z_2)=3z_1(1-z_1),$$

$$\lim_{m_2\to\infty}[m_1]_{q_{m_1}}\mathfrak{D}_{m_2,m_2,q_{m_2},q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}\frac{1}{[m_2]_{q_{m_2}}}}((t-z_1)^2;z_1,z_2)=3z_2(1-z_2).$$

3. Degree of approximation

In what follow, we prove a convergence theorem for these operators, as well as Voronovskaja type theorem.

Theorem 3.1. *Let* (q_{m_i}) *be the sequences of real numbers such that* $0 < q_{m_i} < 1$, *and* $\lim_{m_i\to\infty} q_{m_i} = 1, i = 1, 2$. *Then, the sequences* $\left\{\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h; z_1, z_2)\right\}$ *converge to* $h(z_1, z_2)$ *uniformly on* I^2 *for each* $h \in C(I^2)$.

Proof. By Lemma 2.1, $\lim_{m_1\to\infty, m_2\to\infty} \mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}} = e_{ij}(z_1, z_2)$ uniformly on I^2. The proof of uniform convergence is then completed by applying a Volkov Theorem [18]. □

Theorem 3.2. *For* $h \in C(I^2)$, *we have*

$$\lim_{m_1\to\infty}[m_1]_{q_{m_1}}\left(\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(h; z_1, z_2) - h(z_1, z_2)\right)$$
$$= (1-(a+1)z_1)h'_{z_1}(z_1, z_2) + (1-(a+1)z_2)h'_{z_2}(z_1, z_2) + \frac{3}{2}\{z_1(1-z_1)h_{z_1z_1}(z_1, z_2) + z_2(1-z_2)h_{z_2z_2}(z_1, z_2)\},$$

uniformly on I^2.

Proof. For $z_1, z_2 \in I^2$, by the Taylor's expansion formula, we have

$$\begin{aligned} h(t, s) &= h(z_1, z_2) + h'_{z_1}(z_1, z_2)(t - z_1) + h'_{z_2}(z_1, z_2)(s - z_2)\\ &\quad + \frac{1}{2}\{h''_{z_1z_1}(z_1, z_2)(t - z_1)^2 + 2h_{z_1z_2}(z_1, z_2)(t - z_1)(s - z_2) + h''_{z_2z_2}(z_1, z_2)(s - z_2)^2\}\\ &\quad + \psi(t, s; z_1, z_2)((t - z_1)^2 + (t - z_2)^2), \end{aligned}$$

for $h \in I^2$ where $\psi(t, s; z_1, z_2) \in C(I^2)$ and $\lim_{(t,s)\to(z_1,z_2)} \psi(t, s; z_1, z_2) = 0$. Applying $\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(h; z_1, z_2)$ on above Taylor's expansion formula, we can write

$$\begin{aligned} &\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(h(t, s); z_1, z_2)\\ &= h(z_1, z_2) + h'_{z_1}(z_1, z)\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}((t - z_1); z_1, z_2)\\ &\quad + f'_{z_2}(z_1, z_2)\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}((s - z_2); z_1, z_2) + \frac{1}{2}\{h_{z_1z_1}(z_1, z_2)\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}((t - z_1)^2; z_1, z_2)\\ &\quad + 2h''_{z_1z_2}(z_1, z_2)\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}((t - z_1)(s - z_2); z_1, z_2) + h''_{z_2z_2}(z_1, z_2)\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}((s - z_2)^2; z_1, z_2)\}\\ &\quad + \psi(t, s; z_1, z_2)\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(((t - z_1)^2 + (t - z_2)^2); z_1, z_2)\\ &= h(z_1, z_2) + h'_{z_1}(z_1, z_2)\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((t - z_1); z_1, z_2)\\ &\quad + h'_{z_2}(z_1, z_2)\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((s - z_2); z_1, z_2) + \frac{1}{2}\{h''_{z_1z_1}(z_1, z_2)\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((t - z_1)^2; z_1, z_2)\\ &\quad + 2h''_{z_1z_2}(z_1, z_2)\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((t - z_1); z_1, z_2)\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((s - z_2); z_1, z_2)\\ &\quad + h''_{z_2z_2}(z_1, z_2)\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((s - z_2)^2; z_1, z_2)\} + \psi(t, s; z_1, z_2)\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(((t - z_1)^2 + (t - z_2)^2); z_1, z_2). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we have

$$\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(((t-z_1)^2+(t-z_2)^2);z_1,z_2) \leqslant \left\{\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(\psi^2(t,s;z_1,z_2);z_1,z_2)\right\}^{1/2} \times \left\{\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(((t-z_1)^2+(t-z_2)^2);z_1,z_2)\right\}^2.$$

Hence, by using Theorem 2.4, we get

$$\begin{aligned}\lim_{m_1\to\infty}[m_1]_{q_{m_1}}&\left(\mathfrak{D}_{m_1,m_1,q_{m_1},q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_1}}}}(h;z_1,z_2)-h(z_1,z_2)\right)\\ &=(1-(a+1)z_1)f'_{z_1}(z_1,z_2)+(1-(a+1)z_2)h'_{z_2}(z_1,z_2)\\ &\quad+\frac{3}{2}\left\{z_1(1-z_1)h''_{z_1z_1}(z_1,z_2)+z_2(1-z_2)h''_{z_2z_2}(z_1,z_2)\right\},\end{aligned}$$

which leads us to the required result. □

For $h\in C(I^2)$ and $\delta>0$, the complete modulus of continuity for the function $h(z_1,z_2)$ is defined by

$$\omega(h;\delta_1,\delta_2)=\sup\{|h(u,v)-h(z_1,z_2)|:(u,v),(z_1,z_2)\in I^2,\ |u-z_1|\leqslant\delta_1,|v-z_2|\leqslant\delta_2\},$$

and its partial modulus of continuity with respect to z_1 and z_2 are respectively given by

$$\omega^{(1)}(h;\delta)=\sup_{0\leqslant z_2\leqslant b}\ \sup_{|x_1-x_2|\leqslant\delta}\{|h(x_1,z_2)-h(x_2,z_2)|\},$$

$$\omega^{(2)}(h;\delta)=\sup_{0\leqslant z_1\leqslant a}\ \sup_{|y_1-y_2|\leqslant\delta}\{|h(z_1,y_1)-f(z_1,y_2)|\}.$$

Now, we obtain the rate of convergence of the approximation by the bivariate operators (2.3) defined by means of modulus of continuity of the functions.

Theorem 3.3. *Let $h\in C(I^2)$ and (q_{m_i}) be a sequence of real numbers such that $0<q_{m_i}<1$, and $\lim_{m_i\to\infty}q_{m_i}=1, i=1,2$. Then, for all $(z_1,z_2)\in I^2$, we have*

$$\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right|\leqslant 4\omega\left(h;\sqrt{\frac{3}{[m_1+2]_{q_{m_1}}}\delta_{m_1}(z_1)},\sqrt{\frac{3}{[m_2+2]_{q_{m_2}}}\delta_{m_2}(z_2)}\right),$$

where $\delta_{m_1}(z_1)$ and $\delta_{m_2}(z_2)$ are defined by Lemma 2.3.

Proof. By the linearity and monotonicity of the operators $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)$, we have

$$\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right|\leqslant\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(|h(t_1,t_2)-h(z_1,z_2)|;z_1,z_2).$$

Taking into account the property of monotonicity of $\omega(h;\delta_{m_1},\delta_{m_2})$, we get

$$\begin{aligned}&\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right|\\ &\quad\leqslant\omega\left(h;\sqrt{\delta_{m_1}(z_1)},\sqrt{\delta_{m_2}(z_1)}\right)\left(\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(1;z_1,z_2)+\frac{1}{\sqrt{\delta_{m_1}(z_1)}}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(|t_1-z_1|;z_1,z_2)\right)\end{aligned}$$

$$\times\left(\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(1;z_1,z_2)+\frac{1}{\sqrt{\delta_{m_2}(z_1)}}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(|t_2-z_2|;z_1,z_2)\right).$$

Then, by using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right| &\leqslant \omega(h;\sqrt{\delta_{m_1}(z_1)},\sqrt{\delta_{m_2}(z_1)})\\ &\quad\times\left(1+\frac{1}{\sqrt{\delta_{m_1}}}\left\{\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((t_1-z_1)^2;z_1,z_2)\right\}^{1/2}\right)\\ &\quad\times\left(1+\frac{1}{\sqrt{\delta_{m_2}}}\left\{\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}}((t_2-z_2)^2;z_1,z_2)\right\}^{1/2}\right).\end{aligned}$$

Finally, choosing $\delta_{m_1}=\delta_{m_1}(z_1)$ and $\delta_{m_2}=\delta_{m_2}(z_2)$, we reach the desired result.

We continue by recalling the definition of the Lipschitz class for bivariate function of f. For $0<\gamma\leqslant 1$, the Lipschitz class $\mathrm{Lip}_L(\gamma)$ for bivariate case is as follows:

$$\mathrm{Lip}_L(\gamma):=\left\{h:|h(t_1,t_2)-h(z_1,z_2)|\leqslant L\frac{\|r-s\|^{\gamma}}{(\|r\|+z_1+z_2)^{\gamma/2}}\right\},$$

where $r=(t_1,t_2)$, $s=(z_1,z_2)$ in I^2 and $\|r-s\|=\left\{(t_1-z_1)^2+(t_2-z_2)^2\right\}^{1/2}$ is the Euclidean norm.

Theorem 3.4. *Suppose that* $h\in\mathrm{Lip}_L(\gamma)$*. Then, for every* $(z_1,z_2)\in I^2$*, we have*

$$\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right|\leqslant\frac{L}{(z_1+z_2)^{\gamma/2}}\left\{\frac{3}{[m_1+2]_{q_{m_1}}}\delta^2_{m_1}(z_1)+\frac{3}{[m_2+2]_{q_{m_2}}}\delta^2_{m_2}(z_2)\right\}^{1/2}.$$

Proof. Take $\gamma=1$. Then, for $h\in\mathrm{Lip}_L(\gamma)$ and for each $z_1,z_2\in I^2$, using the monotonicity and linearity of operators, we have

$$\begin{aligned}\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right| &\leqslant \mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(|h(t_1,t_2)-h(z_1,z_2)|;z_1,z_2)\\ &\leqslant L\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}\left(\frac{\|r-s\|}{(\|r\|+z_1+z_2)^{1/2}};z_1,z_2\right)\\ &\leqslant\frac{L}{(z_1+z_2)^{1/2}}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(\|r-s\|;z_1,z_2),\end{aligned}$$

where $r=(t_1,t_2)$ and $s=(z_1,z_2)$. Using the Cauchy-Schwarz inequality and Lemma 2.1, we obtain

$$\begin{aligned}&\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right|\\ &\leqslant\frac{L}{(z_1+z_2)^{1/2}}\left\{\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[n_1]_{q_{m_2}}}}(\|r-s\|^2,z_1,z_2)\right\}^{1/2}\\ &\leqslant\frac{L}{(z_1+z_2)^{1/2}}\left\{\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((t_1-z_1)^2,z_1,z_2)+\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}}((t_2-z_2)^2,z_1,z_2)\right\}^{1/2}\end{aligned}$$

$$\leqslant \frac{L}{(z_1+z_2)^{1/2}}\left\{\frac{3}{[m_1+2]_{q_{m_1}}}\delta^2_{m_1}(z_1)+\frac{3}{[m_2+2]_{q_{m_2}}}\delta^2_{m_2}(z_2)\right\}^{1/2}.$$

Secondly, let $0<\gamma<1$. Then, for $h\in \mathrm{Lip}_L(\gamma)$ and for each $z_1,z_2\in I^2$, we get

$$\begin{aligned}\left|\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(h;z_1,z_2)-h(z_1,z_2)\right| &\leqslant \mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(|h(t_1,t_2)-h(z_1,z_2)|;z_1,z_2)\\ &\leqslant L\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left(\frac{\|r-s\|^\gamma}{(\|r\|+z_1+z_2)^{\gamma/2}};z_1,z_2\right)\\ &\leqslant \frac{L}{(z_1+z_2)^{\gamma/2}}\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(\|r-s\|^\gamma;z_1,z_2).\end{aligned}$$

Now, applying the Hölder's inequality with $u_1=\frac{2}{\gamma}$ and $u_2=\frac{2}{2-\gamma}$, and Lemma 2.1, we get

$$\begin{aligned}&\left|\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(h;z_1,z_2)-h(z_1,z_2)\right|\\ &\leqslant \frac{L}{(z_1+z_2)^{\gamma/2}}\left\{\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(\|r-s\|^2,z_1,z_2)\right\}^{\gamma/2}\\ &\leqslant \frac{L}{(z_1+z_2)^{\gamma/2}}\left\{\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}}_{m_1,q_{m_1}}((t_1-z_1)^2,z_1,z_2)+\mathfrak{D}^{\frac{1}{[m_2]_{q_{m_2}}}}_{m_2,q_{m_2}}((t_2-z_2)^2,z_1,z_2)\right\}^{\gamma/2}\\ &\leqslant \frac{L}{(z_1+z_2)^{1/2}}\left\{\frac{3}{[m_1+2]_{q_{m_1}}}\delta^2_{m_1}(z_1)+\frac{3}{[m_2+2]_{q_{m_2}}}\delta^2_{m_2}(z_2)\right\}^{\gamma/2},\end{aligned}$$

which leads us to the required result. □

Theorem 3.5. *Suppose that $h\in C^1(I^2)$. Then, for every $(z_1,z_2)\in I^2$, we have*

$$\left|\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(h;z_1,z_2)-h(z_1,z_2)\right|\leqslant\| h'_{z_1}\|_{C(I_{ab})}\sqrt{\delta_{m_1}(z_1)}+\| h'_{z_2}\|_{C(I_{ab})}\sqrt{\delta_{m_2}(z_2)},$$

where $\delta_{m_1}(z_1)$ and $\delta_{m_2}(z_2)$ are defined by Lemma 2.3.

Proof. For $(z_1,z_2)\in I^2$ be a fixed point, we obtain,

$$h(t_1,t_2)-h(z_1,z_2)=\int_{z_1}^{t_1}h'_{r_1}(r_1,t_2)dr_1+\int_{z_2}^{t_2}h'_{r_2}(z_1,r_2)dr_2,\ \text{for } (t_1,t_2)\in I^2.$$

Applying the operator defined in (2.3) on both sides, we obtain

$$\begin{aligned}\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(h(t_1,t_2);z_1,z_2)-h(z_1,z_2)&=\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left(\int_{z_1}^{t_1}h'_{r_1}(r_1,t_2)dr_1;z_1,z_2\right)\\ &+\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left(\int_{z_2}^{t_2}h'_{r_2}(z_1,r_2)dr_2;z_1,z_2\right).\end{aligned}$$

Now, by using sup-norm on I^2, we get

$$\left|\int_{z_1}^{t_1} h'_{r_1}(r_1,t_2)dr_1\right| \leqslant \int_{z_1}^{t_1}\left|h'_{r_1}(r_1,t_2)\right||dr_1| \leqslant \| h'_{z_1}\|_{C_{(I^2)}}|t_1-z_1|,$$

and

$$\left|\int_{z_2}^{t_1} h'_{r_2}(z_1,r_2)dr_2\right| \leqslant \int_{z_2}^{t_2}\left|h'_{r_2}(z_1,r_2)\right||dr_2| \leqslant \| h'_{z_2}\|_{C_{(I^2)}}|t_2-z_2|.$$

By using these inequalities, we have

$$\begin{aligned}
&\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]q_{m_1}}\frac{1}{[m_1]q_{m_2}}}(h(t_1,t_2);z_1,z_2)-h(z_1,z_2)\right| \\
&\leqslant \mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]q_{m_1}}\frac{1}{[m_1]q_{m_2}}}\left(\left|\int_{z_1}^{t_1} h'_{r_1}(r_1,t_2)dt\right|;,z_1,z_2\right)+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]q_{m_1}}\frac{1}{[m_1]q_{m_2}}}\left(\left|\int_{z_2}^{t_2} h'_{r_2}(z_1,r_2)dr_2\right|;z_1,z_2\right) \\
&\leqslant \| h'_{z_1}\|_{I^2}\,\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]q_{m_1}}}(|t_1-z_1|;z_1,z_2)+\| h'_{z_2}\|_{I^2}\,\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]q_{m_2}}}(|t_2-z_2|;z_1,z_2).
\end{aligned} \tag{3.1}$$

Hence, applying Hölder inequality, Lemma 2.3, and considering that $\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[n_1]q_{m_1}}\frac{1}{[m_1]q_{m_2}}}(1;z_1,z_2)=1$, we get

$$\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]q_{m_1}}}(|t_1-z_1|;z_1,z_2) \leqslant \left\{\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]q_{m_1}}}((t_1-z_1)^2;z_1,z_2)\times\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]q_{m_1}}}(1;z_1,z_2)\right\}^{1/2} \leqslant \{\delta_{m_1}(z_1)\}^{1/2}. \tag{3.2}$$

Analogously,

$$\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]q_{m_2}}}(|t_2-z_2|;z_1,z_2) \leqslant \left\{\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]q_{m_2}}}((t_2-z_2)^2;z_1,z_2)\times\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]q_{m_2}}}(1;z_1,z_2)\right\}^{1/2} \leqslant \{\delta_{m_2}(z_2)\}^{1/2}. \tag{3.3}$$

Combining equations (3.1)-(3.3), we obtain

$$\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]q_{m_1}}\frac{1}{[m_1]q_{m_2}}}(h;z_1,z_2)-h(z_1,z_2)\right| \leqslant \| h'_{z_1}\|_{C(I^2)}\sqrt{\delta_{m_1}(z_1)}+\| h'_{z_2}\|_{C(I^2)}\sqrt{\delta_{m_2}(z_2)}.$$

This completes the proof. □

Let $C^2(I_{ab})$ be the space of functions h such that $\frac{\partial^i h}{\partial {z_1}^i}, \frac{\partial^i h}{\partial {z_2}^i} \in C(I_{ab}), (i=1,2)$. The norm on the space $C^2(I_{ab})$ is defined as

$$\| h \|_{C^2(I_{ab})}=\| h \|_{C(I_{ab})}+\sum_{i=1}^{2}\left(\left\|\frac{\partial^i h}{\partial {z_1}^i}\right\|_{C(I_{ab})}+\left\|\frac{\partial^i h}{\partial {z_2}^i}\right\|_{C(I_{ab})}\right).$$

For $h \in C(I_{ab})$ and $\delta > 0$, the Peetre's K-functional and the second modulus of smoothness are defined respectively as

$$K(h;\delta)=\inf_{g\in c^2(I_{ab})}\{\| h-g\|_{C(I_{ab})}+\delta\| g\|_{C(I_{ab})}\}$$

and

$$\omega_2(h;\delta)=\sup_{\sqrt{u^2+v^2}\leqslant\delta}\| h(z_1+2u,z_2+2v)-2h(z_1+u,z_2+v)+h(z_1,z_2)\|_{C(I_{ab})},$$

where $\|\cdot\|_{C(I_{ab})}$ is the sup-norm. It is known that ([6], page 192) there exists a positive constant L that is independent of δ and h, such that

$$K(h;\delta) \leqslant L\left\{\omega_2(h;\delta) + \min(1,\delta)\parallel h \parallel_{C(I_{ab})}\right\}.$$

Below, we prove the following order of approximation of q-Durrmeyer-Pólya operators to the function $h \in C^2(I^2)$ by K-functional.

Theorem 3.6. *Let* $h \in C(I^2)$. *Then we have estimates*

$$\begin{aligned}
&\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2) - h(z_1,z_2)\\
&\quad\leqslant L\Big\{\omega_2\left(h;\sqrt{\lambda_{m_1,m_2,p_1,p_2}(q_{n_1},q_{n_2},z_1,z_2)}\right)\\
&\qquad + \min\{1,\eta_{m_1,m_2,p_1,p_2}(q_{m_1},q_{m_2},z_1,z_2)\}\parallel h \parallel_{C(I_{ab})}\Big\} + \omega\left(h;\sqrt{(u_{m_1}-z_1)^2+(v_{m_2}-z_2)^2}\right),
\end{aligned}$$

where $u_{m_1} = \frac{2q_{m_1}[m_1+p_1]_{q_{m_1}}z_1+(1+[2]_{q_{m_1}}\alpha_1)a_{m_1}}{[2]_{q_{m_1}}([m_1+1]_{q_{m_1}}+\beta_1)}$, $v_{m_2} = \frac{2q_{m_2}[m_2+p_2]_{q_{m_2}}z_2+(1+[2]_{q_{m_2}}\alpha_2)b_{m_2}}{[2]_{q_{m_2}}([m_2+1]_{q_{m_2}}+\beta_2)}$, *and*

$$\eta_{m_1,m_2}(h;q_{m_1},q_{m_2},z_1,z_2) = \delta_{m_1}+\delta_{m_2}+(u_{m_1}-z_1)^2+(v_{m_2}-z_2)^2.$$

Proof. We consider modified operators $\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}$ defined by

$$\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2) = \mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2) + h(z_1,z_2) - h(u_{m_1},v_{m_2}), \tag{3.4}$$

where $z_1, z_2 \in I^2$. From (3.4) and by Lemma 2.1, we have $\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(1;z_1,z_2) = 1$, $\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(u-z_1;z_1,z_2) = 0$, and $\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(v-z_2;z_1,z_2) = 0$. Using the Taylor's theorem for $g \in C^2(I^2)$, we may write

$$\begin{aligned}
g(u,v) - g(z_1,z_2) &= \frac{\partial g(z_1,z_2)}{\partial z_1}(u-z_1) + \int_{z_1}^{u}(u-\eta)\frac{\partial^2 g(\eta,y)}{\partial \eta^2}d\eta\\
&\quad + \frac{\partial g(z_1,z_2)}{\partial z_2}(v-z_2) + \int_{z_2}^{v}(v-\xi)\frac{\partial^2 g(x,\xi)}{\partial \xi^2}d\xi.
\end{aligned} \tag{3.5}$$

Applying the operator $\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}$ on both sides of the above equation, we get

$$\begin{aligned}
&\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(g(u,v);z_1,z_2) - g(z_1,z_2)\\
&\quad = \widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}\left(\int_{z_1}^{u}(u-\eta)\frac{\partial^2 g(\eta,z_2)}{\partial \eta^2}d\eta;z_1,z_2\right) + \widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}\left(\int_{z_2}^{v}(v-\xi)\frac{\partial^2 g(z_1,\xi)}{\partial \xi^2}d\xi;z_1,z_2)\right)\\
&\quad = \mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}\left(\int_{z_1}^{u}(u-\eta)\frac{\partial^2 g(\eta,z_2)}{\partial \eta^2}d\eta;z_1,z_2\right) - \int_{z_1}^{u_{m_1}}(u_{m_1}-\eta)\frac{\partial^2 g(z_1,\eta)}{\partial \eta^2}d\eta
\end{aligned}$$

$$+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}\left(\int_{z_2}^{v}(v-\xi)\frac{\partial^2 g(\xi,z_1)}{\partial\xi^2}d\xi;z_1,z_2\right)-\int_{z_2}^{v_{m_2}}(v_{m_2}-\xi)\frac{\partial^2 g(z_1,\xi)}{\partial\xi^2}d\xi.$$

On the other hand, since

$$\begin{aligned}\left|\int_{z_1}^{u}(u-\eta)\frac{\partial^2 g(\eta,z_2)}{\partial\eta^2}d\eta\right| &\leqslant \left|\int_{z_1}^{u}|(u-\eta)|\left|\frac{\partial^2 g(\eta,z_2)}{\partial\eta^2}\right|d\eta\right|\\ &\leqslant \| g\|_{C^2(I^2)}\left|\int_{z_1}^{u}|u-\eta|\left|\frac{\partial^2 g(\eta,z_2)}{\partial\eta^2}\right|d\eta\right| \leqslant \| g\|_{C^2(I^2)}(u-z_2)^2,\end{aligned}$$

and analogously

$$\left|\int_{z_1}^{u_{m_1}}(u_{m_1}-\eta)\frac{\partial^2 g(\eta,z_2)}{\partial\eta^2}d\eta\right| \leqslant (u_{m_1}-z_1)^2\| g\|_{C^2(I^2)},$$

we conclude that

$$\begin{aligned}&\left|\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(g;z_1,z_2)-g(z_1,z_2))\right|\\ &\leqslant \mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}\left(\left|\int_{z_1}^{u}(u-\eta)\frac{\partial^2 g(\eta,z_2)}{\partial\eta^2}d\eta\right|;z_1,z_2)\right)\\ &\quad+\left|\int_{z_1}^{u_{m_1}}(u_{m_1}-\eta)\frac{\partial^2 g(z_1,\eta)}{\partial\eta^2}d\eta\right|+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}\left(\left|\int_{z_2}^{v}(v-\xi)\frac{\partial^2 g(\xi,z_1)}{\partial\xi^2}d\eta\right|;z_1,z_2\right)\\ &\quad+\left|\int_{z_2}^{v_{m_2}}(v_{m_2}-\xi)\frac{\partial^2 g(z_1,\xi)}{\partial\xi^2}d\xi\right| \leqslant \left\{\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}((u-z_1)^2;z_1,z_2)-(u_{m_1}-z_1)^2)\right\}\| g\|_{C^2_{(I^2)}}\\ &\quad+\left\{\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}}((v-z_2)^2;z_1,z_2)+(v_{m_2}-z_2)^2)\right\}\| g\|_{C^2_{(I^2)}}\\ &\leqslant \left\{\delta_{m_1}+\delta_{m_2}+(u_{m_1}-z_1)^2+(v_{m_2}-z_2)^2\right\}\| g\|_{C^2(I^2)}\\ &=\eta_{m_1,m_2}(h;q_{m_1},q_{m_2},z_1,z_2)\| g\|_{C^2(I^2)}.\end{aligned}\tag{3.6}$$

On using Equation (3.4), we see that

$$\left|\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)\right| \leqslant \left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)\right|+|h(z_1,z_2)|+|h(u_{m_1},v_{m_2})| \leqslant 3\| h\|_{C(I^2)}. \tag{3.7}$$

Hence in view of (3.6) and (3.7), we have

$$\begin{aligned}&\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right|\\ &=\left|\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)+h(u_{m_1},v_{m_2})-h(z_1,z_2)\right|\\ &\leqslant \left|\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(h-g;z_1,z_2)\right|+\left|\widehat{\mathfrak{D}}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}(g;z_1,z_2)-g(z_1,z_2)\right|\end{aligned}$$

$$
\begin{aligned}
&+|g(z_1,z_2)-h(z_1,z_2)|+|h(u_{m_1},v_{m_2})-h(z_1,z_2)|\\
&\leqslant 4\parallel h-g\parallel_{C(I_{ab})}+\left|\widehat{\mathfrak{D}}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(g;z_1,z_2)-g(z_1,z_2)\right|+|h(u_{m_1},v_{m_2})-h(z_1,z_2)|\\
&\leqslant\Big(4\parallel h-g\parallel_{C(I^2)}+\eta_{m_1,m_2}(h;q_{m_1},q_{m_2},z_1,z_2)\Big)\parallel g\parallel_{C(I^2)}+\omega\left(h;\sqrt{(u_{m_1}-z_1)^2+(v_{m_2}-z_2)^2}\right)\\
&\leqslant 4K\left(h;\eta_{m_1,m_2,p_1,p_2}(q_{m_1},q_{m_2},z_1,z_2)\right)+\omega\left(h;\sqrt{(u_{m_1}-z_1)^2+(v_{m_2}-z_2)^2}\right)\\
&\leqslant L\Big\{\omega_2\left(h;\sqrt{\eta_{m_1,m_2,p_1,p_2}(q_{m_1},q_{m_2},z_1,z_2)}\right)\\
&\quad+\min\{1,\eta_{m_1,m_2,p_1,p_2}(q_{m_1},q_{m_2},z_1,z_2)\}\parallel h\parallel_{C_{(I^2)}}\Big\}+\omega\left(h;\sqrt{(u_{m_1}-z_1)^2+(v_{m_2}-z_2)^2}\right).
\end{aligned}
$$

This completes the proof. □

Theorem 3.7. *If $h(z_1,z_2)$ has continuous partial derivatives $\frac{\partial h}{\partial z_1}$ and $\frac{\partial h}{\partial z_2}$, as well as $\omega^{(1)}(h'_{z_1};\delta)$ and $\omega^{(2)}(h'_{z_2};\delta)$ denotes the partial modules of continuity of $\frac{\partial h}{\partial z_1}$ and $\frac{\partial h}{\partial z_2}$, respectively, then holds the inequality*

$$
\begin{aligned}
\left|\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(h;z_1,z_2)-h(z_1,z_2)\right|&\leqslant M_1\lambda_{m_1}(z_1)+M_2\lambda_{m_2}(z_2)\\
&\quad+\omega^{(1)}(h'_{z_1};\delta_{m_1})(1+\sqrt{\delta_{m_1}})+\omega^{(2)}(h'_{z_1};\delta_{m_2})(1+\sqrt{\delta_{m_2}}),
\end{aligned}
$$

where M_1, M_2 are the positive constants such that $\left|\frac{\partial h}{\partial z_1}\right|\leqslant M_1$, $\left|\frac{\partial h}{\partial z_1}\right|\leqslant M_2$, $(0\leqslant z_1\leqslant a, 0\leqslant z_2\leqslant b)$, and $\lambda_{m_1}(z_1)=\left|\frac{q_{m_1}[m_1]_{q_{m_1}}-[m_1+2]_{q_{m_1}}}{[m_1+2]_{q_{m_1}}}\right|z_1+\frac{1}{[m_1+2]_{q_{m_1}}}$, $\lambda_{m_2}(z_2)=\left|\frac{q_{m_2}[m_2]_{q_{m_2}}-[m_2+2]_{q_{m_2}}}{[m_2+2]_{q_{m_2}}}\right|z_1+\frac{1}{[m_2+2]_{q_{m_2}}}$.

Proof. From the mean value theorem we have

$$
\begin{aligned}
h(t_1,t_2)-h(z_1,z_2)&=h(t_1,z_2)-h(z_1,z_2)+h(t_1,t_2)-h(t_1,z_2)\\
&=(t_1-z_1)\frac{\partial h(\xi_1,z_2)}{\partial z_1}+(t_2-z_2)\frac{\partial h(z_1,\xi_2)}{\partial z_2}\\
&=(t_1-z_1)\frac{\partial h(z_1,z_2)}{\partial z_1}+(t_1-z_1)\left(\frac{\partial h(\xi_1,z_2)}{\partial z_1}-\frac{\partial h(z_1,z_2)}{\partial z_1}\right)\\
&\quad+(t_2-z_2)\frac{\partial h(z_1,z_2)}{\partial z_2}+(t_2-z_2)\left(\frac{\partial h(z_1,\xi_2)}{\partial z_2}-\frac{\partial h(z_1,z_2)}{\partial z_2}\right),
\end{aligned}
$$

where $z_1<\xi_1<t_1$ and $z_2<\xi_2<t_2$. By using the above identity, we get

$$
\begin{aligned}
\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}(h;z_1,z_2)-h(z_1,z_2)&=\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left((t_1-z_1)\frac{\partial h(z_1,z_2)}{\partial z_1};z_1,z_2\right)\\
&\quad+\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_1]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left((t_1-z_1)\left(\frac{\partial h(\xi_1,z_2)}{\partial z_1}-\frac{\partial h(z_1,z_2)}{\partial z_1}\right);z_1,z_2\right)\\
&\quad+\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left((t_2-z_2)\frac{\partial h(z_1,z_2)}{\partial z_2};z_1,z_2\right)\\
&\quad+\mathfrak{D}^{\frac{1}{[m_1]_{q_{m_1}}}\frac{1}{[m_2]_{q_{m_2}}}}_{m_1,m_2,q_{m_1},q_{m_2}}\left((t_2-z_2)\left(\frac{\partial h(z_1,\xi_2)}{\partial z_2}-\frac{\partial h(z_1,z_2)}{\partial z_2}\right);z_1,z_2\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\right| \\
&\quad\leqslant \left|\frac{\partial h(z_1,z_2)}{\partial z_1}\right|\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_1-z_1);z_1,z_2)\right| \\
&\qquad+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}\left(|t_1-z_1|\left|\frac{\partial h(\xi_1,z_2)}{\partial z_1}-\frac{\partial h(z_1,z_2)}{\partial z_1}\right|;z_1,z_2\right) \\
&\qquad+\left|\frac{\partial h(z_1,z_2)}{\partial z_2}\right|\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_2-z_2);z_1,z_2)\right| \\
&\qquad+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}\left(|t_2-z_2|\left|\frac{\partial h(z_1,\xi_2)}{\partial z_2}-\frac{\partial h(z_1,z_2)}{\partial z_2}\right|;z_1,z_2\right) \\
&\quad\leqslant M_1\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_1-z_1);z_1,z_2)\right|+M_2\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_2-z_2);z_1,z_2)\right| \\
&\qquad+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_1]_{q_{m_2}}}}\left(|t_1-z_1|\,\omega^{(1)}(h'_{z_1};\delta_{m_1})\left(\frac{|t_1-z_1|}{\delta_{m_1}}+1\right);z_1,z_2\right) \\
&\qquad+\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}\left(|t_2-z_2|\,\omega^{(1)}(h'_{z_2};\delta_{m_2})\left(\frac{|t_2-z_2|}{\delta_{m_2}}+1\right);z_1,z_2\right),
\end{aligned}
$$

since $|\xi_1-z_1|<|t_1-z_1|$ and $|\xi_2-z_2|<|t_2-z_2|$. Using last inequalities, we have

$$
\begin{aligned}
&\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2) \\
&\quad\leqslant M_1\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_1-z_1);z_1,z_2)\right| \\
&\qquad+M_2\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_2-z_2);z_1,z_2)\right|+\omega^{(1)}(h'_{z_1};\delta_{m_1})\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}(|t_1-z_1|;z_1,z_2) \\
&\qquad+\frac{\omega^{(1)}(h'_{z_1};\delta_{m_1})}{\delta_{m_1}}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}\left(|t_1-z_1|^2;z_1,z_2\right)+\omega^{(2)}(h'_{z_2};\delta_{m_2})\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}(|t_2-z_2|;z_1,z_2) \\
&\qquad+\frac{\omega^{(2)}(h'_{z_2};\delta_{m_2})}{\delta_{m_2}}\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}\left(|t_2-z_2|^2;z_1,z_2\right).
\end{aligned}
$$

Now, applying the Cauchy-Schwarz inequality

$$
\begin{aligned}
\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}(h;z_1,z_2)-h(z_1,z_2)\leqslant{}& M_1\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_1-z_1);z_1,z_2)\right| \\
&+M_2\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]_{q_{m_1}}}\,\frac{1}{[m_2]_{q_{m_2}}}}((t_2-z_2);z_1,z_2)\right| \\
&+\omega^{(1)}(h'_{z_1};\delta_{m_1})\left\{\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}\left((t_1-z_1)^2;z_1,z_2\right)\right\}^{1/2} \\
&+\frac{\omega^{(1)}(h'_{z_1};\delta_{m_1})}{\delta_{m_1}}\mathfrak{D}_{m_1,q_{m_1}}^{\frac{1}{[m_1]_{q_{m_1}}}}\left((t_1-z_1)^2;z_1,z_2\right) \\
&+\omega^{(2)}(h'_{z_2};\delta_{m_2})\left\{\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]_{q_{m_2}}}}\left((t_2-z_2)^2;z_1,z_2\right)\right\}^{1/2}
\end{aligned}
$$

$$+\frac{\omega^{(2)}(h'_{z_2};\delta_{m_2})}{\delta_{m_2}}\mathfrak{D}_{m_2,q_{m_2}}^{\frac{1}{[m_2]q_{m_2}}}\left((t_2-z_2)^2;z_1,z_2\right).$$

By choosing $\delta_{m_1}=\delta_{m_1}(z_1)$ and $\delta_{m_2}=\delta_{m_2}(z_2)$, we have

$$\left|\mathfrak{D}_{m_1,m_2,q_{m_1},q_{m_2}}^{\frac{1}{[m_1]q_{m_1}}\frac{1}{[m_2]q_{m_2}}}(h;z_1,z_2)-h(z_1,z_2)\right|\leqslant M_1\lambda_{m_1}(z_1)+M_2\lambda_{m_2}(z_2)$$
$$+\omega^{(1)}(h'_{z_1};\delta_{m_1})(1+\sqrt{\delta_{m_1}})+\omega^{(2)}(h'_{z_2};\delta_{m_2})(1+\sqrt{\delta_{m_2}}).$$

This completes the proof. □

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