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On Tribonacci I-convergent sequence spaces

Vakeel A. Khan[a,*], Izhar Ali Khan[a], SK Ashadul Rahaman[a], Ayaz Ahmad[b]

[a] Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India.

[b] Department of Mathematics, National Institute of Technology, Patna-800005, India.

Abstract

In this paper, we use the notion of ideal convergence (I-convergence) to introduce Tribonacci I-convergent sequence spaces, that is, $c_0^I(T), c^I(T)$ and $l_\infty^I(T)$ as a domain of regular Tribonacci matrix $T = (t_{jn})$ (constructed by the Tribonacci sequence). We also present few inclusion relations and prove some topological and algebraic properties based results with respect to these spaces.

Keywords: Tribonacci sequence, regular Tribonacci matrix, Tribonacci I-convergence, Tribonacci I-Cauchy, Tribonacci I-bounded.

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1. Introduction

We use the notatios $\mathbb{R}$, ω, $\mathbb{N}$, l_∞, c, and c_0 to denote the set of real numbers, sequence space, set of natural numbers, space of all bounded sequences, space of all convergent sequences, and space of all null sequences, respectively.

Fast [7] and Steinhaus [24] introduced a generalization of usual convergence, known as statistical convergence. After that in 1999, Kostyrko et al. [22] defined a generalization of statistical convergence, known as I-convergence. Later, Ŝalát et al. [29, 30], Filipów and Tryba [9] and many others [17, 19] further studied the notion of I-convergence and linked with the summability theory. Furthermore, some authors also investigated it from the sequence space point of view. For more details on I-convergence, we refer to [2, 11, 13, 14, 18, 25, 27, 33].

Suppose an infinite real matrix $A = (a_{jn})$ and X & Y are two sequence spaces. Recalling, A be a matrix mapping from X to Y if $\forall z = (z_n)$, the A-transform of z, (i.e., $Az = \{A_j z\}_{j=1}^\infty \in Y$), is

$$A_j z = \sum_n a_{jn} z_n, \quad j \in \mathbb{N}.$$

In [15], a different approach was defined to construct new sequence spaces as follows

$$\lambda(A) := \{z = (z_n) \in \omega : Az \in \lambda\}, \quad \text{where } \lambda \text{ is any sequence space,}$$

and known as the domain of matrix A in λ.

*Corresponding author

Email address: vakhanmaths@gmail.com (Vakeel A. Khan)

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Lemma 1.1 ([34]). *Matrix $A = (a_{jn})_{j,n\in\mathbb{N}}$ is said to be regular iff the following conditions hold:*

(a) $\exists\, M > 0$ s.t $\sum_n |a_{jn}| \leqslant M, \forall j \in \mathbb{N}$;
(b) $\lim_{j\to\infty} a_{jn} = 0, \quad \forall n \in \mathbb{N}$;
(c) $\lim_{j\to\infty} \sum_n a_{jn} = 1$.

In 1963, Mark feinberg [8] was the first who initiated Tribobacci numbers at the age of 14 years. Define the tribonacci sequence by third order recurrence relation

$$t_j = t_{j-1} + t_{j-2} + t_{j-3}, \quad j \geqslant 4 \text{ with } t_1 = t_2 = 1 \text{ and } t_3 = 2.$$

Some important properties of Tribonacci sequence are:

$$\lim_{j\to\infty} \frac{t_j}{t_{j+1}} = 0.54368901\ldots, \qquad \sum_{n=1}^{j} t_n = \frac{t_{j+2} + t_j - 1}{2}, \quad j \geqslant 1.$$

Binet's formula for Tribonacci sequence is given in [32]. For more details, some papers related to Tribonacci sequence are [3–5, 8, 10, 20, 21, 23, 28, 31, 32, 35].

In this research paper, we use the triangle Tribonacci matrix $T = (t_{jn})$ which is defined in [36], as follows:

$$t_{jn} = \begin{cases} \frac{2t_n}{t_{j+2}+t_j-1}, & (1 \leqslant n \leqslant j), \\ 0, & (n > j). \end{cases} \tag{1.1}$$

i.e.,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & 0 & \cdots \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} & \frac{4}{15} & \frac{7}{15} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{1.2}$$

It can be easily verifed that T is a regular matrix (from Lemma 1.1). By using the Tribonacci matrix (1.2) and the notion of I-convergence, we define $c_0^I(T)$, $c^I(T)$, and $l_\infty^I(T)$ as the space of all sequences whose T-transform are in the spaces c_0^I, c^I, and l_∞^I, respectively. Sequence $T_j(z)$, the T-transform of $z = (z_n)$, is defined as

$$T_j(z) = \sum_{n=1}^{j} \frac{2t_n}{t_{j+2} + t_j - 1}, \quad j \in \mathbb{N}. \tag{1.3}$$

Now, recalling few important definitions and lemmas which are used in this paper.

Definition 1.2 ([24]). Natural density of $A = \{a \in A : a \leqslant n\} \subseteq \mathbb{N}$ is defined as

$$d(A) = \lim_{n\to\infty} \frac{1}{n}|A|, \text{ where } |A| \text{ is the cardinality of set } A,$$

whenever the limit exists.

Definition 1.3 ([7]). Statistically convergence of a sequence $(z_n) \in \omega$ to a number $k \in \mathbb{R}$ is defined as if $\forall \epsilon > 0$, the natural density of $A(\epsilon)$ is equal to zero, where

$$A(\epsilon) = \{n \in \mathbb{N} : |z_n - k| \geqslant \epsilon\}.$$

Definition 1.4 ([12]). Let X be any set. I being a subset of $P(X)$ is said to be an ideal if the following conditions holds:

a. $\emptyset \in I$;
b. $E \cup F \in I$ for all $E, F \in I$;
c. $\forall\, E \in I$ and $F \subset E$, then $F \in I$.

If $I \neq 2^X$, then I is called non trivial ideal. If $\{\{x\} : x \in X\} \subset I$, then I is called admissible ideal.

Definition 1.5 ([12]). Let $\mathcal{F}$ is the subset of power set of a set X, then $\mathcal{F}$ is called filter if

a. $\emptyset \notin \mathcal{F}$;
b. $E \cap F \in \mathcal{F}$ for all $E, F \in \mathcal{F}$;
c. $\forall\, E \in \mathcal{F}$ and $E \subset F$, then $F \in \mathcal{F}$.

Definition 1.6 ([22]). Let $I \subset P(\mathbb{N})$ is a non trivial ideal, I-convergence of a sequence $(z_n) \in \omega$ to number $k \in \mathbb{R}$ is defined as if $\forall\, \epsilon > 0$, set $A(\epsilon) \in I$, where

$$A(\epsilon) = \{n \in \mathbb{N} : \mid z_n - k \mid \geqslant \epsilon\},$$

we say that $I-\lim(z_n) = k$.

Definition 1.7 ([22]). Let $I \subset P(\mathbb{N})$ is a non trivial ideal, a sequence $(z_n) \in \omega$ is said to be I-Cauchy if $\forall$ $\epsilon > 0$, there exists a $K = K(\epsilon)$ such that set $A(\epsilon) \in I$, where

$$A(\epsilon) = \{n \in \mathbb{N} : \mid z_n - z_K \mid \geqslant \epsilon\}.$$

Definition 1.8 ([16]). Let $I \subset P(\mathbb{N})$ is a non trivial ideal, a sequence $(z_n) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that set $A \in I$, where

$$A = \{n \in \mathbb{N} : \mid z_n \mid > M\}.$$

Definition 1.9 ([29]). Let $I \subset P(\mathbb{N})$ is a non trivial ideal, for any two sequnces (y_n) and (z_n), we say $y_n = z_n$ for almost all n relative to I if $\{n \in \mathbb{N} : y_n \neq z_n\} \in I$.

Definition 1.10 ([29]). A sequence space X is said to be solid or normal, if $(\alpha_n z_n) \in X$ whenever $(z_n) \in X$ and for any sequence of scalars $(\alpha_n) \in \omega$ with $|\alpha_n| < 1$, for every $n \in \mathbb{N}$.

Lemma 1.11 ([29]). *Every solid space is monotone.*

Lemma 1.12 ([29]). *If $I \subset P(\mathbb{N})$ is a maximal ideal, then for every $K \subset \mathbb{N}$ we have either $K \in I$ or $\mathbb{N} \backslash K$.*

Definition 1.13 ([29]). Let $K = \{n_i \in \mathbb{N} : n_1 < n_2 < \cdots\} \subseteq \mathbb{N}$ and X be a sequence space. A K-step space of X is a sequence space

$$\lambda_K^X = \{(z_{n_i}) \in \omega : (z_n) \in X\}.$$

A canonical pre-image of a sequence $(z_{n_i}) \in \lambda_K^X$ is a sequence $(y_n) \in \omega$ defined as follows:

$$y_n = \begin{cases} z_n, & \text{if } n \in K, \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_K^X is a set of canonical pre-images of all elements in λ_K^X, i.e., y is in canonical pre-image of λ_K^X iff y is canonical pre-image of some element $z \in \lambda_K^X$.

Definition 1.14 ([29]). A sequence space X is said to be monotone, if it contains the canonical pre-images of its step space, (i.e., if for all infinite $K \subseteq \mathbb{N}$ and $(z_n) \in X$ the sequence $(\alpha_n z_n)$, where $\alpha_n = 1$ for $n \in K$ and $\alpha_n = 0$ otherwise, belongs to X).

Definition 1.15 ([29]). A sequence space X is said to be convergence free, if $(z_n) \in X$ whenever $(y_n) \in X$ and $(y_n) = 0$ implies that $(z_n) = 0$ for all $n \in \mathbb{N}$.

Definition 1.16. A map h defined on a domain $D \subset X$, i.e., $h : D \subset X \longrightarrow \mathbb{R}$ is said to satisfy Lipschitz condition if $|h(y) - h(z)| \leqslant M|y - z|$, where M is known as the Lipschitz constant.

2. Main results

By the domain of regular tribonacci matrix (1.1), we define sequence spaces $c_0^I(T)$, $c^I(T)$, and $l_\infty^I(T)$, i.e.,

$$\begin{aligned}
c_0^I(T) &:= \{z=(z_n)\in\omega : \{j\in\mathbb{N} : |T_j(z)|\geqslant\epsilon\}\in I\},\\
c^I(T) &:= \{z=(z_n)\in\omega : \{j\in\mathbb{N} : |T_j(z)-k|\geqslant\epsilon,\ \text{for some } k\in\mathbb{R}\}\in I\},\\
l_\infty^I(T) &:= \{z=(z_n)\in\omega : \text{there exist } M>0 \text{ such that } \{j\in\mathbb{N} : |T_j(z)|\geqslant M\}\in I\},\\
l_\infty(T) &:= \{z=(z_n)\in\omega : \sup_j |T_j(z)|<\infty\}.
\end{aligned}$$

We denote $c_0^I(T)\cap l_\infty(T)$ and $c^I(T)\cap l_\infty(T)$ by $s_0^I(T)$ and $s^I(T)$, respectively. We also present topological properties of these spaces and derive some results such as inclusion relations etc. Throughout this research paper, we assume that I be an admissible ideal in $\mathbb{N}$ and relation between the sequence $z=(z_n)\in\omega$ and $T_j(z)$ is same as given in equation (1.3).

Definition 2.1. A sequence $z=(z_n)\in\omega$ is said to be Tribonacci I-convergent to $k\in\mathbb{R}$ if for every $\epsilon>0$, the set A belongs to I, where

$$A=\{j\in\mathbb{N} : |T_j(z)-k|\geqslant\epsilon\}.$$

Definition 2.2. A sequence $z=(z_n)\in\omega$ is said to be Tribonacci I-Cauchy if for every $\epsilon>0$, there exists $K=K(\epsilon)\in\mathbb{N}$ such that the set A belongs to I, where

$$A=\{j\in\mathbb{N} : |T_j(z)-T_K(z)|\geqslant\epsilon\}$$

Definition 2.3. A sequence $z=(z_n)\in\omega$ is said to be Tribonacci I-bounded if there exists $M>0$ such that the set A belongs to I, where

$$A=\{j\in\mathbb{N} : |T_j(z)|>M\}.$$

Example 2.1. $c^{I_f}(T)=c(T)$, where $I_f=\{Y\subseteq\mathbb{N} : Y \text{ is finite}\}$ is an admissible ideal in $\mathbb{N}$, where $c(T)$ denotes the space of all Tribonacci convergent sequences.

Example 2.2. $c^{I_d}(T)=S(T)$, where $I_d=\{Y\subseteq\mathbb{N} : d(Y)=0\}$ is an admissible ideal in $\mathbb{N}$, where $d(Y)$ denotes the natural density of set Y and $S(T)$ denotes the space of all Tribonacci statistically convergent sequences, i.e.,

$$S(T)=\{z=(z_n)\in\omega : d(j\in\mathbb{N} : |T_j(z)-k|\geqslant\epsilon\})=0,\ \text{for some } k\in\mathbb{R}\}.$$

Remark 2.4. Tribonacci convergence $\implies$ Tribonacci statistically convergence since natural density of all finite subsets of $\mathbb{N}$ is zero. But the converse may not be hold.

Example 2.3. Let $z=(z_n)\in\omega$ such that

$$T_j(z)=\begin{cases}5, & \text{if j is a prime},\\ 0, & \text{otherwise}.\end{cases}$$

Let $k=0$ then clearly $T_j(z)$ is not convergent but it is $T_j(z)$ is statistically convergent to 0 as the natural density of set prime numbers is zero, i.e., $d(\{j\in\mathbb{N} : |T_j(z)-k|\geqslant\epsilon\})=0$. Hence $(z_n)\in S(T)$ but $(z_n)\notin c(T)$.

Theorem 2.5. *The sequence spaces* $c_0^I(T), c^I(T), l_\infty^I(T), s_0^I(T)$, *and* $s^I(T)$ *are linear spaces over* $\mathbb{R}$.

Proof. Suppose that a, b are scalars and $y = (y_n), z = (z_n) \in c^I(T)$, then for every $\epsilon > 0$, there exists $k_1, k_2 \in \mathbb{R}$ such that

$$\left\{j \in \mathbb{N} : |T_j(y) - k_1| \geqslant \frac{\epsilon}{2}\right\} \in I \text{ and } \left\{j \in \mathbb{N} : |T_j(z) - k_2| \geqslant \frac{\epsilon}{2}\right\} \in I.$$

And let

$$L_1 = \left\{j \in \mathbb{N} : |T_j(y) - k_1| < \frac{\epsilon}{2|a|}\right\} \in \mathcal{F}(I) \text{ and } L_2 = \left\{j \in \mathbb{N} : |T_j(z) - k_2| < \frac{\epsilon}{2|b|}\right\} \in \mathcal{F}(I),$$

be such that $L_1^c, L_2^c \in I$. Then

$$\begin{aligned} L_3 &= \{j \in \mathbb{N} : |T_j(ay + bz) - (ak_1 + bk_2)| < \epsilon\} \\ &\supseteq \left\{j \in \mathbb{N} : |T_j(y) - k_1| < \frac{\epsilon}{2|a|}\right\} \cap \left\{j \in \mathbb{N} : |T_j(z) - k_2| < \frac{\epsilon}{2|b|}\right\}. \end{aligned}$$

As in the above equation, the set on the right hand-side belongs to $\mathcal{F}(I)$. So $L_3^c \in I$, which implies that $(ay + bz) \in c^I(T)$. Hence, $c^I(T)$ is linear space. Similarly, we can prove for remaining given spaces. □

Theorem 2.6. *A sequence $z = (z_n) \in \omega$ is Tribonacci I-convergent iff for each $\epsilon > 0$, $\exists K = K(\epsilon) \in \mathbb{N}$ such that*

$$\{j \in \mathbb{N} : |T_j(z) - T_K(z)| < \epsilon\} \in \mathcal{F}(I). \tag{2.1}$$

Proof. Let $z = (z_n)$ is Tribonacci I-convergent to $k \in \mathbb{R}$, so for $\epsilon > 0$, the set

$$L_\epsilon = \left\{j \in \mathbb{N} : |T_j(z) - k| < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I).$$

We fix a natural number $K = K(\epsilon) \in L_\epsilon$. Then, for all $j \in L_\epsilon$

$$|T_j(z) - T_K(z)| \leqslant |T_j(z) - k| + |k - T_K(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, (2.1) holds. Conversely, let for all $\epsilon > 0$, (2.1) holds, then

$$M_\epsilon = \{j \in \mathbb{N} : T_j(z) \in [T_j(z) - \epsilon, T_j(z) + \epsilon]\} \in \mathcal{F}(I), \ \forall\ \epsilon > 0.$$

Let $P_\epsilon = [T_j(z) - \epsilon, T_j(z) + \epsilon]$. Fixing $\epsilon > 0$, then $M_\epsilon \in \mathcal{F}(I)$ and $M_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$. Thus $M_\epsilon \cap M_{\frac{\epsilon}{2}} \in \mathcal{F}(I)$, which implies that

$$P = P_\epsilon \cap P_{\frac{\epsilon}{2}} \neq \emptyset,$$

i.e.,

$$\{j \in \mathbb{N} : T_j(z) \in P\} \in \mathcal{F}(I).$$

Thus,

$$\operatorname{diam}(P) \leqslant \frac{1}{2}\operatorname{diam}(P_\epsilon),$$

where $\operatorname{diam}(P)$ is the length of interval P. Proceeding in this way, by induction we get a sequence of closed intervals $P_\epsilon = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_j \supseteq \cdots$ such that

$$\operatorname{diam}(I_j) \leqslant \frac{1}{2}\operatorname{diam}(I_{j-1}), \text{ for } j = (2, 3, 4, \dots)$$

and

$$\{j \in \mathbb{N} : T_j(z) \in I_j\} \in \mathcal{F}(I).$$

Hence, $\exists$ a number $k \in \cap_{j \in \mathbb{N}} I_j$ and it is common work to check that $k = I\text{-}\lim T_j(z)$. Hence, $z = (z_n)$ is Tribonacci I-convergent sequence. □

Theorem 2.7. *The inclusions* $c_0^I(T) \subset c^I(T) \subset l_\infty^I(T)$ *are strict.*

Proof. It can be easily seen that $c_0^I(T) \subset c^I(T)$. For strictness, take any constant sequence say $z = (z_n) = \alpha$ for all n, where α is any non-zero constant. Then $T_j(z) = \alpha$ for all j. Hence, it is obvious that $T_j(z) \in c^I$ but $T_j(z) \notin c_0^I$, i.e., $z \in c^I(T)$ but $z \notin c_0^I(T)$. Let $z = (z_n) \in c^I(T)$. Then there exists $k \in \mathbb{R}$ such that

$$\{j \in \mathbb{N} : |T_j(z) - k| \geqslant \epsilon\} \in I.$$

We have

$$|T_j(z)| = |T_j(z) - k + k| \leqslant |T_j(z) - k| + |k|.$$

Hence, it can be easily seen that the sequence $(z_n) \in l_\infty^I(T)$. For strictness, take the sequence $z = (z_n) \in \omega$ such that

$$T_j(z) = \begin{cases} \sqrt{j}, & \text{if } j = i^2, \text{ for } i \in \mathbb{N}, \\ 1, & \text{if } j \text{ is odd non-square}, \\ 0, & \text{if } j \text{ is even non-square}. \end{cases}$$

Hence, it is clear that $T_j(z) \in l_\infty^I$ but $T_j(z) \notin c^I$, i.e., $z \in l_\infty^I(T)$ but $z \notin c^I(T)$. This completes the proof. □

Remark 2.8. A Tribonacci bounded sequence is obviously Tribonacci I-bounded as $\emptyset \in I$. But converse part is not always true. For example, let $z = (z_n) \in \omega$ such that

$$T_j(z) = \begin{cases} j^2, & \text{if } j \text{ is prime}, \\ 0, & \text{otherwise}. \end{cases}$$

As $\{j \in \mathbb{N} : |T_j(z)| > 3\} \in I$. Hence, (z_n) is Tribonacci I-bounded but clearly, $T_j(z)$ is not a bounded sequence. Thus $z \in l_\infty^I(T)$ but $z \notin l_\infty(T)$.

Remark 2.9. Tribonacci convergent sequence is obviously Tribonacci I-convergent as I_f is a non-trivial admissible ideal But the converse part may not be always true. Let $z = (z_n) \in \omega$ such that

$$T_j(z) = \begin{cases} \sqrt{j}, & \text{if } j = i^2, \text{ for } i \in \mathbb{N} \\ 0, & \text{otherwise}. \end{cases}$$

Hence, (z_n) is Tribonacci I_d-convergent but not a Tribonacci convergent sequence as $T_j(z)$ is not convergent.

Theorem 2.10. *The sequence spaces* $s^I(T)$ *and* $s_0^I(T)$ *are Banach spaces normed by*

$$||z||_{A(T)} = \sup_j |T_j(z)|, \textit{ where } A \in \{s^I, s_0^I\}.$$

Proof. Take a Cauchy sequence $(z_n^{(i)})$ in $s^I(T) \subset l_\infty(T)$. Then $(z_n^{(i)})$ is convergent in $l_\infty(T)$ and $\lim_{i\to\infty} T_j^{(i)}(z) = T_j(z)$. Assume I-lim $T_j^{(i)}(z) = k_i$ for all $i \in \mathbb{N}$. Now if we prove that (1) $(k_i) \to k$ for some $k \in \mathbb{R}$; (2) I-lim $T_j(z) = k$, then the theorem will be proved.

(1) Since $(z_n^{(i)})$ is a Cauchy sequence, then for every $\epsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that

$$|T_j^{(i)}(z) - T_j^{(m)}(z)| < \frac{\epsilon}{3}, \text{ for all } i, m \geqslant j_0. \tag{2.2}$$

Now, suppose that L_i and L_m are the under-mentioned sets in I:

$$L_i = \left\{ j \in \mathbb{N} : |T_j^{(i)}(z) - k_i| \geqslant \frac{\epsilon}{3} \right\} \tag{2.3}$$

and

$$L_m = \left\{ j \in \mathbb{N} : |T_j^{(m)}(z) - k_m| \geqslant \frac{\epsilon}{3} \right\}. \tag{2.4}$$

Suppose that $i, m \geqslant j_0$ and $j \notin L_i \cap L_m$. By (2.2), (2.3), and (2.4) we have

$$|k_i - k_m| \leqslant |T_j^{(i)}(z) - k_i| + |T_j^{(m)}(z) - k_m| + |T_j^{(i)}(z) - T_j^{(m)}(z)| < \epsilon.$$

Hence, (k_i) is a Cauchy sequence in $\mathbb{R}$ and thus convergent say to k, that is, $\lim_{i\to\infty} k_i = k$.
(2) Suppose $\zeta > 0$, then we can get r_0 such that

$$|k_i - k| < \frac{\zeta}{3}, \text{ for all } i > r_0. \tag{2.5}$$

We have $(z_n^{(i)}) \to (z_n)$ as $i \to \infty$. Thus

$$|T_j^{(i)}(z) - T_j(z)| < \frac{\zeta}{3}, \text{ for all } i > r_0. \tag{2.6}$$

Since $T_j^{(m)}(z)$ is I-convergent to k_m, there exists $U \in I$ such that for all $j \notin U$, we have

$$|T_j^{(m)}(z) - k_m| < \frac{\zeta}{3}. \tag{2.7}$$

Without loss of generality, suppose $m > r_0$, then for each $j \notin U$, we have

$$|T_j(z) - k| \leqslant |T_j(z) - T_j^{(m)}(z)| + |T_j^{(m)}(z) - k_m| + |k_m - k| < \zeta$$

by (2.5), (2.6), and (2.7). Thus (z_n) is Trinonacci I-convergent to k. Hence the space $s^I(T)$ is a Banach space. Similarly, the other case can be proved. □

By Theorem 2.10, we have the following Theorem.

Theorem 2.11. *The spaces* $s^I(T)$ *and* $s_0^I(T)$ *are closed subspaces of* $l_\infty(T)$.

As $s^I(T) \subset l_\infty(T)$ and $s_0^I(T) \subset l_\infty(T)$ are strict and by Theorem 2.11, it is obvious to have following theorem.

Theorem 2.12. *The spaces* $s^I(T)$ *and* $s_0^I(T)$ *are nowhere dense subsets of* $l_\infty(T)$.

Theorem 2.13. *Let* $z = (z_n) \in \omega$. *If there exists a sequence* $y = (y_n) \in c^I(T)$ *such that* $T_j(z) = T_j(y)$ *for almost all* j *relative to* I, *then* $z \in c^I(T)$.

Proof. As we have given that $T_j(z) = T_j(y)$ for almost all j relative to I, that is,

$$\{j \in \mathbb{N} : T_j(z) \neq T_j(y)\} \in I.$$

And suppose $(y_n) \in c^I(T)$ and Tribonacci I–lim $y_n = k$. Then, $\forall \epsilon > 0$, the set

$$\{j \in \mathbb{N} : |T_j(y) - k| \geqslant \epsilon\} \in I.$$

As I is an admissible ideal, we have

$$\{j \in \mathbb{N} : |T_j(z) - k| \geqslant \epsilon\} \subseteq \{j \in \mathbb{N} : T_j(z) \neq T_j(y)\} \cup \{j \in \mathbb{N} : |T_j(y) - k| \geqslant \epsilon\}.$$

Hence, the result is proved. □

Theorem 2.14. *If* I *is not maximal ideal, then the space* $c^I(T)$ *is neither solid nor monotone.*

Proof. Consider a sequence $z = (z_n) \in \omega$ such that $T_j(z) = 1$ for all $j \in \mathbb{N}$, then $(z_n) \in c^I(T)$. Since I is not maximal, by Lemma 1.12 there exists a subset $K \subset \mathbb{N}$ such that $K \notin I$ and $K^c \notin I$. Now, define $y = (y_n)$ by

$$y_n = \begin{cases} z_n, & \text{if } n \in L, \\ 0, & \text{otherwise.} \end{cases}$$

Then (y_n) belongs to the canonical pre-image of the K-step space of $c^I(T)$. But $(y_n) \notin c^I(T)$. Thus $c^I(T)$ is not monotone. Hence, by Lemma 1.11 $c^I(T)$ is not solid. □

Theorem 2.15. *The spaces $c_0^I(T)$ and $s_0^I(T)$ are solid and monotone.*

Proof. For $c_0^I(T)$, let $z = (z_n) \in c_0^I(T)$. Then, for $\epsilon > 0$, we have

$$\{j \in \mathbb{N} : |T_j(z)| \geqslant \epsilon\} \in I. \tag{2.8}$$

Let $\alpha = (\alpha_n)$ be a sequence of scalars with $|\alpha| \leqslant 1$, $\forall n \in \mathbb{N}$. Then

$$|T_j(\alpha z)| = |\alpha T_j(z)| \leqslant |\alpha| \, |T_j(z)| \leqslant |T_j(z)|, \text{ for all } j \in \mathbb{N}.$$

Thus, from the above inequality and (2.8) we have

$$\{j \in \mathbb{N} : |T_j(\alpha z)| \geqslant \epsilon\} \subseteq \{j \in \mathbb{N} : |T_j(z)| \geqslant \epsilon\} \in I$$

implies that

$$\{j \in \mathbb{N} : |T_j(\alpha z)| \geqslant \epsilon\} \in I.$$

Hence, $(\alpha z_n) \in c_0^I(T)$. Therefore, the space $c_0^I(T)$ is solid, and hence by Lemma 1.11 the space $c_0^I(T)$ is monotone. Similarly, the remaining part can be proved. □

Theorem 2.16. *The sequence spaces $c^I(T)$ and $c_0^I(T)$ are not convergence free.*

Proof. Following example will be the proof of this theorem. □

Example 2.4. Let $I = I_d$. Consider (z_n),$(y_n) \in \omega$ such that $T_j(z) = \frac{1}{n}$ and $T_j(y) = n$, $\forall\, j \in \mathbb{N}$. Then (z_n) belongs to $c^I(T)$ and $c_0^I(T)$, but (y_n) does not belongs to $c^I(T)$ and $c_0^I(T)$. Hence the given spaces are not convergence free.

Theorem 2.17. *The sequence spaces $c_0^I(T)$ and $c^I(T)$ are sequence algebras.*

Proof. For $c_0^I(T)$, consider (z_n), $(y_n) \in c_0^I(T)$. Then

$$I\text{–}\lim\ T_j(y) = 0, \quad I\text{–}\lim\ T_j(z) = 0.$$

Thus,

$$I\text{–}\lim\ T_j(y \cdot z) = 0,$$

which implies that $(y_n \cdot z_n) \in c_0^I(T)$. Hence $c_0^I(T)$ is sequence algebra. Similarly, the remaining part can be established. □

Theorem 2.18. *The function $g : s^I(T) \to \mathbb{R}$ defined by $g(z) = |I\text{–}\lim T_j(z)|$, where $s^I(T) = l_\infty(T) \cap c^I(T)$, is a Lipschitz function and hence uniformly continuous.*

Proof. Firstly, we prove that the function is well defined. Let $y, z \in s^I(T)$, such that

$$y = z \Rightarrow I\text{–}\lim T_j(y) = I\text{–}\lim T_j(z) \Rightarrow |I\text{–}\lim T_j(y)| = |I\text{–}\lim T_j(z)| \Rightarrow g(y) = g(z).$$

Thus, g is well defined. Next, let $y = (y_n), z = (z_n) \in s^I(T_j)$, $y \neq z$. Then

$$B_1 = \left\{j \in \mathbb{N} : \left|T_j(y) - g(y)\right| \geqslant |y-z|_*\right\} \in I,$$
$$B_2 = \left\{j \in \mathbb{N} : \left|T_j(z) - g(z)\right| \geqslant |y-z|_*\right\} \in I,$$

where $|y-z|_* = \sup_j \left|T_j(y) - T_j(z)\right|$. Thus

$$C_1 = \left\{j \in \mathbb{N} : \left|T_j(y) - g(y)\right| < |y-z|_*\right\} \in \mathcal{F}(I)$$

and

$$C_2 = \left\{j \in \mathbb{N} : \left|T_j(z) - g(z)\right| < |y-z|_*\right\} \in \mathcal{F}(I).$$

Hence $C = C_1 \cap C_2 \in \mathcal{F}(I)$, so that C is non-empty set. Therefore choosing $j \in B$, we have

$$|g(y) - g(z)| \leqslant \left|g(y) - T_j(y)\right| + \left|T_j(y) - T_j(z)\right| + \left|T_j(z) - g(z)\right| \leqslant 3|y-z|_*.$$

Thus, g is Lipschitz function and hence it is uniformly continuous. □

Theorem 2.19. *If $y = (y_n), z = (z_n) \in s^I(T)$ with $T_j(y \cdot z) = T_j(y) \cdot T_j(z)$, then $(y \cdot z) \in s^I(T)$ and $g(y \cdot z) = g(y) \cdot g(z)$, where $g : s^I(T) \to \mathbb{R}$ is defined by $g(x) = \left|I\text{–}\lim T_j(x)\right|$.*

Proof. For $\epsilon > 0$,

$$A = \left\{j \in \mathbb{N} : \left|T_j(y) - g(y)\right| < \epsilon\right\} \in \mathcal{F}(I), \tag{2.9}$$

and

$$B = \left\{j \in \mathbb{N} : \left|T_j(z) - g(z)\right| < \epsilon\right\} \in \mathcal{F}(I), \tag{2.10}$$

where $\epsilon = |y-z|_* = \sup_j \left|T_j(y) - T_j(z)\right|$. Now, we have

$$\begin{aligned} \left|T_j(y \cdot z) - g(y)g(z)\right| &= \left|T_j(y)T_j(z) - T_j(y)g(z) + T_j(y)g(z) - g(y)g(z)\right| \\ &\leqslant \left|T_j(y)\right|\left|T_j(z) - g(z)\right| + |g(z)|\left|T_j(y) - g(y)\right|. \end{aligned} \tag{2.11}$$

As $s^I(T) \subseteq l_\infty(T)$, there exists an $M \in \mathbb{R}$ such that $|T_j(y)| < M$. Therefore, from the equations (2.9), (2.10), and (2.11), we have

$$\left|T_j(y \cdot z) - g(y)g(z)\right| = \left|T_j(y) \cdot T_j(z) - g(y)g(z)\right| \leqslant M\epsilon + |g(z)|\epsilon = \epsilon_1, \text{ (say)}$$

for all $j \in A \cap B \in \mathcal{F}(I)$. Hence $(y \cdot z) \in s^I(T)$ and $g(y \cdot z) = g(y) \cdot g(z)$. □

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References

[1] M. Aiyub, A. Esi, N. Subramanian, *Poisson Fibonacci binomial matrix on rough statistical convergence on triple sequences and its rate*, J. Intel. Fuzzy Syst., **36** (2019), 3439–3445.
[2] M. J. Bharathi, S. Velmurugan, N. Subramanian, R. Srikanth, *On Triple sequence space of Bernstein operator of rough I_λ-statistical convergence of weighted g(A)*, J. Intel. Fuzzy Syst., **36** (2019), 13–27. 1
[3] I. Bruce, *A modified Tribonacci sequence*, Fibonacci Quart., **22** (1984), 244–246. 1
[4] M. Catalani, *Identities for Tribonacci-related sequences*, (2002).
[5] E. Choi, *Modular Tribonacci numbers by matrix method*, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math., **20** (2013), 207–221. 1
[6] R. G. Cooke, *Infinite matrices and sequence spaces*, Dover, New York, (2014).
[7] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244. 1, 1.3
[8] M. Feinberg, *Fibonacci–Tribonacci*, Fibonacci Quart., **1** (1963), 71–74. 1, 1

[9] R. Filipów, J. Tryba, *Ideal convergence versus matrix summability*, Studia Math., **245** (2019), 101–127. 1
[10] F. T. Howard, *A Tribonacci Identity*, Fibonacci Quart., **39** (2001), 352–357. 1
[11] M. Kemal Ozdemir, A. Esi, N. Subramanian, *Rough convergence of Bernstein fuzzy I-convergent of χ_f^{3FI} and χ_f^{3FI} spaces defined by Orlicz function*, J. Intel. Fuzzy Syst., **37** (2019), 1–7. 1
[12] V. A. Khan, Yasmeen, H. Fatima, H. Altaf, Q.M.D. Lohani, *Intuitionistic fuzzy* I*-convergent sequence spaces defined by compact operator*, Cogent Math., **3** (2016), 8 pages. 1.4, 1.5
[13] V. A. Khan, K. M. A. S. Alshlool, S. A. A. Abdullah, *Spaces of ideal convergent sequences of bounded linear operators*, Numer. Funct. Anal. Optim., **39** (2018), 1278–1290. 1
[14] V. A. Khan, K. M. A. S. Alshlool, S. A. A. Abdullah, R. K. A. Rababah, A. Ahmad, *Some new classes of paranorm ideal convergent double sequences of σ-bounded variation over n-normed spaces*, Cogent Math. Stat., **5** (2018), 13 pages. 1
[15] V. A. Khan, K. M. A. S. Alshlool, M. Alam, *On Hilbert I-convergent sequence spaces*, J. Math. Comput. Sci., **20** (2020), 225–233. 1
[16] V. A. Khan, K. Ebadullah, *I-convergent difference sequence spaces defined by a sequence of moduli*, J. Math. Comput. Sci., **2** (2012), 265–273. 1.8
[17] V. A. Khan, N. Khan, *On Zweier I-convergent double sequence spaces*, Filomat, **30** (2016), 3361–3369. 1
[18] V. A. Khan, A. A. H. Makharesh, K. M. A. S. Alshlool, S. A. A. Abdullah, H. Fatima, *On fuzzy valued lacunary ideal convergent sequence spaces defined by a compact operator*, J. Intel. Fuzzy Syst., **35** (2018), 4849–4855. 1
[19] V. A. Khan, R. K. A. Rababah, K. M. A. S. Alshlool, S. A. A. Abdullah, A. Ahmad, *On Ideal convergence Fibonacci difference sequence spaces*, Adv. Difference Equ., **2018** (2018), 14 pages. 1
[20] E. Kiliç, *Tribonacci sequences with certain indices and their sums*, Ars. Combin., **86** (2008), 13–22. 1
[21] T. Koshy, *Fibonacci and Lucus Numbers with Applications*, Wiley, New York (2001). 1
[22] P. Kostyrko, M. Macaj, T. Ŝalát, *Statistical convergence and I-convergence*, Real Anal. Exch., (1999). 1, 1.6, 1.7
[23] A. Scott, T. Delaney, V. Hoggatt JR, *The Tribonacci sequence*, Fibonacci Quart., **15** (1977), 193–200. 1
[24] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math., **2** (1951), 73–74. 1, 1.2
[25] N. Subramanian, A. Esi, *On Triple sequence of Bernstein operator of weighted rough I_λ-convergence*, J. Class. Anal., **13** (2018), 45–62. 1
[26] N. Subramanian, A. Esi, *Triple Pascal Sequence Spaces*, Asian J. Math. Phys., **3** (2019), 13–22.
[27] N. Subramanian, A. Esi, V. A. Khan, *The Rough Intuitionistic Fuzzy Zweier Lacunary Ideal Convergence of Triple Sequence spaces*, J. Math. Stat., **14** (2018), 72–78. 1
[28] S. Pethe, *Some identities for Tribonacci sequences*, Fibonacci Quart., **26** (1988), 144–151. 1
[29] T. Šalát, B. C. Tripathy, M. Ziman, *On some properties of* I*-convergence*, Tatra Mt. Math. Publ., **28** (2004), 279–286. 1, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15
[30] T. Šalát, B. C. Tripathy, M. Ziman, *On I-convergence field*, Ital. J. Pure Appl. Math., **17** (2005), 45–54. 1
[31] D. V. Shah, *Some Tribonacci identities*, Math. Today, **27** (2011), 1–9. 1
[32] W. R. Spickerman, *Binet's formula for the Tribonacci sequence*, Fibonacci Quart., **20** (1982), 118–120. 1
[33] B. C. Tripathy, B. Hazarika, I*-convergent sequence spaces associated with multiplier sequences*, Math. Inequal. Appl., **11** (2008), 543–548. 1
[34] A. Wilansky, *Summability through functional analysis*, North-Holland Publishing Co., Amsterdam, (1984). 1.1
[35] C. C. Yalavigi, *Properties of Tribonacci numbers*, Fibonacci Quart., **10** (1972), 231–246. 1
[36] T. Yaying, B. Hazarika, *On sequence spaces defined by the domain of a regular tribonacci matrix*, Math. Slovaca, **70** (2020), 697–706. 1