Some new results of fixed point in dislocated quasi-metric spaces

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Abstract

In this paper, we introduce some new fixed point theorems in a dislocated quasi-metric space. We present several fixed point theorems, which generalize and improve some comparable fixed point results. Moreover, we provide some examples to illustrate our results.

Keywords: Fixed point, dislocated quasi-metric spaces, contraction mapping.


1. Introduction

The Banach contraction principle in metric space is the first important result in fixed point theory [4]. Since then, various generalizations have been made in many different forms with different types of spaces [1–3] and [5–7, 10–15]. Some of the well-known generalizations, with useful applications in logical programming and electronics engineering [9], are obtained in the framework of dislocated metric spaces [8], and dislocated quasi-metric spaces [15]. The present paper provides new generalizations of fixed point theorem in the setting of dislocated quasi-metric spaces, which generalize, improve, and fuse the results founded in [2, 12–14] by using a new contraction type and without any continuity requirement.

2. Preliminaries

We introduce here some basic concepts of the theory of dislocated quasi-metric spaces [15].

Definition 2.1. Let \(X\) be a nonempty set and \(d : X \times X \to \mathbb{R}^+\) be a function such that

1. \(d(x, y) = d(y, x) = 0\) implies \(x = y\);
2. \(d(x, y) \leq d(x, z) + d(z, y)\), for all \(x, y, z \in X\).

Then, \(d\) is called dislocated quasi-metric (or simply dq-metric) on \(X\).

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Clearly, for a dq-metric, the self distance of points need not to be zero necessarily and the usual property of symmetry is no longer valid. As example for dq-metric space, we can consider the set \( X = [0, 1] \) endowed with the following dq-metric
\[
d : X \times X \rightarrow \mathbb{R}^+,
\]
\[
d(x, y) = |x - y| + |x|.
\]

**Definition 2.2.** A sequence \( \{x_n\} \) in a dq-metric space \((X, d)\) is called a Cauchy sequence if for every \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that,
\[
d(x_m, x_n) \leq \epsilon \quad \text{or} \quad d(x_n, x_m) \leq \epsilon, \quad \forall \ m, n \geq N.
\]

**Definition 2.3.** A sequence \( \{x_n\} \) is said to be dq-convergent to \( x \) in a dq-metric space \( X \), if
\[
\lim_{n \to \infty} d(x_n, x) = \lim_{n \to \infty} d(x, x_n) = 0.
\]
Here, \( x \) is called dq-limit of sequence \( \{x_n\} \) and we write \( x_n \to x \), as \( n \to \infty \).

**Definition 2.4.** Let \((X, d_1)\) and \((Y, d_2)\) be two dq-metric spaces, the function \( f : X \rightarrow Y \) is said to be continuous if for each sequence \( \{x_n\} \subset X \) which dq-converges to \( x \) in \( X \), the sequence \( \{f(x_n)\} \) is dq-converges to \( f(x) \) in \( Y \).

**Definition 2.5.** A dq-metric space \( (X, d) \) is called complete if every Cauchy sequence in \( X \) is dq-convergent.

We remark here that, in dq-metric space, the dq-limit is unique and a Cauchy sequence which processes a dq-convergent subsequence, is also dq-converges. In the sequel, for simplicity, we omit the prefix “dq” to indicate the limit and convergence.

### 3. Main results

First, we start with the following lemma which we will use in the sequel.

**Lemma 3.1.** If \( x \) is a limit of some sequence \( \{x_n\} \) in a dq-metric space \((X, d)\), then
\[
d(x, x) = 0.
\]

**Proof.** Let \( x \in X \), and \( \{x_n\} \subset X \) a sequence which converges to \( x \). Then
\[
d(x, x) \leq d(x, x_n) + d(x_n, x), \quad \forall \ n \in \mathbb{N}.
\]
Passing to limit, when \( n \to \infty \), we obtain \( d(x, x) \leq 0 \), and therefore
\[
d(x, x) = 0. \quad \square
\]

Next, we state and prove our main fixed point result in complete dq-metric spaces. Unlike various papers which impose the contraction continuity condition [2], the following result provides the same result without continuity condition and under less restrictive condition.

**Theorem 3.2.** Let \((X, d)\) be a complete dq-metric space and \( T \) a self-mapping of \( X \) such that
\[
d(Tx, Ty) \leq \lambda \max \left\{ \frac{2d(x, y)}{d(x, y)} + \frac{2d(x, Ty) \cdot d(y, Tx)}{d(x, y)^2}, \frac{[d(x, Tx) + d(y, Ty)]}{d(x, y)} \right\},
\]
for all \( x, y \in X \) with \( d(x, y) \neq 0 \), and \( \lambda \in [0, \frac{1}{2}) \). Then, \( T \) has a unique fixed point in \( X \).
Proof. Assume \( T : X \to X \) verifies the condition (3.1), we consider
\[
M(x, y) = \max \left\{ 2d(x, y), \frac{2d(x, Tx) d(y, Ty)}{d(x, y)} - d(x, Tx) + d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} - d(x, Tx) + d(y, Ty), \frac{d(x, y)}{2} + d(y, Ty) \right\}.
\]
Then, we distinguish the following different cases.

\( \diamond \) **Case 1:** If \( M(x, y) = \frac{2d(x, Tx) d(y, Ty)}{d(x, y)} \), then
\[
d(Tx, Ty) \leq \lambda \frac{2d(x, Tx) d(y, Ty)}{d(x, y)}, \quad \forall \, x, y \in X.
\]
Taking \( y = Tx \), respectively \( x = Ty \), in the previous inequality, we find
\[
d(Tx, T^2x) \leq 2\lambda d(Tx, T^2x), \quad \forall \, x \in X, \tag{3.2}
\]
\[
d(T^2y, Ty) \leq 2\lambda d(Ty, T^2y), \quad \forall \, y \in X. \tag{3.3}
\]
Since \( 2\lambda \in [0, 1) \), the inequality (3.2) implies that \( d(Tx, T^2x) = 0 \), for all \( x \in X \), which leads also, by taking \( y = x \) in the inequality (3.3) to \( d(T^2x, Tx) = 0 \), for all \( x \in X \). Therefore, we conclude that \( T^2x = Tx \) and thus the mapping \( T \) has a fixed point.

\( \diamond \) **Case 2:** If \( M(x, y) = 2d(x, y) \), then
\[
d(Tx, Ty) \leq 2\lambda d(x, y), \quad \forall \, x, y \in X. \tag{3.4}
\]
We consider a Picard sequence \( x_{n+1} = Tx_n \) with initial guess \( x_0 \in X \). We will show that \( \{x_n\} \) is a Cauchy sequence in \( X \). For that, let \( n \in \mathbb{N}^* \) and use (3.4) to get
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq 2\lambda d(x_{n-1}, x_n) = h d(x_{n-1}, x_n).
\]
We reiterate this process to find \( d(x_n, x_{n+1}) \leq h^n d(x_0, x_1) \) and then we conclude
\[
d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \leq (h^n + h^{n+1} + \cdots + h^{m-1}) d(x_0, x_1) \leq \frac{h^n}{1-h} d(x_0, x_1).
\]
Since \( h = 2\lambda \in [0, 1) \), it follows from (3.5) that \( \{x_n\} \) is a Cauchy sequence in a complete dq-metric space, and therefore there exists \( u \in X \) such that
\[
\lim_{n \to \infty} x_n = u \quad \text{and} \quad \lim_{n \to \infty} Tx_n = u.
\]
From the inequality (3.4), we deduce
\[
d(Tx_n, Tu) \leq 2\lambda d(x_n, u), \quad \forall \, n \in \mathbb{N}.
\]
Then, since \( d(\cdot, Tu), d(\cdot, u) : X \to \mathbb{R} \) are continuous, we deduce
\[
d(u, Tu) = \lim_{n \to \infty} d(Tx_n, Tu) \leq 2\lambda \lim_{n \to \infty} d(x_n, u) = 2\lambda d(u, u).
\]
From Lemma 3.1, \( d(u, u) = 0 \) and thus \( d(u, Tu) = 0 \). On the other hand, we have
\[
d(Tu, u) \leq d(Tu, x_n) + d(x_n, u) = d(Tu, Tx_{n-1}) + d(x_n, u) \leq 2\lambda d(u, x_{n-1}) + d(x_n, u) \xrightarrow{n \to \infty} 0,
\]
which leads to \( d(Tu, u) = 0 \) and then \( d(u, Tu) = d(Tu, u) = 0 \). Therefore, we conclude that \( Tu = u \) and hence \( T \) has a fixed point.
We consider a sequence \( x_{n+1} = Tx_n \) with initial guess \( x_0 \in X \). We will prove that \( \{x_n\} \) is a Cauchy sequence. For that, we use (3.6) to deduce

\[
d(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1}) \\
\leq \lambda [d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})] \\
\leq \lambda d(x_n, x_n) + \lambda d(x_{n+1}, x_{n+1}), \quad \text{for } n \in \mathbb{N},
\]

which implies

\[
d(x_n, x_{n+1}) \leq \frac{\lambda}{1 - \lambda} d(x_n, x_n), \quad \text{for } n \in \mathbb{N}.
\]

Since \( h = \frac{\lambda}{1 - \lambda} \in [0, 1) \), then \( \{x_n\} \) is a Cauchy sequence in the complete space \( X \), and therefore there exists \( u \in X \) such that

\[
\lim_{n \to \infty} x_n = u \quad \text{and} \quad \lim_{n \to \infty} Tx_n = u.
\]

Next, to prove that \( u \) is the fixed point of \( T \), we use (3.6) to obtain

\[
d(Tx_n, Tu) \leq \lambda [d(x_n, Tx_n) + d(u, Tu)] \\
\leq \lambda [d(x_n, u) + d(u, Tx_n) + d(u, Tu)], \quad \forall n \in \mathbb{N}.
\]

Then, by passing to limit for in the above inequality, we find

\[
d(u, Tu) \leq \frac{2\lambda}{1 - \lambda} d(u, u).
\]

Using Lemma 3.1, we get \( d(u, Tu) = 0 \). On the other hand, we have

\[
d(Tu, u) \leq d(Tu, x_n) + d(x_n, u) = d(Tu, Tx_{n-1}) + d(x_n, u) \\
\leq \lambda [d(u, Tu) + d(x_{n-1}, Tx_{n-1})] + d(x_n, u) \\
\leq \lambda [d(x_{n-1}, u) + d(u, Tx_{n-1})] + d(x_n, u) \\
= \lambda [d(x_{n-1}, u) + d(u, x_n)] + d(x_n, u) 
\]

which implies that \( d(Tu, u) = 0 \) and then \( d(u, Tu) = d(Tu, u) = 0 \). Therefore, we conclude that \( Tu = u \) and hence \( T \) has a fixed point.

\[\diamond\text{ Case 4:}\] If \( M(x, y) = \frac{d(x, Ty) + d(y, Tx)}{2} \), then

\[
d(Tx, Ty) \leq \frac{\lambda}{2} [d(x, Ty) + d(y, Tx)], \quad \forall x, y \in X.
\] (3.7)

We consider a sequence \( x_{n+1} = Tx_n \) with initial guess \( x_0 \in X \). Then, we have

\[
d(x_n, x_{n+1}) = d(Tx_n, Tx_{n+1}) \leq \frac{\lambda}{2} [d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\
\leq \frac{\lambda}{2} [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\
\leq \frac{\lambda}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_n)],
\] (3.8)
for all \( n \in \mathbb{N}^* \). Moreover, we have
\[
d(x_n, x_n) = d(Tx_{n-1}, Tx_{n-1}) \leq \frac{\lambda}{2} [d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, Tx_{n-1})] \\
= \lambda d(x_{n-1}, x_n) \\
\leq d(x_{n-1}, x_n), \quad \forall \ n \in \mathbb{N}^*.
\] (3.9)

Then, we combine the two inequalities (3.8) and (3.9) to find
\[
d(x_n, x_{n+1}) \leq \frac{\lambda}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
= \frac{\lambda}{2} [2 d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \quad \forall \ n \in \mathbb{N}^*.
\]

Next, this inequality can be reformulated as follows
\[
d(x_n, x_{n+1}) \leq \frac{\lambda}{1 - \frac{\lambda}{2}} d(x_{n-1}, x_n), \quad \forall \ n \in \mathbb{N}^*.
\]

We can easily see that \( \lambda = \frac{1}{1 - \frac{\lambda}{2}} \in (0, 1) \), and then \( \{x_n\} \) is a Cauchy sequence in the complete space \( X \). Therefore, there exists \( u \in X \) such that
\[
\lim_{n \to \infty} x_n = u \quad \text{and} \quad \lim_{n \to \infty} Tx_n = u.
\]

From the inequality (3.7), we deduce
\[
d(Tx_n, Tu) \leq \frac{\lambda}{2} [d(x_n, Tu) + d(u, Tx_n)], \quad \forall \ n \in \mathbb{N}^*.
\]

Passing to limit, as \( n \to \infty \), we obtain
\[
d(u, Tu) \leq \frac{\lambda}{2} [d(u, Tu) + d(u, u)]. \tag{3.10}
\]

Using Lemma 3.1, it follows from (3.10) that
\[
d(u, Tu) \leq \frac{\lambda}{2} d(u, Tu),
\]
which implies that \( d(u, Tu) = 0 \), since \( \frac{\lambda}{2} \in [0, 1) \). On the other hand, we have
\[
d(Tu, u) \leq d(Tu, x_n) + d(x_n, u) = d(Tu, Tx_{n-1}) + d(x_n, u) \\
\leq \frac{\lambda}{2} [d(u, Tx_{n-1}) + d(x_{n-1}, Tu)] + d(x_n, u) \\
\leq \frac{\lambda}{2} [d(u, x_n) + d(x_{n-1}, u) + d(u, Tu)] + d(x_n, u) \\
= \frac{\lambda}{2} [d(u, x_n) + d(x_{n-1}, u)] + d(x_n, u) \xrightarrow{\ n \to \infty \ } 0,
\]

which implies that \( d(Tu, u) = 0 \) and then \( d(u, Tu) = d(Tu, u) = 0 \). Therefore, we conclude that \( Tu = u \) and hence \( T \) has a fixed point.

\textbf{Case 5: } If \( M(x, y) = d(x, Tx) + d(x, y) \), then
\[
d(Tx, Ty) \leq \lambda [d(x, Tx) + d(x, y)], \quad \forall \ x, y \in X.
\]
We define a sequence \( x_{n+1} = Tx_n \) with initial guess \( x_0 \in X \). Then, we have
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda [d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)]
\]
\[
= \lambda [d(x_{n-1}, x_n) + d(x_{n-1}, x_n)]
\]
\[
= 2\lambda d(x_{n-1}, x_n), \quad \forall \ n \in \mathbb{N}^*.
\]
Since \( 2\lambda \in [0, 1) \), \( \{x_n\} \) is a Cauchy sequence in the complete space \( X \). Then, there exists \( u \in X \) such that \( \{x_n\} \) and \( \{Tx_n\} \) converge to \( u \) in \( X \). Next, for all \( n \in \mathbb{N}^* \), we have
\[
d(Tx_n, Tu) \leq \lambda [d(x_n, Tx_n) + d(x_n, u)]
\]
\[
\leq \lambda [d(x_n, u) + d(u, Tx_n) + d(x_n, u)], \quad (3.11)
\]
\[
d(Tu, Tx_n) \leq \lambda [d(u, Tu) + d(u, x_n)]. \quad (3.12)
\]
We pass to limit in (3.11) to get \( d(u, Tu) = 0 \) and then in (3.12) to conclude
\[
d(u, Tu) = d(Tu, u) = 0.
\]
Therefore, we have \( Tu = u \) and thus \( u \) is a fixed point of \( T \).

\( \diamond \) **Case 6:** If \( M(x, y) = d(y, Ty) + d(x, y) \), then
\[
d(Tx, Ty) \leq \lambda [d(y, Ty) + d(x, y)], \quad \forall \ x, y \in X.
\]
We consider a sequence \( x_{n+1} = Tx_n \) with a given \( x_0 \in X \). Then, we have
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda [d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)]
\]
\[
\leq \lambda [d(x_{n-1}, x_n) + d(x_{n-1}, x_n)],
\]
for all \( n \in \mathbb{N}^* \). Then, for \( h = \frac{\lambda}{1-\lambda} \in [0, 1) \), we conclude that
\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n), \quad \forall \ n \in \mathbb{N}^*.
\]
Therefore, \( \{x_n\} \) is a Cauchy sequence in the complete space \( X \), and then there exists \( u \in X \) such that \( \{x_n\} \) and \( \{Tx_n\} \) converge to \( u \) in \( X \). Furthermore, we have
\[
d(Tu, Tx_n) \leq \lambda [d(x_n, Tx_n) + d(u, x_n)]
\]
\[
\leq \lambda [d(x_n, u) + d(u, Tx_n) + d(u, x_n)], \quad (3.13)
\]
\[
d(Tx_n, Tu) \leq \lambda [d(u, Tu) + d(x_n, u)]. \quad (3.14)
\]
We pass to limit in (3.13) to get \( d(Tu, u) = 0 \), and then in (3.14) to conclude
\[
d(u, Tu) \leq \lambda d(u, Tu).
\]
Therefore, since \( \lambda \in [0, 1) \), we deduce that \( d(u, Tu) = 0 \). Finally, we conclude that
\[
d(Tu, u) = d(u, Tu) = 0,
\]
which leads to \( Tu = u \) and thus the self-mapping \( T \) has a fixed point \( u \).

\( \diamond \) **Case 7:** If \( M(x, y) = \frac{2\lambda}{3} [d(x, Ty) + d(x, y)] \), then
\[
d(Tx, Ty) \leq \frac{2\lambda}{3} [d(x, Ty) + d(x, y)], \quad \forall \ x, y \in X. \quad (3.15)
\]
Let $x_0 \in X$ given, we consider a sequence $x_{n+1} = Tx_n$, then from (3.15), it follows
\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)
\leq \frac{2\lambda}{3} [d(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)]
= \frac{2\lambda}{3} [d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]
\leq \frac{2\lambda}{3} [d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_{n-1}, x_n)],
\]
for all $x, y \in X$ and $n \in \mathbb{N}^*$. This inequality implies that
\[
d(x_n, x_{n+1}) \leq \frac{4\lambda}{1 - \frac{2\lambda}{3}} d(x_{n-1}, x_n), \quad \forall x, y \in X. \tag{3.16}
\]
Since $h = \frac{4\lambda}{1 - \frac{2\lambda}{3}} \in [0, 1)$, the inequality (3.16) implies that \{x_n\} is a Cauchy sequence in the complete space $X$. Therefore, there exists $u \in X$ such that,
\[
\lim_{n \to \infty} x_n = u \quad \text{and} \quad \lim_{n \to \infty} Tx_n = u. \tag{3.17}
\]
Thus, the continuity of $d(\cdot, u)$, $d(\cdot, Tu) : X \to \mathbb{R}$ and (3.15) lead to
\[
d(u, Tu) = \lim_{n \to \infty} d(Tx_n, Tu)
\leq \lim_{n \to \infty} \frac{2\lambda}{3} [d(x_n, Tu) + d(x_n, u)]
= \frac{2\lambda}{3} d(u, Tu),
\]
which implies that $d(u, Tu) = 0$, since $\frac{2\lambda}{3} \in [0, 1)$. Moreover, we have
\[
d(Tu, Tx_n) \leq \frac{2\lambda}{3} [d(u, Tx_n) + d(u, x_n)]. \tag{3.18}
\]
Keeping in mind (3.17), we deduce that $d(Tu, u) = 0$ and finally, we conclude that
\[
d(u, Tu) = d(Tu, u) = 0,
\]
which implies that $Tu = u$, and hence $u$ is a fixed point of $T$. Therefore, the existence part of Theorem 3.2 has been established. For the uniqueness part, we consider two fixed points $u, v \in X$ of a self-mapping $T$ where $u$ is the unique limit of the Picard sequence $x_{n+1} = Tx_n$ with a given initial guess $x_0 \in X$. Then, it comes from (3.1) that
\[
d(Tu, Tv) \leq \lambda \max \left\{ \frac{2 d(u, v)}{d(u, Tu)} d(Tu, Tu) + d(u, v), \frac{d(u, v)}{d(v, Tv)} d(Tu, u) + d(u, v), \frac{d(u, v)}{d(v, Tv)} d(Tu, v) + d(u, v), \frac{d(u, v)}{d(v, Tv)} d(Tu, v) + d(u, v) \right\},
\]
Using Lemma 3.1 and the fact $\lambda \in [0, 1/2)$, we deduce the following results.

\begin{itemize}
  \item [\textcircled{1}] \textbf{Case 1}: Recalling that $1 - 2\lambda > 0$, then we have
    \[
d(u, v) = d(Tu, Tv) \leq 2\lambda d(u, v) \Rightarrow (1 - 2\lambda) d(u, v) \leq 0
    \Rightarrow d(u, v) = 0.
    \]
  \item [\textcircled{2}] \textbf{Case 2}: Keeping in mind that $d(u, u) = 0$, then we have
    \[
d(u, v) = d(Tu, Tv) \leq \frac{2\lambda d(u, Tu)}{d(u, v)} d(v, Tv) = \frac{2\lambda d(u, u) d(v, v)}{d(u, v)} = 0
    \Rightarrow d(u, v) = 0.
\end{itemize}
The assumptions of Theorem 3.2 are satisfied, and then 0 is the unique fixed point of a mapping $T$. We can show that

\[ d(u, v) = d(Tu, Tv) \leq \lambda [d(u, Tu) + d(v, Tv)] = \lambda [d(u, u) + d(u, v)] = 0 \]

\[ \Rightarrow (1 - \lambda) d(u, v) \leq 0 \]

\[ \Rightarrow d(u, v) = 0. \]

Case 4: Recalling that $\frac{\lambda}{1 - \lambda^2} \in [0, 1)$, then we have

\[ d(u, v) = d(Tu, Tv) \leq \lambda [d(u, Tu) + d(v, Tu)] = \lambda [d(u, u) + d(u, v)] \]

\[ \Rightarrow d(u, v) \leq \frac{\lambda}{1 - \lambda^2} d(v, u) \]

\[ \Rightarrow d(u, v) \leq \left( \frac{\lambda}{1 - \lambda^2} \right)^2 d(u, v) \]

\[ \Rightarrow d(u, v) = 0. \]

Case 5: Since $d(u, u) = 0$ and $1 - \lambda > 0$, then we have

\[ d(u, v) = d(Tu, Tv) \leq \lambda [d(u, Tu) + d(u, v)] = \lambda [d(u, u) + d(u, v)] \]

\[ \Rightarrow (1 - \lambda) d(u, v) \leq 0 \]

\[ \Rightarrow d(u, v) = 0. \]

Case 6: Recalling that $\frac{\lambda}{1 - 2\lambda} \in [0, 1)$, then we have

\[ d(u, v) = d(Tu, Tv) \leq \lambda [d(v, Tv) + d(u, v)] = \lambda [d(v, v) + d(u, v)] \]

\[ \leq \lambda [d(v, u) + d(u, v) + d(u, v)] \]

\[ \Rightarrow d(u, v) \leq \frac{\lambda}{1 - 2\lambda} d(v, u) \]

\[ \Rightarrow d(u, v) \leq \left( \frac{\lambda}{1 - 2\lambda} \right)^2 d(u, v) \]

\[ \Rightarrow d(u, v) = 0. \]

Case 7: Keeping in mind that $1 - \frac{4\lambda}{3} > 0$, then we have

\[ d(u, v) = d(Tu, Tv) \leq \frac{2\lambda}{3} [d(u, Tu) + d(u, v)] = \frac{2\lambda}{3} [d(u, u) + d(u, v)] \]

\[ \Rightarrow (1 - \frac{4\lambda}{3}) d(u, v) \leq 0 \]

\[ \Rightarrow d(u, v) = 0. \]

Hence, we have proved that in all the cases, that $d(u, v) = 0$. In addition, by using the same techniques, we can show that $d(v, u) = 0$, and therefore, we can conclude that $u = v$. \qed

Now, we illustrate our result by the following example.

Example 3.3. Consider the set $X = \{0, 10, \frac{1}{2}\}$ endowed with the metric $d$ defined by

\[ d(x, y) = x + 2y, \quad \forall x, y \in X. \]

We construct a self-mapping $T$ by $T(0) = 0$, $T(10) = \frac{1}{2}$ and $T(\frac{1}{2}) = 0$. For $\lambda = \frac{1}{3}$, we can easily see that all the assumptions of Theorem 3.2 are satisfied, and then 0 is the unique fixed point of a mapping $T$. 


As a consequence of Theorem 3.2, we may state the following corollary.

**Corollary 3.4 (12, Theorem 3.1).** Let \((X,d)\) be a complete dq-metric space and \(T : X \rightarrow X\) a continuous self-mapping. If the following condition holds

\[
d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(x, Tx) d(y, Ty)}{d(x, y)} + a_3 [d(x, Tx) + d(y, Ty)] + a_4 [d(x, Ty) + d(y, Tx)] + a_5 [d(x, Tx) + d(x, y)] + a_6 [d(y, Ty) + d(x, y)] + a_7 [d(x, Ty) + d(y, Tx)],
\]

for all \(x, y \in X\) with \(d(x, y) \neq 0\), and where \(\{a_i\}_{i=1, \ldots, 7} \subset \mathbb{R}^+\) satisfying

\[
0 < a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1,
\]

then, the self-mapping \(T\) has a unique fixed point.

Note here that the above corollary requires continuity of a mapping \(T\). In the next theorem, we provide a comparable result without any the continuity condition.

**Theorem 3.5.** Let \((X,d)\) be a complete dq-metric space and \(T : X \rightarrow X\) a self-mapping. If

\[
d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(x, Tx) d(y, Ty)}{d(x, y)} + a_3 [d(x, Tx) + d(y, Ty)] + a_4 [d(x, Ty) + d(y, Tx)] + a_5 [d(x, Tx) + d(x, y)] + a_6 [d(y, Ty) + d(x, y)] + a_7 [d(x, Ty) + d(y, Tx)],
\]

holds for all \(x, y \in X\) with \(d(x, y) \neq 0\), and where \(\{a_i\}_{i=1, \ldots, 7} \subset \mathbb{R}^+\) satisfying

\[
0 < a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 < 1,
\]

then, the self-mapping \(T\) has a unique fixed point.

**Proof.** Let \(T\) be self-mapping of \(X\) verifying assumptions of Theorem 3.5 and consider

\[
M(x, y) = \max \left\{ \frac{2d(x, y)}{d(x, y) + d(y, Ty)}, \frac{2d(x, Tx) d(y, Ty)}{d(x, y)}, d(x, Tx) + d(y, Ty), \frac{2d(x, Ty)}{d(x, y) + d(y, Ty)} \right\}.
\]

Using the inequality (3.18) and the definition (3.19) of \(M\), we find

\[
d(Tx, Ty) \leq \frac{a_1}{2} M(x, y) + \frac{a_2}{2} M(x, y) + a_3 M(x, y) + 2a_4 M(x, y) + a_5 M(x, y) + a_6 M(x, y) + \frac{3a_7}{2} M(x, y).
\]

We set \(\lambda = \frac{a_1}{2} + \frac{a_2}{2} + a_3 + 2a_4 + a_5 + a_6 + \frac{3a_7}{2} \in [0, \frac{1}{2}]\), then, the previous inequality implies

\[
d(Tx, Ty) \leq \lambda M(x, y), \quad \text{with} \quad \lambda \in [0, 1).
\]

Hence, Theorem 3.2 concludes the proof of Theorem 3.5.
We now provide another result, which generalizes our main result stated Theorem 3.2.

**Theorem 3.6.** Let \((X, d)\) be a complete dq-metric space and \(T\) a self-mapping of \(X\) such that
\[
d(Tx, Ty) \leq \lambda \max \left\{ \frac{\alpha d(x, y)}{d(x, y)} \cdot \frac{d(x, Tx) + d(y, Ty)}{d(x, y)}, \frac{d(x, Ty) + d(y, Tx)}{d(x, y)} + \frac{\alpha d(x, y)}{d(x, y)} \cdot \frac{d(x, y)}{d(x, y)} \right\},
\]
for all \(x, y \in X\) with \(d(x, y) \neq 0\), \(\lambda \in (0, \frac{1}{\alpha})\) and \(\alpha \geq 2\). Then, \(T\) has a unique fixed point in \(X\).

**Proof.** Theorem 3.6 can be proved in a similar way to the proof of Theorem 3.2.

Here, an illustrative example for which our main result Theorem 3.6 is applicable.

**Example 3.7.** Consider the set \(X = \{0, \frac{1}{7}, 30\}\) endowed with the following dq-metric
\[
d(x, y) = x + 2y, \quad \forall x, y \in X.
\]
Next, we construct a self-mapping \(T\) given by \(T(0) = 0\), \(T(30) = \frac{1}{7}\) and \(T(\frac{1}{7}) = 0\). For \(\lambda = \frac{1}{3}\) and \(\alpha = 3\), we can see that all the assumptions of Theorem 3.6 are satisfied, and hence 0 is the unique fixed point of the mapping \(T\).

4. Conclusion

In this paper, we gave some new results of fixed point theorems in complete dislocated quasi-metric space. These results generalize the results founded in [12]. However, their proof of their results seems not to be correct, see page 4698. More precisely, the authors have considered the inequality
\[
d(\xi_n, \xi_n) \leq d(\xi_{n-1}, \xi_n) + d(\xi_n, \xi_{n+1}),
\]
which is not always true in dq-metric spaces. As example, we consider the set \(X = \{0, 1, 2\}\) and the mapping \(d : X \times X \to \mathbb{R}^+\) defined by
\[
d(0, 0) = 0, \quad d(1, 1) = 7, \quad d(2, 2) = 6,
\]
\[
d(0, 1) = 5, \quad d(1, 0) = 2, \quad d(1, 2) = 3,
\]
\[
d(2, 1) = 4, \quad d(0, 2) = 1, \quad d(2, 0) = 5.
\]
We can easily verify that \(d\) is a dq-metric on \(X\) for which the previous inequality is not always valid, as we can see for \(\xi_{n-1} = 2, \xi_n = 1\) and \(\xi_{n+1} = 0\).

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References


