J. Math. Computer Sci., 24 (2022), 186–200

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On quasi bi-slant Lorentzian submersions from LP-Sasakian manifolds

Rajendra Prasad[a], Fatemah Mofarreh[b], Abdul Haseeb[c,*], Sandeep Kumar Verma[a]

[a]*Department of Mathematics and Astronomy, University of Lucknow, Lucknow, India.*

[b]*Mathematical Science Department, Faculty of Science, Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia.*

[c]*Department of Mathematics, College of Science, Jazan University, Jazan-2097, Kingdom of Saudi Arabia.*

Abstract

At this work, quasi bi-slant Lorentzian submersions from LP-Sasakian manifolds onto Riemannian manifolds have been studied. Further, the geometry of leaves of the distributions, integrability conditions and totally geodesic conditions have also been discussed. Finally, we construct some examples of this setting.

Keywords: LP-Sasakian manifolds, slant submersions, Lorentzian submersions, quasi bi-slant Lorentzian submersions.

2020 MSC: 53C12, 53C15, 53C25, 53C50, 55D15.

1. Introduction

Differential geometry is one of the most popular branch of mathematics and physics from ancient days. There are several topics in differential geometry which have very important applications in both, mathematics and physics [2, 14, 20]. Immersions and submersions are some of them. The properties of Riemannian submersions become an interesting subject in complex geometry as well as in contact geometry.

The theory of Riemannian submersions was first established by O′Neill [24] and Gray [8]. In 1976, Watson [32] introduced almost Hermitian submersions within almost Hermitian manifolds. In 1985, Chinea [5] generalized the idea of almost Hermitian submersion to different sub-classes of the almost contact manifolds. There are so many important and interesting results about Riemannian and almost Hermitian submersion which are studied at [4, 6, 30]. Recently, slant submersions, semi-invariant submersions as well as semi-slant submersions from almost Hermitian manifolds on Riemannian manifolds have been studied in [21, 27, 28], respectively. Several types of Riemannian submersions between Riemannian manifolds endowed with various constructures were investigated by several geometers ([1, 3, 12, 13, 19, 26, 29]). In 2016, Sahin et al. [31] proved decomposition theorems for hemi-slant Riemannian submersions from Hermitian manifolds on Riemannian manifolds.

*Corresponding author

Email addresses: `rp.manpur@rediffmail.com` (Rajendra Prasad), `fyalmofarrah@pnu.edu.sa` (Fatemah Mofarreh), `malikhaseeb80@gmail.com`, `haseeb@jazanu.edu.sa` (Abdul Haseeb), `skverma1208@gmail.com` (Sandeep Kumar Verma)

doi: 10.22436/jmcs.024.03.01

Received: 2020-12-23 *Revised:* 2021-01-23 *Accepted:* 2021-01-24

Magid [16] and Falcitelli et al. [7], introduced the theory of Lorentzian submersions. Matsumoto [17] started the idea of LP-Sasakian manifolds, while in 1992, related subject is investigated by Mihai and Rosca [18]. Recently, Gunduzalp [9] and Gunduzalp and Sahin [10] studied paracontact and Lorentzian almost paracontact structures. Kumar et al. [15] defined and studied conformal semi-slant submersions from LP-Sasakian manifolds onto Riemannian manifolds. Very recently, Prasad et al. [23] introduced the concept of quasi bi-slant submersions from Kaehler manifold on the Riemannian manifold.

In this research we undertake our work as follows. In Section 2, we present several main informations relating to quasi bi-slant Lorentzian submersion. At Section 3, certain interesting outcomes on quasi bi-slant Lorentzian submersions from an LP-Sasakian manifold onto the Riemannian manifold are obtained and studied the geometry of leaves of distributions that are included at this submersion. In the same section, certain conditions are obtained of similar submersions to become totally geodesic. Finally, some non-trivial examples for such submersions have constructed.

2. Preliminaries

The n-dimension smooth manifold $\mathcal{M}$ admitting φ the $(1,1)$-tensor field, ζ : the structural vector field, η : the 1-form and g : the Lorentzian metric named the Lorentzian para Sasakian (in brief, LP-Sasakian) manifold [11, 25] satisfies:

$$\varphi^2 = I + \eta \otimes \zeta, \ \ \varphi \circ \zeta = 0, \ \ \eta \circ \varphi = 0, \tag{2.1}$$

$$\eta(\zeta) = -1, \ \ g(\cdot, \zeta) = \eta(\cdot), \tag{2.2}$$

$$g(\varphi\cdot, \varphi\cdot) = g + \eta \otimes \eta, \ \ g(\varphi\cdot, \cdot) = g(\cdot, \varphi\cdot), \tag{2.3}$$

$$\nabla\zeta = \varphi, \tag{2.4}$$

$$(\nabla_X \varphi)Y = \eta(Y)X + g(X, Y)\zeta + 2\eta(X)\eta(Y)\zeta, \tag{2.5}$$

choosing X, Y at $\mathcal{M}$, where ∇ denotes Levi-Civita connection respecting to Lorentzian metric g.

In the LP-Sasakian manifold, clearly

$$\operatorname{rank}(\varphi) = n - 1.$$

Now, in case

$$\Phi(X, Y) = \Phi(Y, X)$$

for all X, Y on $\mathcal{M}$, then Φ is called symmetric $(0, 2)$ tensor field, where $\Phi(X, Y) = g(X, \phi Y)$.

Lemma 2.1. *Suppose $\mathcal{W}$ is a subspace of dimension $\geqslant 1$ in the Lorentz vector space. Then the following are equivalent:*

1. *$\mathcal{W}$ is timelike, hence is itself a Lorentz vector space;*
2. *$\mathcal{W}$ includes two linearly independent null vectors;*
3. *$\mathcal{W}$ contains a timelike vector.*

Lemma 2.2. *Suppose $\mathcal{W}$ is a subspace of Lorentz vector space V and Suppose g is the metric (scalar product) of V, therefore the possible cases for $\mathcal{W}$ are:*

1. *$g\big|_{\mathcal{W}}$ is positive definite, then $\mathcal{W}$ is the inner product space;*
2. *$g\big|_{\mathcal{W}}$ is non-degenerate of index 1, therefore $\mathcal{W}$ is timelike;*
3. *$g\big|_{\mathcal{W}}$ is degenerate, therefore $\mathcal{W}$ is lightlike.*

Lemma 2.3. *Let $\mathcal{Z}$ be the subspace spanned by the timelike vector in Lorentz vector space V, therefore the subspace $\mathcal{Z}^{\perp}$ is spacelike and V is a direct sum of $\mathcal{Z}$ and $Z^{\perp}$.*

This argument shows, more generally, that the subspace $\mathcal{W}$ is timelike if and only if $\mathcal{W}^{\perp}$ is spacelike. Since $\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$.

$\mathcal{W}$ is lightlike if and only if $\mathcal{W}^{\perp}$ is lightlike.

Lemma 2.4. *For the subspace $\mathcal{W}$ of the Lorentz vector space, the coming statements are equivalent:*

1. *$\mathcal{W}$ is lightlike, that is, degenerate;*
2. *$\mathcal{W}$ includes the null vector but not timelike vector;*
3. *$\mathcal{W} \cap \mathcal{A} = \mathcal{L} - \mathcal{O}$, where $\mathcal{L}$ is the one dimensional subspace and $\mathcal{A}$ is the null cone of V, which means*

$$\mathcal{L} = \mathcal{W} \cap \mathcal{W}^{\perp}.$$

Note that we denote $(M, \varphi, \zeta, \eta, g_M)$: the almost contact metric manifold, $(\aleph, g_\aleph)$: the Riemannian manifold and $\ker h_*$: the vertical distribution of h in M. To use later, we recall the following definitions.

Definition 2.5 ([22])**.** The Riemannian submersion $h : (M, \varphi, \zeta, \eta, g_M) \to (\aleph, g_\aleph)$ is named an invariant Riemannian submersion in case

$$\varphi(\ker\ h_*) = \ker\ h_*.$$

Definition 2.6 ([19])**.** Suppose $h : (M, \varphi, \zeta, \eta, g_M) \to (\aleph, g_\aleph)$ is a Riemannian submersion such that $(\mathrm{in\ brief, s.t.})$ $\varphi(\ker\ h_*) \subseteq (\ker\ h_*)^{\perp}$. Therefore, h is called the anti-invariant Riemannian submersion.

Definition 2.7 ([1])**.** The Riemannian submersion $h : (M, \varphi, \zeta, \eta, g_M) \to (\aleph, g_\aleph)$ is called the semi-invariant Riemannian submersion in case there is the distribution $\mathfrak{D}_1 \subseteq \ker\ h_*$, s.t.,

$$\ker\ h_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus < \zeta >, \quad \text{and} \quad \varphi\left(\mathfrak{D}_1\right) = \mathfrak{D}_1, \varphi\left(\mathfrak{D}_2\right) \subseteq (\ker\ h_*)^{\perp},$$

where $\mathfrak{D}_2$ is orthogonal complementary distribution to $\mathfrak{D}_1$ at $\ker\ h_*$.

Suppose the complementary orthogonal subbundle to $\varphi(\ker\ h_*)$ in $(\ker\ h_*)^{\perp}$ is denoted by μ. Therefore we get

$$(\ker\ h_*)^{\perp} = \varphi\left(\mathfrak{D}_2\right) \oplus \mu.$$

Clearly, μ is the invariant subbundle of $(\ker\ h_*)^{\perp}$ respecting to the almost contact constructor φ.

Definition 2.8 ([9])**.** The Riemannian submersion $h : (M, \varphi, \zeta, \eta, g_M) \to (\aleph, g_\aleph)$ is called a slant submersion, in case for all $X(\neq 0) \in (\ker\ h_*)_p$, $p \in M$, the angle $\theta(X)$ within φX and the space $(\ker\ h_*)_p$ is constant. The angle θ is called the slant angle of the submersion and in case $\theta \in (0, \frac{\pi}{2})$, therefore h is named the proper slant submersion.

Definition 2.9 ([22])**.** The Riemannian map $h : (M, \varphi, \zeta, \eta, g_M) \to (\aleph, g_\aleph)$ named the semi-slant Riemannian map in case there are three orthogonal complementary distributions $\mathfrak{D}_1, \mathfrak{D}_2$ and $< \zeta >$ in $\ker h_*$, s.t.,

$$\ker h_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus < \zeta >, \ \varphi(\mathfrak{D}_1) = \mathfrak{D}_1,$$

and the angle $\theta = \theta(X)$ (called a semi-slant angle) between φX as well as the space $(\mathfrak{D}_2)_p$ is constant of $X(\neq 0) \in (\mathfrak{D}_2)_p$ for $p \in M$, where $\mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus < \zeta >$ is an orthogonal decomposition for $\ker h_*$.

Definition 2.10 ([31])**.** Suppose (M, g_M, J) is the almost Hermitian manifold and $(\aleph, g_\aleph)$ is the Riemannian manifold. The Riemannian submersion $h : (M, g_M, J) \to (\aleph, g_\aleph)$ named the hemi-slant submersion in case

$$\ker h_* = \mathfrak{D}^{\theta} \oplus \mathfrak{D}^{\perp}.$$

The distribution $\mathfrak{D}^{\theta}$ is slant with an angle θ (named a hemi-slant angle) and $\mathfrak{D}^{\perp}$ is anti-invariant.

Definition 2.11 ([9]). Suppose (M, g_M) be a Lorentzian manifold and $(\mathcal{B}, g_{\mathcal{B}})$ a Riemannian manifold. A Lorentzian submersion is a map $h : (M, g_M) \to (\mathcal{B}, g_{\mathcal{B}})$ which is onto and satisfies the following three conditions.

(A_1) h_{*p} is onto for all $p \in M$.
(A_2) The fibers $h^{-1}(b)$ are semi-Riemannian (Lorentzian) submanifolds of M for each $b \in \mathcal{B}$.
(A_3) h_* preserves scalar products of horizontal vectors.

Now, the concept of a quasi bi-slant Lorentzian submersion from LP-Sasakian manifolds onto Riemannian manifolds is introduced:

Definition 2.12. Suppose $(\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}})$ is the LP-Sasakian manifold as well as $(\aleph, g_{\aleph})$ is the Riemannian manifold. The Lorentzian submersion

$$h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$$

named the quasi bi-slant Lorentzian submersion in case there are four mutually orthogonal distributions D, D_1, D_2 and $< \zeta >$, s.t.,

(i) $\ker h_* = D \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} < \zeta >$;
(ii) $\varphi(D) = D$, which means D is invariant;
(iii) $\varphi(D_1) \perp D_2$ and $\varphi(D_2) \perp D_1$;
(iv) for any $X(\neq 0) \in (D_1)_p$, $p \in M$, the angle θ_1 within φX and $(D_1)_p$ is constant and independent of the choice of point p and X in $(D_1)_p$;
(v) for all $Z(\neq 0) \in (D_2)_q$, $q \in M$, the angle θ_2 within φZ and $(D_2)_q$ is constant and independent of the choice of point q and Z in $(D_2)_q$.

The angles θ_1 and θ_2 named slant angles of h, where D, D_1 and D_2 are spacelike subspaces and $\ker h_*$ is Lorentzian subspace.

Thus it is noted that:

(a) In case $\dim D \neq 0$ and $\dim D_1 = \dim D_2 = 0$, therefore h is invariant submersion.
(b) In case $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, therefore h is proper semi-slant submersion.
(c) In case $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, therefore h is slant submersion with slant angle θ_1.
(d) In case $\dim D = \dim D_1 = 0$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, therefore h is slant submersion with slant angle θ_2.
(e) In case $\dim D_1 \neq 0$, $\dim D = 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, therefore h is the anti-invariant submersion.
(f) In case $\dim D_1 \neq 0$, $\dim D \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $\dim D_2 = 0$, therefore h is semi-invariant submersion.
(g) In case $\dim D_1 \neq 0$, $\dim D = 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, therefore h is the hemi-slant submersion.
(h) In case $\dim D_1 \neq 0$, $\dim D = 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, therefore h is the bi-slant submersion.
(i) In case $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $\theta_2 = \frac{\pi}{2}$, therefore h can be called a quasi-hemi-slant submersion.
(j) In case $\dim D \neq 0, \dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, therefore h is proper quasi bi-slant submersion.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_E L = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}L + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}L, \tag{2.6}$$
$$\mathcal{T}_E L = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}L + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}L \tag{2.7}$$

for all vector fields E, L at $\mathcal{M}$, where ∇ defines Levi-Civita connection of $g_{\mathcal{M}}$. Clearly, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators at the tangent bundle of $\mathcal{M}$ reversing vertical and horizontal distributions. Using equations (2.6) and (2.7), results in

$$\nabla_X Y = \mathcal{T}_X Y + \mathcal{V}\nabla_X Y, \tag{2.8}$$
$$\nabla_X V = \mathcal{T}_X V + \mathcal{H}\nabla_X V, \tag{2.9}$$
$$\nabla_V X = \mathcal{A}_V X + \mathcal{V}\nabla_V X, \tag{2.10}$$
$$\nabla_V W = \mathcal{H}\nabla_V W + \mathcal{A}_V W, \tag{2.11}$$

for all $X, Y \in \Gamma(\ker h_*)$ and $V, W \in \Gamma(\ker h_*)^\perp$, where $\mathcal{H}\nabla_X V = \mathcal{A}_V X$, in case V is basic. It can be easily observed that $\mathcal{T}$ works at the fibers as the second fundamental form, where $\mathcal{A}$ works on horizontal distribution and measures obstruction to the integrability of the same distribution.

Clearly, for $q \in \mathcal{M}$, $U \in \mathcal{V}_q$ and $Z \in \mathcal{H}_q$

$$\mathcal{A}_U,\ \mathcal{T}_Z : T_q\mathcal{M} \to T_q\mathcal{M}$$

are skew-symmetric, such that

$$g_M(\mathcal{A}_U E, L) = -g_M(E, \mathcal{A}_U L) \text{ and } g_M(\mathcal{T}_Z E, L) = -g_M(E, \mathcal{T}_Z L)$$

for each $E, L \in T_q\mathcal{M}$. Since $\mathcal{T}_Z$ is skew-symmetric, therefore it is observed that h has totally geodesic fibres if and only if $\mathcal{T} \equiv 0$.

Definition 2.13. Let M and M' be two smooth manifolds. Let ∇ and ∇' be connections on M and M', respectively. A smooth map $h : M \to M'$ is called connection preserving map if

$$h_*(\nabla_X Y) = \nabla'_{h_* X}(h_* Y)$$

for all vector fields X, Y on M.

A smooth map $h : M \to M'$ is called geodesic preserving map if for each geodesic σ in M, $h \circ \sigma$ is geodesic in M'.

It is known that if a map is connection preserving then it is also the geodesic preserving. Geodesic preserving map is also called totally geodesic map.

We also know if M and M' be two smooth manifolds and h be a diffeomorphism from M onto M', then for a connection ∇' on M' there exist unique connection ∇ on M such that h is connection preserving map.

Suppose $(\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}})$ is an LP-Sasakian manifold, $(\aleph, g_\aleph)$ is the Riemannian manifold and $h : \mathcal{M} \to \aleph$ is a smooth map. Therefore the second fundamental form of h is

$$(\nabla h_*)(U, V) = \nabla^h_U h_* V - h_*(\nabla_U V), \quad \text{for } U, V \in \Gamma(T_p\mathcal{M}),$$

where ∇ denotes Levi-Civita connection of the metrices $g_{\mathcal{M}}$ and $g_\aleph$ and ∇^h is the pullback connection.

The differentiable map $h : \mathcal{M} \to \aleph$ is totally geodesic in case

$$(\nabla h_*)(U, V) = 0, \text{ for all } U, V \in \Gamma(TM).$$

Now the following lemma can be proved as in [3].

Lemma 2.14. *Suppose h is the Lorentzian submersion from the LP-Sasakian manifold $(\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}})$ on Riemannian manifold $(\aleph, g_\aleph)$, therefore we get*

(i) $(\nabla h_*)(V, W) = 0$;
(ii) $(\nabla h_*)(X, Z) = -h_*(\mathcal{T}_X Z) = -h_*(\nabla_X Z)$;
(iii) $(\nabla h_*)(V, X) = -h_*(\nabla_V X) = -h_*(\mathcal{A}_V X)$, *where V, W are horizontal vector fields and X, Z are vertical vector fields.*

3. Quasi Bi-Slant Lorentzian submersions

Throughout this section, we take $(\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}})$ be a LP-Sasakian manifold and $(\aleph, g_{\aleph})$ be a Riemannian manifold.

Suppose $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ is the quasi bi-slant Lorentzian submersion. Therefore, we get

$$T\mathcal{M} = \ker h_* \oplus_{\text{orth}} (\ker h_*)^{\perp}.$$

Here, for all vector field $Z \in \Gamma(\ker h_*)$, we choose

$$Z = PZ + QZ + RZ - \eta(Z)\zeta, \tag{3.1}$$

where P, Q and R indicates to the projection morphisms of $\ker h_*$ on D, D_1 and D_2, in the same order.

Choosing $Z \in \Gamma(\ker h_*)$, we set

$$\varphi Z = \psi Z + \omega Z, \tag{3.2}$$

where $\psi Z \in \Gamma(\ker h_*)$ and $\omega Z \in \Gamma(\omega D_1 \oplus \omega D_2)$. From (3.1) and (3.2), we get

$$\varphi Z = \psi(PZ) + \omega(PZ) + \psi(QZ) + \omega(QZ) + \psi(RZ) + \omega(RZ).$$

Since $\varphi D = D$, therefore $\omega PZ = 0$. Hence we obtain

$$\varphi Z = \psi(PZ) + \psi QZ + \omega QZ + \psi RZ + \omega RZ.$$

Thus we have

$$\varphi(\ker h_*) = D \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2),$$

where $\oplus$ defines orthogonal direct sum.

Moreover, Suppose $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$, therefore $g_M(V, W) = 0$. Now from the Definition 2.12 (iii), we have $g_M(\varphi V, W) = g_{\mathcal{M}}(V, \varphi W) = 0$. Now, we consider

$$g_{\mathcal{M}}(\psi V, W) = g_{\mathcal{M}}(\varphi V - \omega V, W) = g_{\mathcal{M}}(\varphi V, W), = 0.$$

In Similar way, we have $g_{\mathcal{M}}(V, \psi W) = 0$. Suppose $Z \in \Gamma(D)$ and $Y \in \Gamma(D_1)$. Therefore we get

$$g_{\mathcal{M}}(\psi Y, Z) = g_{\mathcal{M}}(\varphi Y - \omega Y, Z) = g_{\mathcal{M}}(\varphi Y, Z) = -g_{\mathcal{M}}(Y, \varphi Z) = 0,$$

as D is invariant, which means $\varphi Z \in \Gamma(D)$. Similarly, for $Z \in \Gamma(D)$ and $X \in \Gamma(D_2)$, we obtain $g_{\mathcal{M}}(\psi X, Z) = 0$. From above equations, we have

$$g_{\mathcal{M}}(\psi Z, \psi W) = 0, \quad \text{and} \quad g_{\mathcal{M}}(\omega Z, \omega W) = 0$$

for any $Z \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$. So, we can write $\psi D_1 \cap \psi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}$. If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and D_2 is anti-invariant, which means $\varphi(D_2) \subseteq (\ker h_*)^{\perp}$. Here we present D_2 as $D^{\perp}$. In addition, we have

$$\varphi(\ker h_*) = D \oplus \psi D_1 \oplus \omega D_1 \oplus \varphi D^{\perp},$$

where $\oplus$ defines orthogonal direct sum. Since $\omega D_1 \subseteq (\ker h_*)^{\perp}$, $\omega D_2 \subseteq (\ker h_*)^{\perp}$, so it is obtained that

$$(\ker h_*)^{\perp} = \omega D_1 \oplus \omega D_2 \oplus \mu,$$

where μ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ at $(\ker h_*)^{\perp}$. Also for all $V \in \Gamma(\ker h_*)^{\perp}$, we set

$$\varphi V = CV + BV, \tag{3.3}$$

where $CV \in \Gamma(\mu)$ and $BV \in \Gamma(\ker h_*)$.

$\mathrm{Span}\{\zeta\} = \langle\zeta\rangle$ determines timelike vector field distribution. In case the spacelike vector field X is orthogonal to ζ, therefore $g(\varphi X, \varphi X) = g(X, X) > 0$, thus φX is spacelike and hence ψX is also spacelike.

Wirtinger angle θ is written as

$$\cos\theta = \frac{g(\varphi X, \psi X)}{|\varphi X||\psi X|}.$$

Since $g|_{\ker h_*}$ is non-degenerate metric of index 1 at all points of $\mathcal{M}$, therefore $(\ker h_*)_x$ is timelike subspace of $T_x\mathcal{M}$ at any point of $\mathcal{M}$, and so $(\ker h_*)_x^\perp$ is spacelike subspace of $T_x\mathcal{M}$ at all points $x \in \mathcal{M}$.

Lemma 3.1. *Let $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ be the quasi bi-slant Lorentzian submersion. Therefore we got*

$$\psi^2 V + B\omega V = V + \eta(V)\zeta, \qquad \omega\psi V + C\omega V = 0, \qquad \omega B W + C^2 W = W, \qquad \psi B W + BCW = 0,$$

for all $V \in \Gamma(\ker h_*)$ and $W \in \Gamma(\ker h_*)^\perp$.

Proof. By making use of the equations (2.1), (3.2), and (3.3), Lemma 3.1 follows. □

Lemma 3.2. *Let $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ be the quasi bi-slant Lorentzian submersion. Therefore, we got*

(i) $\psi^2 V = (\cos^2\theta_1)V$,
(ii) $g_{\mathcal{M}}(\psi V, \psi W) = \cos^2\theta_1 g_{\mathcal{M}}(V, W)$,
(iii) $g_{\mathcal{M}}(\omega V, \omega W) = \sin^2\theta_1 g_{\mathcal{M}}(V, W)$,

for all $V, W \in \Gamma(D_1)$.

Proof.

(i) Let $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ be the quasi bi-slant Lorentzian submersion with the quasi bi-slant angle θ_1. Therefore, for $V(\neq 0) \in \Gamma(D_1)$, we have

$$\cos\theta_1 = \frac{|\psi V|}{|\varphi V|}, \tag{3.4}$$

and

$$\cos\theta_1 = \frac{g_{\mathcal{M}}(V, \psi V)}{|V||\psi V|}.$$

By making use of (2.1), (2.3), and (3.2), we have

$$\cos\theta_1 = \frac{g_{\mathcal{M}}(\psi V, \psi V)}{|\varphi V||\psi V|}, \tag{3.5}$$

$$\cos\theta_1 = \frac{g_{\mathcal{M}}(V, \psi^2 V)}{|\varphi V||\psi V|}.$$

From the equations (3.4) and (3.5), we get $\psi^2 V = (\cos^2\theta_1)V$, for $V \in \Gamma(D_1)$.

(ii) For all $V, W \in \Gamma(D_1)$, by the use of equations (2.1), (2.3), (3.2), and Lemma 3.2 (i), we have

$$g_{\mathcal{M}}(\psi V, \psi W) = g_{\mathcal{M}}(\varphi V - \omega V, \psi W) = g_{\mathcal{M}}(V, \psi^2 W) = \cos^2\theta_1 g_{\mathcal{M}}(V, W).$$

(iii) By using the equations (2.3), (3.2), and Lemma 3.2 (i) and (ii), Lemma 3.2 (iii) follows. □

Similarly, the coming Lemma is obtained.

Lemma 3.3. *Suppose $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ is the quasi bi-slant Lorentzian submersion. Therefore, we have*

(i) $\psi^2 Z = (\cos^2\theta_2)Z$;

(ii) $g_{\mathcal{M}}(\psi Z, \psi U) = \cos^2\theta_2 g_{\mathcal{M}}(Z, U)$;
(iii) $g_{\mathcal{M}}(\omega Z, \omega U) = \sin^2\theta_2 g_{\mathcal{M}}(Z, U)$;

for all $Z, U \in \Gamma(D_2)$.

Lemma 3.4. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the quasi bi-slant Lorentzian submersion. Therefore, we get*

$$\mathcal{V}\nabla_X\psi Y + \mathcal{T}_X\omega Y - \psi\mathcal{V}\nabla_X Y - B\mathcal{T}_X Y = g_{\mathcal{M}}(X, Y)\zeta + \eta(Y)X + 2\eta(X)\eta(Y)\zeta, \tag{3.6}$$
$$\mathcal{T}_X\psi Y + \mathcal{H}\nabla_X\omega Y = \omega\mathcal{V}\nabla_X Y + C\mathcal{T}_X Y, \tag{3.7}$$
$$\mathcal{V}\nabla_U BV + \mathcal{A}_U CV - g_{\mathcal{M}}(CU, V)\zeta = \psi\mathcal{A}_U V + B\mathcal{H}\nabla_U V, \tag{3.8}$$
$$\mathcal{A}_U BV + \mathcal{H}\nabla_U CV = \omega\mathcal{A}_U V + C\mathcal{H}\nabla_U V, \tag{3.9}$$
$$\mathcal{V}\nabla_X BU + \mathcal{T}_X CU = \psi\mathcal{T}_X U + B\mathcal{H}\nabla_X U, \tag{3.10}$$
$$\mathcal{T}_X BU + \mathcal{H}\nabla_X CU = \omega\mathcal{T}_X U + C\mathcal{H}\nabla_X U, \tag{3.11}$$
$$\mathcal{V}\nabla_V\psi X + \mathcal{A}_V\omega X = B\mathcal{A}_V X + \psi\mathcal{V}\nabla_V X, \tag{3.12}$$
$$\mathcal{A}_V\psi X + \mathcal{H}\nabla_V\omega X - \eta(X)V = C\mathcal{A}_V X + \omega\mathcal{V}\nabla_V X, \tag{3.13}$$

for all $X, Y \in \Gamma(\ker h_*)$ *and* $U, V \in \Gamma(\ker h_*)^{\perp}$.

Proof. Using equations (2.1), (2.2), (2.5), (2.8)-(2.11), we can easily get the equations (3.6)-(3.13). □

Now, we define

$$(\nabla_V\psi)W = \mathcal{V}\nabla_V\psi W - \psi\mathcal{V}\nabla_V W, \tag{3.14}$$
$$(\nabla_V\omega)W = \mathcal{H}\nabla_V\omega W - \omega\mathcal{V}\nabla_V W, \tag{3.15}$$
$$(\nabla_X C)Y = \mathcal{H}\nabla_X CY - C\mathcal{H}\nabla_X Y, \tag{3.16}$$
$$(\nabla_X B)Y = \mathcal{V}\nabla_X BY - B\mathcal{H}\nabla_X Y, \tag{3.17}$$

for all $V, W \in \Gamma(\ker h_*)$ and $X, Y \in \Gamma(\ker h_*)^{\perp}$.

Lemma 3.5. *Let* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *be the quasi bi-slant Lorentzian submersion. Therefore, we get*

$$\begin{aligned}
(\nabla_V\varphi)W &= B\mathcal{T}_V W - \mathcal{T}_V\omega W + g_{\mathcal{M}}(V, W)\zeta + 2\eta(V)\eta(W)\zeta + \eta(W)V,\\
(\nabla_V\omega)W &= C\mathcal{T}_V W - \mathcal{T}_V\psi W,\\
(\nabla_X C)Y &= \omega\mathcal{A}_X Y - \mathcal{A}_X BY,\\
(\nabla_X B)Y &= \psi\mathcal{A}_X Y - \mathcal{A}_X CY + g_{\mathcal{M}}(X, Y)\zeta,
\end{aligned}$$

for all $V, W \in \Gamma(\ker h_*)$ *and* $X, Y \in \Gamma(\ker h_*)^{\perp}$.

Proof. By the use of equations (3.6)-(3.9) and (3.14)-(3.17), Lemma 3.5 follows. □

Now, in case tensors φ and ω are parallel respecting to ∇ at $\mathcal{M}$, therefore

$$B\mathcal{T}_V W = \mathcal{T}_V\omega W - g_{\mathcal{M}}(V, W)\zeta - 2\eta(V)\eta(W)\zeta - \eta(W)V,$$

and

$$C\mathcal{T}_V W = \mathcal{T}_V\psi W$$

for all $V, W \in \Gamma(T\mathcal{M})$.

Theorem 3.6. *Let* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore, the invariant distribution* D *is integrable if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_X\varphi Y - \mathcal{T}_Y\varphi X, \omega QZ + \omega RZ) = -g_{\mathcal{M}}(\mathcal{V}\nabla_X\varphi Y - \mathcal{V}\nabla_Y\varphi X, \psi QZ + \psi RZ)$$

for all $X, Y \in \Gamma(D)$ *and* $Z \in \Gamma(D_1 \oplus D_2 \oplus <\zeta>)$.

Proof. For $X, Y \in \Gamma(D)$, and $Z \in \Gamma(D_1 \oplus D_2 \oplus <\zeta>)$, by the use of equations (2.1)-(2.5), (2.8), (3.1), and (3.2), we have

$$\begin{aligned} g_{\mathcal{M}}([X,Y],Z) &= g_{\mathcal{M}}(\nabla_X \varphi Y, \varphi Z) - g_{\mathcal{M}}(\nabla_Y \varphi X, \varphi Z) - \eta(Z)\eta(\nabla_X Y) + \eta(Z)\eta(\nabla_Y X), \\ &= g_{\mathcal{M}}(\nabla_X \varphi Y, \varphi Z) - g_{\mathcal{M}}(\nabla_Y \varphi X, \varphi Z), \\ &= g_{\mathcal{M}}(\mathcal{T}_X \varphi Y - \mathcal{T}_Y \varphi X, \omega RZ + \omega QZ) + g_{\mathcal{M}}(-\mathcal{V}\nabla_Y \varphi X + \mathcal{V}\nabla_X \varphi Y, \psi QZ + \psi RZ), \end{aligned}$$

this proof is completed. □

Theorem 3.7. *Let $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ is the proper quasi bi-slant Lorentzian submersion. Then the slant distribution D_1 is integrable if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_W \omega\psi Z - \mathcal{T}_Z \omega\psi W, U) = g_{\mathcal{M}}(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z, \varphi PU + \psi RU) + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z, \omega RU)$$

for all $Z, W \in \Gamma(D_1)$ as well as $U \in \Gamma(D \oplus D_2 \oplus <\zeta>)$.

Proof. For any $Z, W \in \Gamma(D_1)$ and $U \in \Gamma(D \oplus D_2 \oplus <\zeta>)$, we have

$$g_{\mathcal{M}}([Z,W],U) = g_{\mathcal{M}}(\nabla_Z W, U) - g_{\mathcal{M}}(\nabla_W Z, U).$$

By the use of equations (2.1)-(2.5), (2.8), (2.9), (3.1), and (3.2) and Lemma 3.2, it is obtained that

$$\begin{aligned} g_{\mathcal{M}}([Z,W],U) &= g_{\mathcal{M}}(\varphi\nabla_Z W, \varphi U) - g_{\mathcal{M}}(\varphi\nabla_W Z, \varphi U), \\ &= g_{\mathcal{M}}(\nabla_Z \varphi W, \varphi U) - g_{\mathcal{M}}(\nabla_W \varphi Z, \varphi U), \\ &= g_{\mathcal{M}}(\nabla_Z \psi W, \varphi U) + g_{\mathcal{M}}(\nabla_Z \omega W, \varphi U) - g_{\mathcal{M}}(\nabla_W \psi Z, \varphi U) - g_{\mathcal{M}}(\nabla_Z \omega W, \varphi U), \\ &= \cos^2\theta_1 g_{\mathcal{M}}(\nabla_Z W, U) - \cos^2\theta_1 g_{\mathcal{M}}(\nabla_W Z, U) + g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W - \mathcal{T}_W \omega\psi Z, U) \\ &\quad + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W + \mathcal{T}_Z \omega W, \varphi PU + \psi RU + \omega RU) \\ &\quad - g_{\mathcal{M}}(\mathcal{H}\nabla_W \omega Z + \mathcal{T}_W \omega Z, \varphi PU + \psi RU + \omega RU). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2\theta_1 g_{\mathcal{M}}([Z,W],U) &= g_{\mathcal{M}}(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z, \varphi PU + \psi RU) + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z, \omega RU) \\ &\quad + g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W - \mathcal{T}_W \omega\psi Z, U), \end{aligned}$$

This proof is completed. □

Similarly, the coming theorem is presented.

Theorem 3.8. *Let $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ is the proper quasi bi-slant Lorentzian submersion. Therefore the slant distribution D_2 is integrable if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_Y \omega\psi X - \mathcal{T}_X \omega\psi Y, Z) = g_{\mathcal{M}}(\mathcal{H}\nabla_X \omega Y - \mathcal{H}\nabla_Y \omega X, \omega QZ) + g_{\mathcal{M}}(\mathcal{T}_X \omega Y - \mathcal{T}_Y \omega X, \varphi PZ + \psi QZ)$$

for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D \oplus D_1 \oplus <\zeta>)$.

Proposition 3.9. *Suppose $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ is the proper quasi bi-slant Lorentzian submersion. Therefore the vertical distribution $(\ker h_*)$ does not determines the totally geodesic foliation at $\mathcal{M}$.*

Proof. Suppose we have $X \in \Gamma(\ker h_*)$ and $Z \in \Gamma(\ker h_*)^\perp$, by the use of (2.4) we get

$$g_{\mathcal{M}}(\nabla_X \zeta, Z) = g_{\mathcal{M}}(\varphi X, Z),$$

as $g_{\mathcal{M}}(\varphi X, Z) \neq 0$, so $g_{\mathcal{M}}(\nabla_X \zeta, Z) \neq 0$ for some X and Z. Hence, $(\ker h_*)$ is not defining a totally geodesic foliation at $\mathcal{M}$. □

Theorem 3.10. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore the distribution* $(\ker h_*) - <\zeta>$ *determines the totally geodesic foliation at* $\mathcal{M}$ *if and only if*

$$\begin{aligned} &g_{\mathcal{M}}(\mathcal{T}_U PV + \cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV, X) \\ &\quad = -g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi RV, X) g_{\mathcal{M}}(\mathcal{T}_U \omega V, BX) - g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega V, CX) \end{aligned}$$

for any $U, V \in \Gamma(\ker h_*) - <\zeta>$ *and* $X \in \Gamma(\ker h_*)^{\perp}$.

Proof. For all $U, V \in \Gamma(\ker h_*) - <\zeta>$ and $X \in \Gamma(\ker h_*)^{\perp}$, by the use of equations (2.2), (2.3), and (3.1), we have

$$g_{\mathcal{M}}(\nabla_U V, X) = g_{\mathcal{M}}(\nabla_U \varphi PV, \varphi X) + g_{\mathcal{M}}(\nabla_U \varphi QV, \varphi X) + g_{\mathcal{M}}(\nabla_U \varphi RV, \varphi X).$$

Using equations (2.3), (2.10), (2.11), (3.1), (3.2), Lemma 3.2, and Lemma 3.3, we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_U V, X) &= g_{\mathcal{M}}(\mathcal{T}_U PV, X) + \cos^2\theta_1 g_{\mathcal{M}}(\mathcal{T}_U QV, X) + \cos^2\theta_2 g_{\mathcal{M}}(\mathcal{T}_U RV, X) \\ &\quad + g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV, X) + g_{\mathcal{M}}(\nabla_U(\omega PV + \omega QV + \omega RV), \varphi X). \end{aligned}$$

Since $\omega PV + \omega QV + \omega RV = \omega V$ and $\omega PV = 0$, thus we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_U V, X) = g_{\mathcal{M}}(&\mathcal{T}_U PV + \cos^2\theta_1 \mathcal{T}_U QV + \cos^2\theta_2 \mathcal{T}_U RV, X) \\ &+ g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega\psi PV + \mathcal{H}\nabla_U \omega\psi QV + \mathcal{H}\nabla_U \omega\psi RV, X) \\ &+ g_{\mathcal{M}}(\mathcal{T}_U \omega V, BX) + g_{\mathcal{M}}(\mathcal{H}\nabla_U \omega V, CX), \end{aligned}$$

this proof is completed. □

Theorem 3.11. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore, the horizontal distribution* $(\ker h_*)^{\perp}$ *does not demonstrates the totally geodesic foliation at* $\mathcal{M}$.

Proof. Suppose $Z, V \in \Gamma(\ker h_*)^{\perp}$, and by the use of equation (2.4), we got

$$g_{\mathcal{M}}(\nabla_Z V, \zeta) = -g_{\mathcal{M}}(V, \nabla_Z \zeta) = -g_{\mathcal{M}}(V, \varphi Z),$$

as $g_{\mathcal{M}}(V, \varphi Z) \neq 0$, therefore $g_{\mathcal{M}}(\nabla_Z V, \zeta) \neq 0$ for some V and Z. Hence, $(\ker h_*)^{\perp}$ does not demonstrates a totally geodesic foliation at $\mathcal{M}$. □

Proposition 3.12. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution* D *does not demonstrates the totally geodesic foliation on* $\mathcal{M}$.

Proof. For all $U, V \in \Gamma(D)$, using equation (2.4), we got

$$g_{\mathcal{M}}(\nabla_U V, \zeta) = -g_{\mathcal{M}}(V, \varphi U),$$

since $g_{\mathcal{M}}(V, \varphi U) \neq 0$, so $g_{\mathcal{M}}(\nabla_U V, \zeta) \neq 0$ for some U and V. Hence D is not defining the totally geodesic foliation on $\mathcal{M}$. □

Theorem 3.13. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution* $D \oplus <\zeta>$ *demonstrates the totally geodesic foliation if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_X \varphi PY, \omega RZ + \omega QZ) = -g_{\mathcal{M}}(\mathcal{V}\nabla_X \varphi PY, \psi QZ + \psi RZ),$$

and

$$g_{\mathcal{M}}(\mathcal{T}_X \varphi PY, CV) = -g_{\mathcal{M}}(\mathcal{V}\nabla_X \varphi PY, BV),$$

for all $X, Y \in \Gamma(D \oplus <\zeta>)$, $Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$ *and* $V \in \Gamma(\ker h_*)^{\perp}$.

Proof. For all $X, Y \in \Gamma(D \oplus <\zeta>), Z = QZ + RZ \in \Gamma(D_1 \oplus D_2)$ and $V \in \Gamma(\ker h_*)^\perp$, the use of equations (2.1)-(2.5), (2.8), (3.1), and (3.2), gives

$$\begin{aligned} g_{\mathcal{M}}(\nabla_X Y, Z) = g_{\mathcal{M}}(\nabla_X \varphi Y, \varphi Z), &= g_{\mathcal{M}}(\nabla_X \varphi PY, \varphi QZ + \varphi RZ) \\ &= g_{\mathcal{M}}(\mathcal{T}_X \varphi PY, \omega RZ + \omega QZ) + g_{\mathcal{M}}(\mathcal{V}\nabla_X \varphi PY, \psi QZ + \psi RZ). \end{aligned}$$

Now, again the use of equations (2.1)-(2.5), (2.8), (3.1), and (3.3), leads to

$$g_{\mathcal{M}}(\nabla_X Y, V) = g_{\mathcal{M}}(\nabla_X \varphi Y, \varphi V), = g_{\mathcal{M}}(\nabla_X \varphi PY, BV + CV) = g_{\mathcal{M}}(\mathcal{V}\nabla_X \varphi PY, BV) + g_{\mathcal{M}}(\mathcal{T}_X \varphi PY, CV),$$

this proof is completed. □

Proposition 3.14. *Suppose $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_\aleph)$ is the proper quasi bi-slant Lorentzian submersion. Therefore the distribution D_i does not defines a totally geodesic foliation at $\mathcal{M}$, where $i = 1, 2$.*

Proof. For any $Z, V \in \Gamma(D_i)$, by the use of equation (2.4) we have

$$g_{\mathcal{M}}(\nabla_Z V, \zeta) = -g_{\mathcal{M}}(Z, \varphi V),$$

since $g_{\mathcal{M}}(Z, \varphi V) \neq 0$, so $g_{\mathcal{M}}(\nabla_Z V, \zeta) \neq 0$ for some V and Z. Hence D_i is not defining the totally geodesic foliation at $\mathcal{M}$, where $i = 1, 2$. □

Theorem 3.15. *Suppose $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_\aleph)$ is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution $D_1 \oplus <\zeta>$ demonstrates the totally geodesic foliation if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W, X) = -g_{\mathcal{M}}(\mathcal{T}_Z \omega W, \varphi PX + \psi RX) - g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, \omega RX) + \eta(W) g_{\mathcal{M}}(Z, \varphi PX + \psi RX),$$

and

$$g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega\psi W, V) = -g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, CV) - g_{\mathcal{M}}(\mathcal{T}_Z \omega W, BV) + \eta(W) g_{\mathcal{M}}(Z, BV),$$

for all $Z, W \in \Gamma(D_1 \oplus <\zeta>), X \in \Gamma(D \oplus D_2)$ and $V \in \Gamma(\ker h_*)^\perp$.

Proof. For every $Z, W \in \Gamma(D_1 \oplus <\zeta>), X \in \Gamma(D \oplus D_2)$ and $V \in \Gamma(\ker h_*)^\perp$, the use of equations (2.1)-(2.5), (2.9), (3.1), (3.2), and Lemma 3.2 gives

$$\begin{aligned} g_{\mathcal{M}}(\nabla_Z W, X) &= g_{\mathcal{M}}(\nabla_Z \varphi W, \varphi X) - \eta(W) g_{\mathcal{M}}(Z, \varphi X) \\ &= g_{\mathcal{M}}(\nabla_Z \psi W, \varphi X) + g_{\mathcal{M}}(\nabla_Z \omega W, \varphi X) - \eta(W) g_{\mathcal{M}}(Z, \varphi PX + \psi RX), \\ &= \cos^2\theta_1 g_{\mathcal{M}}(\nabla_Z W, X) + g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W, X) \\ &\quad + g_{\mathcal{M}}(\mathcal{T}_Z \omega W, \varphi PX + \psi RX) + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, \omega RX) - \eta(W) g_{\mathcal{M}}(Z, \varphi PX + \psi RX). \end{aligned}$$

Now, we have

$$\begin{aligned} \sin^2\theta_1 g_{\mathcal{M}}(\nabla_Z W, X) = g_{\mathcal{M}}(\mathcal{T}_Z \omega\psi W, X) &+ g_{\mathcal{M}}(\mathcal{T}_Z \omega W, \varphi PX + \psi RX) \\ &+ g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, \omega RX) - \eta(W) g_{\mathcal{M}}(Z, \varphi PX + \psi RX). \end{aligned}$$

Next, from equations (2.1)-(2.5), (2.9), (3.2), (3.3), and Lemma 3.2, we have

$$\begin{aligned} g_{\mathcal{M}}(\nabla_Z W, V) &= g_{\mathcal{M}}(\nabla_Z \varphi W, \varphi V) - \eta(W) g_{\mathcal{M}}(Z, \varphi V) \\ &= g_{\mathcal{M}}(\nabla_Z \psi W, \varphi V) + g_{\mathcal{M}}(\nabla_Z \omega W, \varphi V) - \eta(W) g_{\mathcal{M}}(Z, \varphi V) \\ &= \cos^2\theta_1 g_{\mathcal{M}}(\nabla_Z W, V) + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega\psi W, V) \\ &\quad + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, CV) + g_{\mathcal{M}}(\mathcal{T}_Z \omega W, BV) - \eta(W) g_{\mathcal{M}}(Z, BV). \end{aligned}$$

Now, we have

$$\sin^2\theta_1 g_{\mathcal{M}}(\nabla_Z W, V) = g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega\psi W, V) + g_{\mathcal{M}}(\mathcal{H}\nabla_Z \omega W, CV) + g_{\mathcal{M}}(\mathcal{T}_Z \omega W, BV) - \eta(W) g_{\mathcal{M}}(Z, BV),$$

this proof is completed. □

Similarly, we can easily prove the coming theorem.

Theorem 3.16. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution* $D_2 \oplus < \zeta >$ *demonstrates the totally geodesic foliation if and only if*

$$g_{\mathcal{M}}(\mathcal{T}_X \omega\psi Y, Z) = g_{\mathcal{M}}(\mathcal{T}_X \omega QY, \varphi PZ + \varphi RZ) + g_{\mathcal{M}}(\mathcal{H}\nabla_X \omega QY, \omega RZ) + \eta(Y) g_{\mathcal{M}}(X, \varphi PZ + \psi RZ),$$

and

$$g_{\mathcal{M}}(\mathcal{H}\nabla_X \omega\psi Y, V) = -g_{\mathcal{M}}(\mathcal{H}\nabla_X \omega Y, CV) - g_{\mathcal{M}}(\mathcal{T}_X \omega Y, BV) + \eta(Y) g_{\mathcal{M}}(X, BV),$$

for all $X, Y \in \Gamma(D_2 \oplus < \zeta >), Z \in \Gamma(D \oplus D_1)$ *and* $V \in \Gamma(\ker h_*)^{\perp}$.

By the use of Proposition 3.9 and Theorem 3.11 one can give the coming theorem.

Theorem 3.17. *Suppose* $h : (\mathcal{M}, \varphi, \zeta, \eta, g_{\mathcal{M}}) \to (\aleph, g_{\aleph})$ *is the proper quasi bi-slant Lorentzian submersion. Therefore, the map* h *is not a totally geodesic map.*

Example 3.18. Consider the differentiable manifold R^{11} with coordinates $(x^1, \ldots, x^5, y^1 \ldots.., y^5, z)$ and base field $\{E_i, E_{5+i}, \zeta\}$ where $E_i = 2\frac{\partial}{\partial y^i}, E_{5+i} = 2(\frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z})$, $i = 1, \ldots, 5$ and contravariant vector field $\zeta = 2\frac{\partial}{\partial z}$. Define Lorentzian almost paracontact structure on R^{11} as follows:

$$\varphi(\sum_{i=1}^{5}(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = -\sum_{i=1}^{5} Y_i \frac{\partial}{\partial x^i} - \sum_{i=1}^{5} X_i \frac{\partial}{\partial y^i} + \sum_{i=1}^{5} Y_i y^i \frac{\partial}{\partial z},$$

$$\eta = -\frac{1}{2}(dz - \sum_{i=1}^{5} y^i dx^i),$$

$$g = -\eta \otimes \eta + \frac{1}{4}(\sum_{i=1}^{5} dx^i \otimes dx^i + \sum_{i=1}^{5} dy^i \otimes dy^i),$$

where X_i, Y_i and Z are C^{∞} functions on R^{11}. Then $(R^{11}, \varphi, \zeta, \eta, g)$ is the LP-Sasakian manifold. Suppose R^4 is the Riemannian manifold with the Riemannian metric tensor field g_{R^4} defined as

$$g_{R^4} = \frac{1}{4}\sum_{i=1}^{4}(dv^i \otimes dv^i)$$

on R^4, where (v^1, v^2, v^3, v^4) is local coordinate system on R^4.

Let $h : R^{11} \to R^4$ is the map written as

$$h(x^1, \ldots, x^5, y^1 \ldots., y^5, z) = (x^2, \sin\theta_1 x^3 + \cos\theta_1 x^4, \sin\theta_2 y^1 - \cos\theta_2 y^2, y^4)$$

that is quasi bi-slant Lorentzian submersion map which satisfies

$$\bar{X}_1 = \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}, \qquad \bar{X}_2 = \cos\theta_1(\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}) - \sin\theta_1(\frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z}),$$

$$\bar{X}_3 = \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, \qquad \bar{X}_4 = \cos\theta_2 \frac{\partial}{\partial y^1} + \sin\theta_2 \frac{\partial}{\partial y^2},$$

$$\bar{X}_5 = \frac{\partial}{\partial y^3}, \qquad \bar{X}_6 = \frac{\partial}{\partial y^5},$$

$$\bar{X}_7 = \zeta = 2\frac{\partial}{\partial z}, \qquad (\ker h_*) = (D \oplus D_1 \oplus D_2 \oplus < \zeta >),$$

where

$$
\begin{aligned}
D &=< \bar{X}_3 = \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, & \bar{X}_6 &= \frac{\partial}{\partial y^5}, >, \\
D_1 &=< \bar{X}_2 = \cos\theta_1 (\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}) - \sin\theta_1 (\frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z}), & \bar{X}_5 &= \frac{\partial}{\partial y^3} >, \\
D_2 &=< \bar{X}_1 = \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}, & \bar{X}_4 &= \cos\theta_2 \frac{\partial}{\partial y^1} + \sin\theta_2 \frac{\partial}{\partial y^2} >, \\
< \zeta > &=< \bar{X}_7 = 2\frac{\partial}{\partial z} >,
\end{aligned}
$$

and

$$
\begin{aligned}
(\ker h_*)^{\perp} =< V_1 &= \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial z}, & V_2 &= \sin\theta_1 (\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}) + \cos\theta_1 (\frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z}), \\
V_3 &= \sin\theta_2 \frac{\partial}{\partial y^1} - \cos\theta_2 \frac{\partial}{\partial y^2}, & V_4 &= \frac{\partial}{\partial y^4} >,
\end{aligned}
$$

with bi-slant angles θ_1 and θ_2. Also by direct computations, we obtain

$$
h_* V_1 = \frac{\partial}{\partial v^1}, \; h_* V_2 = \frac{\partial}{\partial v^2}, \; h_* V_3 = \frac{\partial}{\partial v^3}, \; h_* V_4 = \frac{\partial}{\partial v^4}.
$$

Example 3.19. Consider R^{11} and R^4 has same structure as in Example 3.18. Suppose R^4 is the Riemannian manifold with the Riemannian metric tensor field g_{R^4} defined as

$$
g_{R^4} = \frac{1}{4} \sum_{i=1}^{4} (dv^i \otimes dv^i)
$$

on R^4, where (v^1, v^2, v^3, v^4) is local coordinate system on R^4. Let $h : R^{11} \to R^4$ be the map determined as

$$
h(x^1, \dots, x^5, y^1, \dots y^5, z) = (\frac{\sqrt{3}x^1 + x^2}{2}, x^4, y^1, \frac{y^3 - y^4}{\sqrt{2}})
$$

that is quasi bi-slant Lorentzian submersion map which satisfies

$$
\begin{aligned}
\bar{X}_1 &= (\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}) - \sqrt{3}(\frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial z}), & \bar{X}_2 &= \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}, & \bar{X}_3 &= \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, \\
\bar{X}_4 &= \frac{\partial}{\partial y^2}, & \bar{X}_5 &= (\frac{\partial}{\partial y^3} + \frac{\partial}{\partial y^4}), & \bar{X}_6 &= \frac{\partial}{\partial y^5}, \\
\bar{X}_7 &= \zeta = 2\frac{\partial}{\partial z}, & (\ker h_*) &= (D \oplus D_1 \oplus D_2 \oplus < \zeta >),
\end{aligned}
$$

where

$$
\begin{aligned}
D &=< \bar{X}_3 = \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, & \bar{X}_6 &= \frac{\partial}{\partial y^5} >, \\
D_1 &=< \bar{X}_1 = (\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}) - \sqrt{3}(\frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial z}), & \bar{X}_4 &= \frac{\partial}{\partial y^2} >, \\
D_2 &=< \bar{X}_5 = (\frac{\partial}{\partial y^3} + \frac{\partial}{\partial y^4}), & \bar{X}_2 &= \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z} >, \\
< \zeta > &=< \bar{X}_7 = 2\frac{\partial}{\partial z} >,
\end{aligned}
$$

and

$$(\ker h_*)^\perp =< V_1 = \sqrt{3}(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}) + (\frac{\partial}{\partial x^2} + y^2\frac{\partial}{\partial z}),\ \ V_2 = \frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial z},\ \ V_3 = \frac{\partial}{\partial y^1},\ \ V_4 = (\frac{\partial}{\partial y^3} - \frac{\partial}{\partial y^4}) >,$$

with bi-slant angles $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{\pi}{4}$. Also by direct computations, we obtain

$$h_*V_1 = 2\frac{\partial}{\partial v^1},\quad h_*V_2 = \frac{\partial}{\partial v^2},\quad h_*V_3 = \frac{\partial}{\partial v^3},\quad h_*V_4 = \sqrt{2}\frac{\partial}{\partial v^4}.$$

It can be easily seen that Theorem 3.11, and Propositions 3.12 and 3.14 are satisfied by the Examples 3.18 and 3.19.

Acknowledgment

This research was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program. The authors would also like to thank the referees for their helpful comments and suggestions.

References

[1] M. A. Akyol, R. Sarı, E. Aksoy, *Semi-invariant $\xi^\perp$-Riemannian submersions from almost contact metric manifolds*, Int. J. Geom. Methods Mod. Phys., **14** (2017), 17 pages. 1, 2.7
[2] C. Altafini, *Redundant robotic chains on Riemannian submersions*, IEEE Trans. Robot. Autom., **20** (2004), 335–340. 1
[3] P. Baird, J. C. Wood, *Harmonic Morphism between Riemannian Manifolds*, Oxford University Press, (2003). 1, 2
[4] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Birkhäuser Boston, Boston, (2002). 1
[5] D. Chinea, *Almost contact metric submersions*, Rend. Circ. Mat. Palermo, **34** (1985), 89–104. 1
[6] M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian submersions and related topics*, World Scientific, (2004). 1
[7] M. Falcitelli, S. Ianus, A. M. Pastore, *Riemannian Submersions and Related Topics*, World Scientific Publishing Co., River Edge, NJ, (2004). 1
[8] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech., **16** (1967), 715–738. 1
[9] Y. Gündüzalp, *Slant submersions from Lorentzian almost paracontact manifolds*, Gulf J. Math., **3** (2015), 18–28. 1, 2.8, 2.11
[10] Y. Gündüzalp, B. Şahin, *Paracontact semi-Riemannian submersions*, Turkish J. Math., **37** (2013), 114–128. 1
[11] A. Haseeb, R. Prasad, *Certain curvature conditions in Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection*, Int. J. Maps Math., **3** (2020), 85–99. 2
[12] S. Ianuş, A. M. Ionescu, R. Mocanu, G. E. Vîlcu, *Riemannian submersions from Almost contact metric manifolds*, Abh. Math. Semin. Univ. Hambg., **81** (2011), 101–114. 1
[13] S. Ianuş, R. Mazzocco, G. E. Vîlcu, *Riemannian submersion from quaternionic manifolds*, Acta Appl. Math., **104** (2008), 83–89. 1
[14] S. Ianus, M. Visinescu, *Space-time compactication and Riemannian submersions*, In: The mathematical heritage of C. F. Gauss, World Sci. Publ., River Edge, NJ, (1991). 1
[15] S. Kumar, R. Prasad, P. K. Singh, *Conformal semi-slant submersions from Lorentzian para Sasakian manifolds*, Commun. Korean Math. Soc., **34** (2019), 637–655. 1
[16] M. A. Magid, *Submersions from anti-de Sitter space with totally geodesic fibers*, J. Differential Geom., **16** (1981), 323–331. 1
[17] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci., **12** (1989), 151–156. 1
[18] I. Mihai, R. Roşca, *On Lorentzian P-Sasakian manifolds*, Classical Analysis, World Sci. Publ., River Edge, NJ, (1992). 1
[19] C. Murathan, I. K. Erkena, *Anti-invariant Riemannian submersions from cosymplectic manifolds onto Riemannian manifolds*, Filomat, **29** (2015), 1429–1444. 1, 2.6
[20] M. T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys., **41** (2000), 6918–6929. 1
[21] K.-S. Park, R. Prasad, *Semi-slant submersions*, Bull. Korean Math. Soc., **50** (2013), 951–962. 1
[22] R. Prasad, S. Kumar, *SSemi-slant Riemannian maps from almost contact metric manifolds into Riemannian manifolds*, Tbilisi Math. J., **11** (2018), 19–34. 2.5, 2.9
[23] R. Prasad, S. S. Shukla, S. Kumar, *On quasi-bi-slant submersions*, Mediterr. J. Math., **16**, (2019), 18 pages. 1
[24] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J., **33** (1966), 459–469. 1
[25] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, San Diego, (1983). 2
[26] B. Şahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Cent. Eur. J. Math., **8** (2010), 437–447. 1

[27] B. Şahin, *Slant submersions from Almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie, **54** (2011), 93–105. 1

[28] B. Şahin, *Semi-invariant submersions from almost Hermitian manifolds*, Canad. Math. Bull., **56** (2013), 173–183. 1

[29] B. Şahin, *Riemannian submersion from almost Hermitian manifolds*, Taiwanese J. Math., **17** (2013), 629–659. 1

[30] B. Şahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*, Elsevier/Academic Press, London, (2017). 1

[31] H. M. Taştan, B. Şahin, S. Yanan, *Hemi-slant submersions*, Mediterr. J. Math., **13** (2016), 2171–2184. 1, 2.10

[32] B. Watson, *Almost Hermitian submersions*, J. Differential Geometry, **11** (1976), 147–165. 1